Department of Mathematics
University of Utah

Attainability of the singular Wiener bound and leaf venation patterns

Presented by Zhonggan Huang

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Contact: `zhonggan@math.utah.edu`

Outline

Leaf venation patterns I

Leaf veins are vital transport media in trees. A leaf venation pattern is the arrangement of leaf veins within a leaf. It usually has a hierarchical structure, with the veins being classified by *lower-order* (major) veins or *higher-order* (minor) ones [2, 3].

Figure: Swamp Cottonwood Vein Pattern

Why does the higher-order leaf vein pattern contain so many redundant loops (also called reticulate pattern)?

Leaf venation patterns II

Biologists believe that the spatial and temporal hydraulic fluctuations lead to the reticulate patterns, but currently no math explanations.

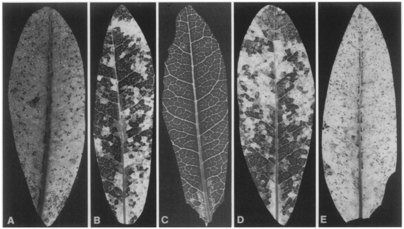

Figure: Backlit lower leaf surface of *Arbutus unedo* of different times of a day [1]. (Light area: infiltrated area; dark area: non-infiltrated area. A: 9 a.m.; B: 10 a.m.; C: 12 p.m.; D: 4 p.m.; E: 5 p.m.)

A meme picture

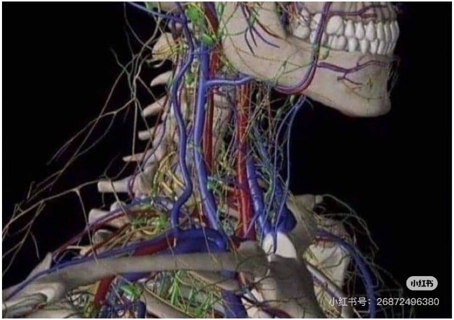

Figure: It looks nasty!

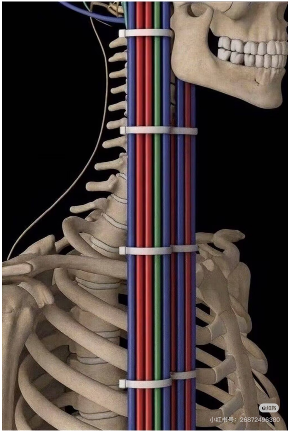

Figure: Much better!

Ideal leaf venation pattern

Figure: Higher-order vein pattern

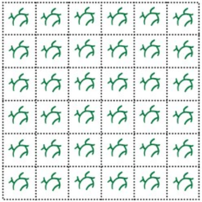

Figure: An ideal periodic network

We model the higher order leaf vein patterns by a periodic network.

Question: What are the periodic networks that have better conductivity properties?

How do we evaluate the effect of the geometry of the networks on the conductivity property?

Outline

Effective tensor and the Wiener bound I

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Based on standard homogenization theory, we describe the microscale design of a composite material by a periodic positive definite matrix field $A : \mathbb{T}^2 \to \mathbb{R}^{2\times 2}$. For each $x \in \mathbb{T}^2$, the matrix $A(x)$ represents the local conductance of the material.

To model the higher-order vein patterns, a convenient set-up is

$$A_\delta(x) := \begin{cases} \frac{1}{\delta} I_{2\times 2} & \text{dist}(x, \Gamma) \leq \delta/2 \\ \delta I_{2\times 2} & \text{elsewhere,} \end{cases}$$

where $\delta > 0$ is a small parameter and $\Gamma \subset \mathbb{T}^2$ is a periodic network.

Effective tensor and the Wiener bound II

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The corresponding large scale conductance $Q = Q(A)$, called **effective tensor** can be computed through the following quadratic form

$$p \cdot Q(A)p := \inf_{\varphi \in C^\infty(\mathbb{T}^2)} \int_{\mathbb{T}^2} (p + \nabla\varphi) \cdot A(x)(p + \nabla\varphi) dx.$$

For each $p \in \mathbb{R}^2$, the value $p \cdot Q(A)p$ represents the *effective conductance* of the microscale design A in direction p.

Effective tensor and the Wiener bound III

In 1912, Wiener discovered that, when A satisfies for some $\lambda > 0$

$$\lambda^{-1} \leq A(x) \leq \lambda \text{ for all } x \in \mathbb{T}^2,$$

then the effective tensor $Q(A)$ has an explicit range

$$\left(\int_{\mathbb{T}^2} A^{-1}(x) dx \right)^{-1} \leq Q(A) \leq \int_{\mathbb{T}^2} A(x) dx.$$

The upper bound is attained if and only if $\nabla \cdot A = 0$; the lower bound is attained if and only if curl $A^{-1} = 0$.

Effective tensor and the Wiener bound IV

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The attainability result becomes much harder as A becomes singular, especially when we consider $A = A_\delta$ and send $\delta \to 0$.

Indeed, as $\delta \to 0$, the function A_δ, when viewed as a measure

$$A_\delta(x)dx \rightharpoonup I_{2\times 2} d\mathcal{H}^1\big|_\Gamma =: dA_\infty$$

converges weakly to the restriction of the 1-D Hausdorff measure restricted to the network Γ as $\delta \to 0$.

In this limit case A_∞, we have formally the Wiener bound degenerates to

$$0 \le Q(A_\infty) \le \mathcal{H}^1(\Gamma) I_{2\times 2} = A_\infty(\mathbb{T}^2).$$

Effective tensor and the Wiener bound V

In this limit case A_∞, the Wiener bound formally degenerates to

$$0 \le Q(A_\infty) \le A_\infty(\mathbb{T}^2).$$

The upper attainability equation

$$\nabla \cdot A_\infty = 0$$

can still be interpreted (at least heuristically) in distributional sense.
The lower attainability equation

$$\text{curl}\, A_\infty^{-1} = 0$$

does not make sense even in distributional sense, as A_∞^{-1} is supposed to be infinity for Lebesgue almost every point.

The singular Wiener bound

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We replace $A(x)dx$ in the classical setting by matrix-valued measures

$$d\theta(x) := \sigma(x)dw(x),$$

where w is a Radon measure on $\mathbb{T}^n$ and σ is a positive semi-definite matrix field such that $\mathrm{Tr}\,(\sigma(x)) = 1$ for w-a.e. $x \in \mathbb{T}^n$.
Similarly, the effective tensor $Q = Q(\theta)$ takes the form

$$p \cdot Q(\theta)p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} (p + \nabla\varphi) \cdot \sigma(x)(p + \nabla\varphi)dw(x).$$

The Wiener bound now becomes

$$0 \leq Q(\theta) \leq \theta(\mathbb{T}^n).$$

Outline

The appearance of loops I

Theorem (Lower attainability, formal)

A planar periodic network is resilient to hydraulic fluctuations if and only if it is reticulate.

The appearance of loops II

Let us begin with the definition of **reticulate**.
Recall that a closed path (also called loop) in $\mathbb{T}^n$ based at $x_0 \in \mathbb{T}^n$ is a continuous map $\gamma : [0, 1] \to \mathbb{T}^n$ such that

$$\gamma(0) = \gamma(1) = x_0.$$

Two closed paths γ_0, γ_1 based at x_0 are called homotopic if there is a continuous map $F : [0, 1]^2 \to \mathbb{T}^n$ such that

$$F(0, t) = \gamma_0(t),\ F(1, t) = \gamma_1(t) \text{ and } F(s, 0) = F(s, 1) = x_0$$

for all $s, t \in [0, 1]$.

The appearance of loops III

For two closed paths γ_0, γ_1 based at $x_0 \in \mathbb{T}^n$, one can define path composition (addition)

$$\gamma_0\gamma_1(t) := \begin{cases} \gamma_0(2t) & 0 \le t < 1/2 \\ \gamma_1(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The first fundamental group $\pi_1(\mathbb{T}^n, x_0)$ is defined as the additive group of closed paths based at x_0 up to homotopy equivalence.

Theorem

There is a canonical group isomorphism

$$i_{x_0} : \pi_1(\mathbb{T}^n, x_0) \to \mathbb{Z}^n.$$

The appearance of loops IV

For each $x_0 \in \mathbb{T}^n$ and $E \subset \mathbb{T}^n$, we can define $\pi_1(E, x_0)$ as the subgroup of $\pi_1(\mathbb{T}^n, x_0)$, which is composed of the homotopy classes of closed paths that have image in E and basepoint x_0.

To describe the abundance of loops in E, we denote

$$H_E := \bigcup_{x_0 \in E} i_{x_0}(\pi_1(E, x_0)) \subset \mathbb{Z}^n.$$

Definition

Call $E \subset \mathbb{T}^n$ **loopy** if H_E spans the whole $\mathbb{R}^n$. Call E to be **reticulate**, if E has a loopy connected component.

The appearance of loops V

It is not difficult to observe that loopiness is equivalent to reticulation in dimension $n = 2$.

Lemma

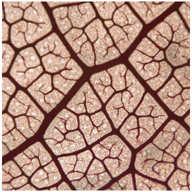
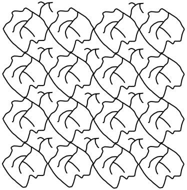
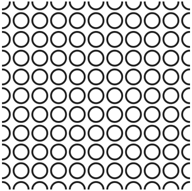
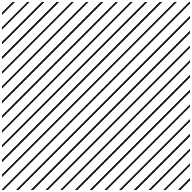
In dimension $n = 2$, a set $E \subset \mathbb{T}^2$ is loopy if and only if it is reticulate.

The equivalence is not true in dimension $n \geq 3$. To see this one can construct the following set in $\mathbb{T}^3$

$$\left(\mathbb{T}^1 \times \{0\} \times \{1/2\}\right) \cup \left(\{0\} \times \{1/2\} \times \mathbb{T}^1\right) \cup \left(\{1/2\} \times \mathbb{T}^1 \times \{0\}\right),$$

which is loopy but not reticulate.

The appearance of loops VI

From left to right. We can see heuristically the conductivity properties of these networks.

1. Higher-order veins in *Ampelocera ruizii*;
2. Reticulate network, and the homotopy classes span $\mathbb{R}^2$;
3. Non-reticulate network, and the homotopy classes are all trivial;
4. Non-reticulate network, and the homotopy classes span a line.

The appearance of loops VII

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Let us now define networks and the notion of resilience.

We consider isotropic media and so $\sigma \equiv I_{n\times n}$. To model the networks we consider a Radon measure w on $\mathbb{T}^n$ that satisfies the following *network-like* condition

$$0 < \limsup_{r\to 0} \frac{w(B_r(x))}{2r} < \infty$$

for w-a.e. $x \in \mathbb{T}^n$. We also require the following *coercivity* condition

$$\limsup_{r\to 0} \frac{w(B_r(x))}{2r} > c > 0$$

for some constant c and $\mathcal{H}^1$-a.e. x in the support $\Gamma := \operatorname{Spt} w$.

The appearance of loops VIII

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An equivalent formulation is that

$$dw(x) = a(x)d\mathcal{H}^1\big|_{\Gamma}(x)$$

for some closed subset $\Gamma \subset \mathbb{T}^n$ with $\mathcal{H}^1(\Gamma) < \infty$ and $a \in L^1_+(\mathcal{H}^1\big|_{\Gamma})$ satisfies the coercivity condition

$$a(x) \geq \lambda > 0$$

for $\mathcal{H}^1$-a.e. $x \in \Gamma$.

The appearance of loops IX

Recall that for each $p \in \mathbb{R}^n$, the following quantity

$$p \cdot Q(w)p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} |\nabla\varphi(x) + p|^2 dw(x),$$

represents the effective conductance of w in direction p.

Definition

Call w to be **resilient** to fluctuations if the effective conductance $p \cdot Q(w)p > 0$ for all $p \in \mathbb{R}^n$, i.e., $Q(w)$ is positive definite.

The appearance of loops X

It turns out that $Q(w) > 0$ is enough to derive reticulation of the network.

Theorem (Lower attainability)

Suppose w is network-like and coercive. Then the kernel of the effective tensor satisfies

$$\ker Q(w) = H_\Gamma^\perp,$$

where $\Gamma = \mathrm{Spt}\, w$, and $H_\Gamma^\perp$ is the $\mathbb{R}$-orthogonal complement of the homotopy classes of loops in Γ. In particular, $Q(w)$ is positive definite if and only if $\Gamma = \mathrm{Spt}\, w$ is loopy (reticulate if planar).

The mechanism I

There are three ingredients in proving the lower attainability theorem.

1. A countable decomposition of $Q(w)$ in terms of connected components of w. This reduces to the case where the support $\Gamma = \text{Spt}\, w$ is connected. Because $\mathcal{H}^1(\Gamma) < \infty$, the set Γ is also 1-rectifiable.
2. By the Ważewski parametrization, there is a continuous surjective path $\gamma : [0, 1] \to \Gamma$. Combining path lifting properties, we prove $H_\Gamma^\perp \subset \ker Q(w)$.
3. We finish the proof by proving the following lower bound for $p \in H_\Gamma$

$$p \cdot Q(w)p \geq C|p|^2,$$

for some $C > 0$ depending only on the ambient dimension n, the geometry of Γ and the coercivity constant λ.

The mechanism II

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Theorem (Countable decomposition)

Let w be a Radon measure such that $\mathcal{H}^1(\operatorname{Spt} w) < \infty$. Then there exist countably many 1-rectifiable connected components $E_i \subset \operatorname{Spt} w$ such that the measures $w_i := w\big|_{E_i}$ satisfy

$$Q(w) = \sum_{i=1}^{\infty} Q(w_i). \tag{1}$$

Note that there is generally no additivity

$$Q(w_1 + w_2) \neq Q(w_1) + Q(w_2)$$

especially when the supports of w_1 and w_2 intersect.

The mechanism III

The countable decomposition theorem is proved via an induction argument on selecting “nice” components.

Theorem

Let w be a measure with $Q(w) \neq 0$*, then the support* Spt w *is not totally disconnected, that is, there is a nonsingleton component in* Spt w*. This implies that the 1-D Hausdorff measure*

$$\mathcal{H}^1(\mathrm{Spt}\, w) > 0.$$

In particular, the Hausdorff dimension $\dim_{\mathcal{H}}(\mathrm{Spt}\, w) \geq 1$.

This also shows that Cantor-type sets have zero conductance.

The mechanism IV

The countable decomposition theorem reduces the problem to the case where the support $\Gamma = \operatorname{Spt} w$ is connected and

$$\mathcal{H}^1(\Gamma) < \infty.$$

Recall the formula of $Q(w)$

$$p \cdot Q(w)p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} |\nabla\varphi(x) + p|^2 dw(x),$$

To prove $H_\Gamma^\perp \subset \ker Q(w)$, it suffices to construct a function $\varphi_p \in C^\infty(\mathbb{T}^n)$ such that $\varphi_p = -p \cdot x$ near Γ for $p \in H_\Gamma^\perp$.

The mechanism V

Consider $H \leq \mathbb{Z}^n$ the smallest subgroup that contains H_Γ, we can construct the quotient space

$$\mathbb{R}^n / H.$$

Note that the standard projection $\pi : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ can be factored as

$$\begin{array}{ccc} & \mathbb{R}^n/H & \\ {}^{\pi_H}\nearrow & & \searrow^{\pi^H} \\ \mathbb{R}^n & \xrightarrow[\pi]{} & \mathbb{T}^n \end{array}$$

The linear function $p \cdot x$ can be constructed on $\mathbb{R}^n$ and $\mathbb{R}^n/H$, but not on $\mathbb{T}^n$.

The mechanism VI

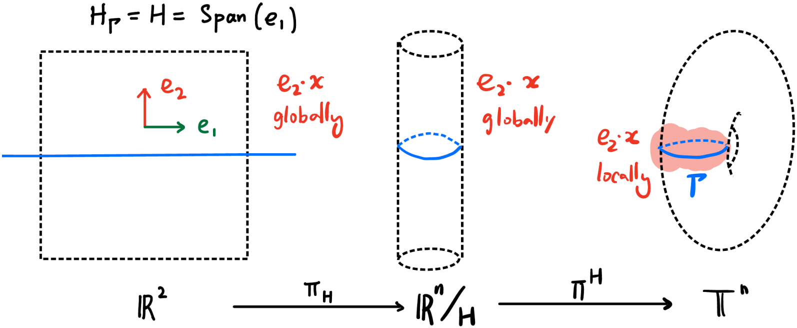

Figure: The linear function $e_2 \cdot x$ can be globally defined on $\mathbb{R}^2$ and $\mathbb{R}^2/H$, but can only be defined locally near Γ on $\mathbb{T}^2$. The whole proof requires the Ważewski parametrization

The mechanism VII

Let us now discuss the reverse inclusion $\ker Q(w) \subset H_\Gamma^\perp$. Let $q \in \ker Q(w)$, then by the previous result, we have

$$q = q_1 + q_2,\ q_1 \in \mathrm{Span}_{\mathbb{R}}(H_\Gamma), q_2 \in H_\Gamma^\perp.$$

Because $H_\Gamma^\perp \subset \ker Q(w)$ we have

$$Q(w)q_1 = Q(w)q = 0.$$

We want to show that this implies that $q_1 = 0$.

The mechanism VIII

To show that $q_1 = 0$, we just need to prove

$$p \cdot Q(w)p \geq C|p|^2$$

for some $C > 0$ and all $p \in H_\Gamma$. Indeed, because $\{p/|p| \; ; \; p \in H_\Gamma\}$ is dense in $\text{Span}_{\mathbb{R}}(H_\Gamma) \cap \partial B_1$, we have

$$q_1 \cdot Q(w)q_1 \geq C|q_1|^2,$$

which forces $q_1 = 0$ as $Q(w)q_1 = Q(w)q = 0$.

The mechanism IX

Lemma (Reparametrization)

Suppose $\gamma : [0,1] \to \Gamma$ *is a constant speed Lipschitz path with*

$$m(\gamma) := \sup_{x \in \Gamma} \# \gamma^{-1}(x) < \infty \text{ and path length } \ell(\gamma) = |\gamma'|,$$

then we have

$$q \cdot Q(w) q \geq \frac{1}{m(\gamma)\ell(\gamma)} \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_0^1 \left| \frac{d}{dt} [\varphi(\gamma(t))] + q \cdot \gamma'(t) \right|^2 a(\gamma(t)) dt. \tag{2}$$

The mechanism X

Apply the lemma to a closed path γ_q that belongs to the homotopy class $q \in H_\Gamma \subset \mathbb{Z}^n$

$$
\begin{aligned}
q \cdot Q(w) q &\geq \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \frac{1}{m(\gamma_q)\ell(\gamma_q)} \int_0^1 \left| \frac{d}{dt} [\varphi(\gamma_q(t))] + q \cdot \gamma_q'(t) \right|^2 a(\gamma_q(t)) dt \\
&\geq \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \frac{\lambda}{m(\gamma_q)\ell(\gamma_q)} \left(\int_0^1 \frac{d}{dt} [\varphi(\gamma_q(t))] + q \cdot \gamma_q'(t) dt \right)^2 \\
&= \frac{\lambda |q|^4}{m(\gamma_q)\ell(\gamma_q)},
\end{aligned}
$$

where $\lambda > 0$ is the coercivity constant of $a(x)$, $m(\gamma_q)$ is the maximal multiplicity and $\ell(\gamma_q)$ is the total length of the path γ_q (multiply counted).

The mechanism XI

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The proof is done by selecting an appropriate γ_q for a fixed $q \in H_\Gamma$ so that

$$m(\gamma_q) \lesssim_{\Gamma,n} |q| \text{ and } \ell(\gamma_q) \lesssim_{\Gamma,n} |q|.$$

A quick example is the geodesic $(0,k) \in \mathbb{Z}^2$. It has length $|k|$ and multiplicity $|k|$. Note that now

$$q \cdot Q(w)q \geq \frac{\lambda |q|^4}{m(\gamma_q)\ell(\gamma_q)} \geq \lambda C|q|^2,$$

for some constant $C = C(\Gamma, n) > 0$.

Outline

Conductance maximality and area criticality I

What is the geometry of the network when the upper bound is attained?

The answer can be summarized as the following identity:

$$\text{Conductance Maximality} \;=\; \text{Area Criticality.}$$

Conductance maximality and area criticality II

We model material mixtures by nonnegative matrix-valued measures θ of the form

$$d\theta = \sigma dw$$

where w is a Radon measure and σ is a positive semi-definite matrix field.

The effective tensor $Q(\theta)$ is defined as

$$p \cdot Q(\theta) p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\nabla \varphi(x) + p) \cdot \sigma(x) (\nabla \varphi(x) + p) dw,$$

We are interested in such θ on $\mathbb{T}^n$ that attain the upper bound

$$Q(\theta) = \theta(\mathbb{T}^n).$$

Conductance maximality and area criticality III

To understand these measures, we need to introduce the notion of **stationary varifolds**. They are weak notions of stationary mean curvature flows, that is, the critical points of the area functional.

For integers $1 \leq k \leq n$, a k-varifold μ on $\mathbb{T}^n$ is a Radon measure on $\mathbb{T}^n \times G(k,n)$, where $G(k,n)$ is the Grassmannian manifold consisting of k-dimensional subspaces of $\mathbb{R}^n$.

Conductance maximality and area criticality IV

For integers $1 \leq k \leq n$, a k-varifold μ on $\mathbb{T}^n$ is a Radon measure on $\mathbb{T}^n \times G(k,n)$, where $G(k,n)$ is the Grassmannian manifold consisting of k-dimensional subspaces of $\mathbb{R}^n$.

Call μ to be stationary if for any smooth vector fields $\Phi \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$ we have

$$\int_{\mathbb{T}^n \times G(k,n)} \mathrm{Tr}\,(P_\tau D_x \Phi) d\mu(x,\tau) = 0,$$

where for each k-dimensional subspace $\tau \in G(k,n)$, P_τ is the orthogonal projection onto τ.

Conductance maximality and area criticality V

Theorem (Upper attainability)

There is a continuous (w.r.t. weak topology) surjective map $\mathcal{T}$ from the space of all varifolds on $\mathbb{T}^n$ to the space of positive semi-definite matrix-valued Radon measures (p.s.d. measures) . The following statements are equivalent for a fixed p.s.d. measure θ:

1. *The p.s.d. measure θ attains the upper Wiener bound $Q(\theta) = \theta(\mathbb{T}^n)$.*
2. *The p.s.d. measure θ satisfies $\nabla \cdot \theta = 0$ in the distributional sense.*
3. *All varifold realizations $\mu \in \mathcal{T}^{-1}(\theta)$ are stationary.*
4. *There exists a stationary varifold $\mu \in \mathcal{T}^{-1}(\theta)$.*

Conductance maximality and area criticality VI

The proof will be presented in four steps:

1. Construct the continuous map $\mathcal{T}$ from the space of all varifolds to the space of all p.s.d. matrix-valued measures;
2. Show that μ is a stationary varifold if and only if $\nabla \cdot \mathcal{T}(\mu) = 0$ in the distributional sense;
3. Show that a p.s.d. measure θ satisfies $Q(\theta) = \theta(\mathbb{T}^n)$ if and only if $\nabla \cdot \theta = 0$;
4. Show that $\mathcal{T}$ is surjective.

The first two steps are known facts [De Philippis-De Rosa-Ghiraldin, 2018]. The key improvement here is the surjectivity.

Conductance maximality and area criticality VII

For a varifold μ on $(x, \tau) \in \mathbb{T}^n \times G(k, n)$, we can disintegrate (also known as conditional probability):

$$d\mu(x, \tau) = d\rho_x(\tau) d \|\mu\| (x),$$

where ρ_x is a Borel selection of probability measures on $G(k, n)$ and $\|\mu\| := i_{\#}\mu$ is the pushforward of μ under the projection $i(x, \tau) = x$.

We define the map $\mathcal{T}$ as

$$\mathcal{T}(\mu)(x) := \frac{1}{k} \int_{G(k,n)} P_\tau d\rho_x(\tau) \, d \|\mu\| (x)$$

One can observe the continuity of this map immediately from the formula.

Lemma

A varifold μ on $\mathbb{T}^n$ is stationary if and only if $\nabla \cdot \mathcal{T}(\mu) = 0$ in the distributional sense.

Proof sketch

Note that by the definition of $\mathcal{T}(\mu)$

$$\frac{1}{k}\int_{\mathbb{T}^n \times G(k,n)} \mathrm{Tr}\,(P_\tau D_x \Phi) d\mu(x,\tau) = \int_{\mathbb{T}^n} \frac{1}{k} \mathrm{Tr}\,(P_\tau d\rho_x(\tau) D_x \Phi)\, d\,\|\mu\|\,(x)$$

$$= \int_{\mathbb{T}^n} D_x \Phi : d\mathcal{T}(\mu),$$

where we have denoted $A : B := \mathrm{Tr}\,(AB)$. Q.E.D.

Conductance maximality and area criticality IX

Lemma

A p.s.d. matrix-valued measure θ on $\mathbb{T}^n$ satisfies $Q(\theta) = \theta(\mathbb{T}^n)$ if and only if $\nabla \cdot \theta = 0$.

Proof sketch

Note that $Q(\theta) = \theta(\mathbb{T}^n)$ if and only if $\operatorname{Tr} Q(\theta) = \operatorname{Tr} \theta(\mathbb{T}^n)$.

$$\begin{aligned} \operatorname{Tr} Q(\theta) &= \sum_{i=1}^{n} e_i \cdot Q(\theta) e_i \\ &= \sum_{i=1}^{n} \inf_{\phi^i \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\nabla \phi^i(x) + e_i) \otimes (\nabla \phi^i(x) + e_i) : d\theta \\ &= \inf_{\Phi \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)} \int_{\mathbb{T}^n} (D_x \Phi + I)^T (D_x \Phi + I) : d\theta. \end{aligned}$$

Conductance maximality and area criticality X

Note that $\operatorname{Tr} Q(\theta) = \operatorname{Tr} \theta(\mathbb{T}^n)$ if and only if

$$\int_{\mathbb{T}^n} (D_x\Phi + I)^T (D_x\Phi + I) : d\theta \geq \operatorname{Tr} \theta(\mathbb{T}^n)$$

for all $\Phi \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$. The formula simplifies to

$$-\int_{\mathbb{T}^n} D_x\Phi^T D_x\Phi : d\theta \leq \int_{\mathbb{T}^n} D_x\Phi : d\theta.$$

The proof is done by replacing Φ by $h\Phi$ for some $h \neq 0$, and then sending $h \to 0$. Q.E.D.

It then suffices to show that the map $\mathcal{T}$ is surjective.

Conductance maximality and area criticality XI

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Lemma (Pointwise realizability)

A positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$ *takes the form*

$$\frac{A}{\operatorname{Tr} A} = \frac{1}{k}\int_{G(k,n)} P_\tau d\rho(\tau) \tag{3}$$

for some probability measure ρ *on* $G(k,n)$ *if and only if*

$$\frac{\operatorname{Tr} A}{\lambda_{\max}(A)} \geq k,$$

where $\lambda_{\max}(A)$ *is the maximal eigenvalue of* A*. In particular,* A *can always be written in the form* (3) *when* $k = 1$*.*

Proof sketch of the pointwise realizability

It is easy to see the "only if" part. To show the reverse, we observe that after rotations of coordinates, it suffices to prove for $\operatorname{Tr} A = 1$ that are diagonal: $A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. This can be done by applying the Krein-Milman theorem and the fact that the extreme points of

$$F := \left\{ 0 \le \lambda_i \le 1/k, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

are permutations of

$$\lambda_1 = 1/k, \ldots, \lambda_k = 1/k, \lambda_{k+1} = 0, \ldots, \lambda_n = 0.$$

Q.E.D.

Conductance maximality and area criticality XIII

Lemma (Surjectivity)

There is a k-varifold in $\mathcal{T}^{-1}(\theta)$ for a p.s.d. matrix-valued measure $d\theta = \sigma dw$ if and only if

$$\frac{\operatorname{Tr}\sigma(x)}{\lambda_{\max}(\sigma(x))} \geq k$$

for some integer $1 \leq k \leq n$ and w-a.e. $x \in \mathbb{T}^n$. In particular, the map $\mathcal{T}$ is surjective as $\frac{\operatorname{Tr}\sigma}{\lambda_{\max}(\sigma)} \geq 1$ always holds.

Proof sketch of Surjectivity

The “only if” part is easy. To show the “if” part, we denote the full w-measure set

$$F := \left\{ x \in \mathrm{Spt}\,\theta\ ;\ \frac{\mathrm{Tr}\,\sigma(x)}{\lambda_{\max}(\sigma(x))} \geq k \right\},$$

Denote $\mathcal{P}_k$ the space of probability measures on $G(k,n)$ and consider

$$K := \left\{ (x,\rho) \in F \times \mathcal{P}_k\ ;\ \frac{\sigma(x)}{\mathrm{Tr}\,\sigma(x)} = \frac{1}{k} \int_{G(k,n)} P_\tau d\rho(\tau) \right\} \subset F \times \mathcal{P}_k.$$

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Because the relation $\dfrac{\sigma(x)}{\operatorname{Tr}\sigma(x)} = \dfrac{1}{k}\displaystyle\int_{G(k,n)} P_\tau d\rho(\tau)$ is Borel in x and continuous in ρ, the set K is Borel. By the pointwise realizability, for each $x \in F$ the slice

$$K_x := \left\{ \rho\,;\ \frac{\sigma(x)}{\operatorname{Tr}\sigma(x)} = \frac{1}{k}\int_{G(k,n)} P_\tau d\rho(\tau) \right\}$$

is compact and nonempty in $\mathcal{P}_k$. By the Kuratowski-Ryll-Nardzewski measurable selection theorem (and its variants), there is a Borel selector $R : F \to \mathcal{P}_k$ such that $R(x) \in K_x$ for all $x \in F$. This establishes the existence of a k-varifold in $\mathcal{T}^{-1}(\theta)$. Q.E.D.

Stationary networks I

Let us discuss a special case of stationary 1-varifolds that have a.e. 1-D density 1, which are simple while containing sufficient richness in math.

A (planar periodic) **stationary network** is a graph embedding Γ consisting of finitely many nodes $\mathcal{N} \subset \mathbb{T}^2$, and straight line segments in $\mathbb{T}^2$ as edges, denoted by $\mathcal{E}$, where

1. the nodes are the endpoints of the edges;
2. for each $x_0 \in \mathcal{N}$, the edges $\overline{x_0 x_i}$ joining at x_0 satisfy the following balance condition
$$\sum_{x_i \sim x_0} \frac{x_i - x_0}{|x_i - x_0|} = 0.$$

Stationary networks II

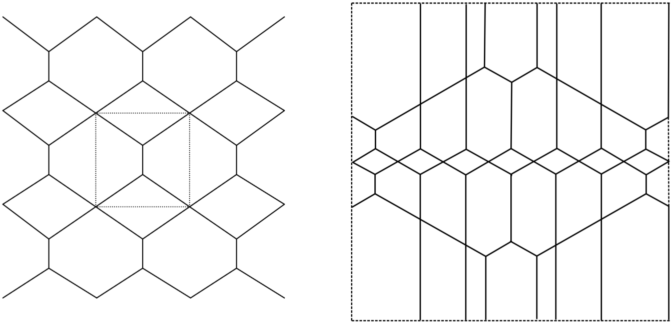

Figure: Two stationary networks.

Stationary networks III

A (planar periodic) stationary network is called **irreducible** if it is not the union of two distinct stationary networks.

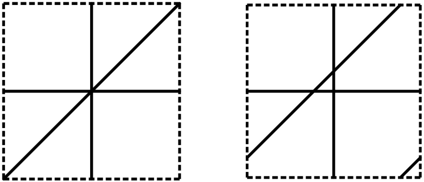

Figure: An irreducible stationary network can be infinitesmally perturbed.

Open question: For an irreducible stationary network, is the maximal number of edges joining at one node bounded by 4?

Fractional monotonicity and dimension bound I

It is well-known that for a stationary k-varifold μ, the density quotient

$$\frac{\|\mu\|\left(B_r(x)\right)}{r^k}$$

is monotone increasing for all $r > 0$ and $x \in \operatorname{Spt}\|\mu\|$. We shall slightly improve this result.

Definition

For a positive semi-definite matrix $A \in \mathbb{R}^{n \times n}$, call

$$\dim_r(A) := \frac{\operatorname{Tr} A}{\lambda_{\max}(A)}$$

the **realizable dimension** of A.

Fractional monotonicity and dimension bound II

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Definition

For a p.s.d. matrix-valued measure θ such that $d\theta = \sigma dw$, we define the **lower realizable dimension** as the lower semi-continuous envelop

$$\underline{\dim_r}(\theta)(x) := \sup_{\delta > 0} \operatorname{ess\,inf}\{\dim_r(\sigma(y)) \; ; \; |y - x| \le \delta, \, y \in \operatorname{Spt}\theta\},$$

where "ess inf" is taken with respect to the Radon measure w. We also define the **lower local dimension**

$$\underline{\dim_{\mathrm{loc}}}(\theta)(x) := \liminf_{r \to 0^+} \frac{\log w(B_r(x))}{\log r}.$$

One can also define the upper version symmetrically.

Fractional monotonicity and dimension bound III

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Theorem (Pointwise dimension bound)

Suppose θ attains the upper Wiener bound, then

$$\underline{\dim}_r(\theta)(x) \leq \underline{\dim}_{\mathrm{loc}}(\theta)(x)$$

for every $x \in \operatorname{Spt}\theta$. In particular, $\underline{\dim}_{\mathrm{loc}}(\theta)(x) \geq 1$ for all $x \in \operatorname{Spt}\theta$.

Lemma (Fractional monotonicity formula)

Given $d\theta = \sigma\, dw$ attains the upper bound, $x_0 \in \operatorname{Spt}\theta$ and $\alpha \in [1, \underline{\dim}_r(\theta)(x_0))$ then

$$\frac{w(B_r(x_0))}{r^\alpha} \tag{4}$$

is monotone nondecreasing for all small $r > 0$.

Proof sketch of the fractional monotonicity formula

We assume without loss $x_0 = 0$ and $r > 0$ is small.
Apply x to the equation $\nabla \cdot \theta = 0$ and do integration by parts, we obtain

$$\begin{aligned} w(B_r) &= \int_{\partial B_r} \frac{x \cdot \sigma(x) x}{|x|} dw(x) \\ &\leq r \int_{\partial B_r} \frac{\operatorname{Tr} \sigma(x)}{\dim_r(\sigma(x))} dw(x) \\ &\leq \frac{r}{\alpha} \frac{d}{dr} w(B_r), \end{aligned}$$

which finishes the proof. Q.E.D.

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