GRADUATE PDE: NOTES

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写这部笔记的目的在于方便我自己以及同学们学习交流。如果你正在上研究生偏微分方程的课，请记得自己做好笔记，勿要依赖这部笔记。

这部笔记在 2019 年十二月的时候敲成 LaTeX，然后在 2021 年二月到三月的时候做了适当补充。笔记的主要部分，也即是前六章，为 2018 秋到 2019 夏由王学锋教授在南科大所教授的内容。我将课上提到了但是没有详细讲述的内容：L^p 理论、Schauder 理论、de Giorgi-Nash-Moser 估计等添加进了附录中。这部笔记应该还有少量疏漏谬误，如有找到，请发送到我的邮箱，非常感谢。另外附录中关于分布解的内正则性的证明是我自己斗胆写下的，如果有感兴趣的读者阅读了我的证明，欢迎与我讨论。

目录

Chapter 1

Maximum Principles

1.1 Second Order Elliptic Differential Equations and Some Examples

There are already tons of introductions on second order elliptic equations, and here we simply list the definitions and several examples.

Definition 1.1. *Suppose $\Omega \subset \mathbb{R}^n$ is an open and connected domain. $\mathbf{x} = (x_1, \cdots, x_n) \in \Omega$, and we define an operator*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} + c(\mathbf{x})u \tag{1.1.1}$$

with u a proper real-valued function on Ω, and $(a_{ij}(x))_{n\times n}$ is symmetric.

Definition 1.2. *We say the operator L **elliptic** on Ω if $(a_{ij}(x))_{n\times n}$ is positive definite for arbitrary $x \in \Omega$. We also say that L is **strictly elliptic** if there exists a positive lower bound for the eigenvalues.*

Definition 1.3. *An **elliptic equation** is one of the form*

$$Lu(x) = f(x),\ x \in \Omega, \tag{1.1.2}$$

*where f is given, and u is unknown. u is called an **upper solution** to (1.1.2) if $Lu \geq f$ almost everywhere, and in the reverse case it is called a **lower solution**.*

EXAMPLES:

1. (Lower Harmonic Function) Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on $\Omega \subset \mathbb{R}^2$, then we have by Cauchy-Riemann Equation we see u, v satisfy $\Delta u = 0,\ \Delta v = 0$, that is, they are harmonic. Now, the absolute value of f becomes a lower harmonic function, which is obtained by a direct computation.

2. (Electrostatics) Let $f(x)$ be a function that represents the density of the electric materials, then the solution to the Cauchy equation

$$-\Delta u = f$$

gives the static electric field the materials produce, where the operator $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ will be called *Laplacian* throughout the note.

3. (Mean Curvature) By mean curvature we mean half of the trace of the metric tensor of a surface. We now consider the surface $z = u(x, y), (x, y) \in \Omega$. Computations show that its mean curvature is $H(x, y) = \frac{1}{2}\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$. In the theory of minimal surfaces, a surface that minimizes area locally if and only if its mean curvature vanishes, i.e.

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

This is equation gives

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0.$$

And it can be shown that the thermal tensor

$$\begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_x u_y & 1 + u_x^2 \end{pmatrix} \succcurlyeq I.$$

4. (Steady States of Heat Equation) Let $u(\mathbf{x}, t)$ be the temperature at point $\mathbf{x} \in \mathbb{R}^n$, at time t. Then the heat energy density will be $E(\mathbf{x}, t) = c\rho u(\mathbf{x}, t)$, with c the specific heat and ρ the density of mass. Suppose c, ρ are constants, and Ω a domain the material possesses. Then the rate of change of total heat energy in Ω will be modelled by

$$\frac{d}{dt}\int_\Omega E(\mathbf{x}, t)d\mathbf{x} = \text{rate in} - \text{rate out } ☻$$

To formulate ☻, we assume that $\mathbf{N}$ is the unit outer normal vector field on the surface $S = \partial\Omega$, $\mathbf{V}$ the heat transfer velocity vector field. Then the net rate at which mass/heat crosses surface S in the direction $\mathbf{N}$ is

$$\int_S \rho\mathbf{V} \cdot \mathbf{N} dS,$$

where $\mathbf{F} = \rho\mathbf{V}$ is called the *flux*, and so

$$☻ = \int_S \mathbf{F} \cdot \mathbf{N} dS.$$

Now, what is $\mathbf{F}$? According to Fourier's Law, $\mathbf{F}$ should have angle less than $90°$ with $-\nabla u$, i.e. the flux should be approximately in the diffusing direction of the temperature. This observation (although not mathematical) forces $\mathbf{F} = A(\mathbf{x}, t)(-\nabla u(\mathbf{x}, t))$, with A symmetric and positive definite. It is of wide interest to study this thermal tensor

A, which actually represents various heat-transfer properties of different matters. A rough classification of such matrices is to call $A = kI,\ k \in \mathbb{R}$, isotropic, and otherwise anisotropic.

With the integral equation established successfully, we may deduce by divergence theorem

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} E(\mathbf{x},t)d\mathbf{x} &= -\int_{\partial\Omega} A\nabla u \cdot \mathbf{N} dS \\ &= \int_{\Omega} \nabla \cdot (A\nabla u)\, d\mathbf{x},\end{aligned}$$

where Ω can be replaced by any smooth sub-region. Thus the integrands should satisfy the equation

$$\frac{dE(\mathbf{x},t)}{dt} = \nabla \cdot (A\nabla u)$$

or

$$c\rho u_t = \nabla \cdot (A\nabla u)\,.$$

The *steady state* of the above heat equation is a solution u that is independent of time t, i.e.

$$-\nabla \cdot (A\nabla u) = 0.$$

5. (Irrotational and Incompressible fluid) Let $\mathbf{V}$ be a velocity vector field in a simply connected domain Ω. "Irrotational" means that $\mathbf{V} = \nabla\phi$ for some function ϕ, and "Incompressible" means that $div\mathbf{V} = 0$. These two conditions imply that $\Delta\phi = 0$.

1.2 Weak Maximum Principle for Second Order Elliptic Differential Equations

Baby Example: $Lu = u''$ on the interval (a,b). Suppose $u \in C^2(a,b)\cap C^0[a,b]$ satisfying $u'' \geq 0$ in the interior, then $\max_{[a,b]} u = \max_{\{a,b\}} u$.

Question: Is it still true for the case $Lu \geq 0$, in Ω, that

$$\max_{\Omega} u = \max_{\partial\Omega} u\ ?$$

Theorem 1.2.1. Weak Maximum Principle ($c \equiv 0$) *Suppose L is strictly elliptic on a bounded domain Ω and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω, then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u,$$

provided b_i's are bounded on Ω.

证明. **Special case**: $Lu > 0$ in Ω

Since u is continuous on $\bar{\Omega}$, then there should be some $x_0 \in \Omega$ such that $u(\mathbf{x}_0) = \max_{\bar{\Omega}} u$.

Case 1. $\mathbf{x}_0 \in \partial\Omega$, then the proof is done;

Case 2. $\mathbf{x}_0 \in \Omega$ implies that $\nabla u(\mathbf{x}_0) = \mathbf{0}$. By considering the Hessian matrix evaluated at $\mathbf{x}_0$, we see

$$D_x^2 u(\mathbf{x}_0) = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix} \preccurlyeq 0,$$

by maximality. Observing that

$$0 < Lu(\mathbf{x}_0) = a_{ij}u_{ij}(\mathbf{x}_0) + b_i u_i(\mathbf{x}_0) = a_{ij}u_{ij} = Tr(AB), \text{ with } B = D_x^2 u(\mathbf{x}_0),$$

we take orthogonal matrix P such that

$$P^T AP = diag\{\lambda_i\},$$

where λ_i's are eigenvalues of A and $\lambda_i \geq \lambda_0$ for all $1 \leq i \leq n$.

Now, we have

$$\begin{aligned} Tr(AB) &= Tr(P^T ABP) \\ &= Tr(P^T APP^T BP) \\ &= Tr(diag(\lambda_i)\tilde{B}) \\ &= \sum \lambda_i \tilde{b}_{ii} \\ &\leq 0, \end{aligned}$$

since $B \preccurlyeq 0$, and so is $\tilde{B}$, which implies $\tilde{b}_{ii} \leq 0$. Thus the contradiction is established.

General case: For $\epsilon > 0$, we define

$$v(\mathbf{x}) = u(\mathbf{x}) + \epsilon e^{\alpha x_1}.$$

Then we have

$$\begin{aligned} Lv &= Lu + L\left(\epsilon e^{\alpha x_1}\right) \\ &\geq \epsilon \left[\alpha^2 a_{11} + b_1\alpha\right] e^{\alpha x_1} \\ &\geq \left[\alpha^2 \lambda_0 - M\alpha\right] e^{\alpha x_1} \\ &> 0, \quad \text{if } \alpha \text{ is taken large}, \end{aligned}$$

where M is the upper bound for the b_i's. Applying the special case to v and letting $\epsilon \to 0$, we are done (here we need the boundedness of Ω). $\square$

Remark:

1. If "$Lu \geq 0$" is replaced by "$Lu \leq 0$", then we obtain weak minimum principle

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

2. Physical meaning of $Lu \geq 0$ in Ω is that $-Lu(\mathbf{x}) = f(\mathbf{x}) \leq 0$ the creation-degradation rate is negative, which means the material is reducing heat, like a refrigerator. One non-example is that the following equation

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ \frac{\partial u}{\partial \vec{n}}\big|_{\partial\Omega} = 0 \end{cases}$$

has no solution, according to divergence theorem. Physically, an isolated box which is loosing heat everywhere cannot reach a steady state.

Question: What if $c(\mathbf{x}) \not\equiv 0$?

Bad News: We consider

$$Lu = u'' + u, \quad \text{on } (0, \pi)$$

then $L(\sin x) = 0$, but it does not satisfy the maximum principle.

Theorem 1.2.2. Weak Maximum Principle $(c \leq 0)$ *Suppose L is strictly elliptic on a bounded domain Ω and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω, then*

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

provided b_i's are bounded on Ω, and $c \leq 0$ in Ω.

证明. Let $\Omega^+ = \{x \in \Omega;\ u(x) > 0\}$ be a sub-domain, then there are two cases.

Case 1. $\Omega^+ = \emptyset$, trivial.

Case 2. $\Omega^+ \neq \emptyset$,

sub-case 1. $\overline{\Omega^+} \subset \Omega$, then on Ω^+, we have

$$0 \leq a_{ij}u_{ij} + b_i u_i + cu,$$

which implies

$$a_{ij}u_{ij} + b_i u_i \geq -c(x)u \geq 0, \text{ on } \Omega^+.$$

Applying weak maximum principle for $c \equiv 0$ to u on Ω^+, we obtain

$$\max_{\overline{\Omega^+}} u = \max_{\partial\Omega^+} u = 0,$$

which is a contradiction.

sub-case 2. $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$, then

$$\begin{aligned} \max_{\bar{\Omega}} u &= \max_{\overline{\Omega^+}} u \\ &= \max_{\partial\Omega^+} u \\ &= \max_{\partial\Omega^+ \cap \partial\Omega} \\ &\leq \max_{\partial\Omega} u^+ \end{aligned}$$

$\square$

Caution: Without "+", we will have a counter-example

$$Lu = u'' - u, \ \Omega = (-1, 1),$$

and $u(x) = -(x^2 + 100)$, then $Lu = 100 - 2 + x^2 > 0$ on Ω, and $\max_{\bar{\Omega}} u = -100$, while $\max_{\partial\Omega} u = -101$.

Remark:

1. If we replace "$Lu \geq 0$" by "$Lu \leq 0$", then we have a minimum principle

$$\min_{\bar{\Omega}} \geq \min_{\partial\Omega} u^-,$$

where $u^- = \min\{0, u\}$.

2. If we have "$Lu = 0$", then there are two cases

Case 1.

$$\begin{aligned}\max_{\bar{\Omega}} |u| &= \max_{\bar{\Omega}} u \\ &\leq \max_{\partial\Omega} u^+ \\ &\leq \max_{\partial\Omega} |u| \\ &\leq \max_{\bar{\Omega}} |u|,\end{aligned}$$

and so $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u^+$.

Case 2.

$$\begin{aligned}\min_{\bar{\Omega}} |u| &= -\min_{\bar{\Omega}} u \\ &\leq -\min_{\partial\Omega} u^- \\ &\leq \max_{\partial\Omega} |u| \\ &\leq \max_{\bar{\Omega}} |u|,\end{aligned}$$

and so $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u^-$.

3. **Question:** What if Ω is unbounded?

Bad News: $u(x, y) = y$ is harmonic on the upper half plane satisfying zero boundary condition, then its maximum value is infinity while on the boundary it's constant 0.

Solution: This can be saved if $\lim\limits_{\Omega \ni x \to (\pm)\infty} u(x)$ exists.

Theorem 1.2.3. *Suppose Ω is unbounded, and for every $R > 0$, b_i's are bounded in $\Omega \cap B_R(\boldsymbol{0})$. L is strictly elliptic on $\Omega \cap B_R(\boldsymbol{0})$, and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies*

$$\begin{cases} Lu \geq 0 \ \text{in } \Omega; \\ u(\infty) := \lim_{\Omega \ni x \to (\pm)\infty} u(x) \, \text{exists.} \end{cases}$$

Then

i) $\sup_{\bar{\Omega}} u = \max\{\sup_{\partial\Omega} u,\ u(\infty)\}$ *if* $c \equiv 0$*;*

ii) $\sup_{\bar{\Omega}} u \le \max\{\sup_{\partial\Omega} u^+,\ u(\infty)\}$ *if* $c \le 0$ *in* Ω*.*

证明. Applying WMP on $\Omega \cap B_R(0)$, we obtain

$(c \equiv 0)$

$$\begin{aligned}\sup_{\Omega\cap B_R} u &= \max_{\partial(\Omega\cap B_R)} u \\ &= \max\left(\max_{\partial\Omega_1^R} u, \max_{\partial\Omega_2^R} u\right).\end{aligned}$$

Sending $R \to \infty$, we obtain

$$\sup_{\bar{\Omega}} u = \max\{\sup_{\partial\Omega} u,\ u(\infty)\}.$$

$(c \le 0)$ Similar proof.

$\square$

Theorem 1.2.4. Comparison Principle *Assume Ω is bounded, L is strictly elliptic on Ω, b_i's are bounded and $c \le 0$ on Ω. Suppose*

$$\begin{cases} Lu \ge Lv, & \text{in } \Omega; \\ u \le v, & \text{on } \partial\Omega. \end{cases}$$

$u,\ v \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then $u \le v$ in $\bar{\Omega}$.

证明. Simple application of WMP($c \le 0$). $\square$

This immediately gives the following corollary.

Corollary 1.2.1. *Assume conditions in CP on Ω and L. Then the Dirichlet Boundary Value problem has at most one classical solution.*

Example for non-solution:

$$\begin{cases} u'' + u = 1, & \text{on } \Omega = (0, \pi); \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

has no solution. This can be verified by multiplying $\sin x$ to the equation and integrate it over Ω.

Applications of WMP:

1.
$$(\text{DBVP}) \begin{cases} \Delta u + f(u) = 0, & \text{in bounded } \Omega; \\ u\big|_{\partial\Omega} = 0, \end{cases}$$

with $f \in C^1(\mathbb{R})$ and decreasing, then (DBVP) has at most one solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$.

证明. Suppose u_1, u_2 are two classical solution, then we have

$$\Delta(u_1 - u_2) + f(u_1) - f(u_2) = 0$$

We write $f(u_1) - f(u_2) = C(x)(u_1 - u_2)$, then C is non-positive. According to WMP($c \leq 0$) we see

$$\max_{\bar{\Omega}} |u_1 - u_2| = \max_{\partial\Omega} |u_1 - u_2|,$$

and thus $u_1 \equiv u_2$ in $\bar{\Omega}$. □

2. Let Ω be bounded in $\mathbb{R}^n$, $n \geq 3$. Suppose $u \in C^2(\Omega^c) \cap C^0(\overline{\Omega^c})$ satisfying

$$\begin{cases} \Delta u = 0, & \text{in } \Omega^c; \\ u(\infty) := & \lim_{\mathbf{x}\to\infty} u(x) = 0, \end{cases}$$

then, there exists a constant $C > 0$ such that

$$|u(x)| \leq \frac{C}{|\mathbf{x}|^{n-2}},\ \forall \mathbf{x} \in \Omega^c.$$

证明. Take a large $M > 0$ such that $M\Gamma(\mathbf{x} - \mathbf{x}_0) \geq |u|(\mathbf{x})$, for all $\mathbf{x} \in \partial\Omega^c$, where $\mathbf{x}_0$ is in Ω and Γ is the fundamental solution. Letting $v(\mathbf{x}) = M\Gamma(\mathbf{x} - \mathbf{x}_0) - u(\mathbf{x})$, and applying WMP on unbounded domain, we obtain

$$\inf_{\Omega^c} v \geq \min\left(\inf_{\partial\Omega^c} v^-, v(\infty)\right) = 0,$$

which implies $|u(\mathbf{x})|$ is dominated by $1/|\mathbf{x}|^{n-2}$ up to a multiplicative constant. Similar proof show that $u(\mathbf{x})$ decays exactly at this rate. □

3. Let Ω be bounded, L be strictly elliptic on Ω, i.e.

$$a_{ij}(\mathbf{x})\xi_i\xi_j \geq \lambda_0|\xi|^2,\ \forall \xi \in \mathbb{R}^n, \mathbf{x} \in \Omega,$$

where $\lambda_0 > 0$ is some constant. b_i's are bounded by $M > 0$ and $c \leq 0$. Suppose that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu(\mathbf{x}) = f(\mathbf{x})$, then

$$\max_{\bar{\Omega}} |u| \leq \max_{\partial\Omega} |u| + K \sup_{\Omega} |f|,$$

where $K > 0$ is some constant depending only on λ_0 and M.

Structural Stability: Let f_a and ϕ_a be approximations to f and $u\big|_{\partial\Omega}$, then the above result gives global stability of the solution, i.e. the corresponding approximate solution u_a satisfies

$$\max_{\bar{\Omega}} |u - u_a| \leq \max_{\partial\Omega} |u - \phi_a| + K \sup_{\Omega} |f - f_a|,$$

证明. We WLOG assume that $||f||_\infty = \sup_\Omega |f| < \infty$. Now, we define $\bar{u}(\mathbf{x}) = \max_{\partial\Omega} |u| + ||f||_\infty \left(e^{\alpha d} - e^{\alpha x_1}\right)$, where $d > 0$ and $\Omega \subset (0, d) \times \mathbb{R}^{n-1}$. Applying L to $\bar{u}$ we have

$$\begin{aligned} L\bar{u} &= -a_{11}\alpha^2 e^{\alpha x_1} ||f||_\infty - b_1 \alpha e^{\alpha x_1} ||f||_\infty + c\bar{u} \\ &\leq ||f||_\infty \left(-a_{11}\alpha^2 e^{\alpha x_1} - b_1 \alpha e^{\alpha x_1}\right) \\ &\leq \left(-\lambda_0 \alpha^2 + M\alpha\right) e^{\alpha x_1} ||f||_\infty \\ &\leq f, \end{aligned}$$

where α is taken so large that $\left(-\lambda_0\alpha^2 + M\alpha\right) e^{\alpha x_1} \leq -1$. According to CP, we see that $\bar{u} \geq u$ in Ω. Similarly, $L\bar{u} \leq -f(x) = -Lu$, and $\bar{u}\big|_{\partial\Omega} \geq -u\big|_{\partial\Omega}$, which implies that $\bar{u} \geq -u$ in Ω. Above all, we obtain

$$\begin{aligned} \max_{\bar{\Omega}} |u| &\leq \max_{\partial\Omega} |\bar{u}| \\ &\leq \max_{\partial\Omega} |u| + \left(e^{\alpha d} - 1\right) ||f||_\infty. \end{aligned}$$

□

1.3 Strong Maximum Principle for Second Order Elliptic Differential Equations

Baby Example: $Lu = u'' \geq 0$ on $(0, 1)$, and assume there is a local maximum point x_0 of u in $(0, 1)$, then $u \equiv u(x_0)$ on $(0, 1)$.
Physical Intuition: There should be no hot pots within a refrigerator.

1.3.1 Hopf Boundary Point Lemma

Definition 1.4. *We say Ω satisfies **interior sphere** condition at $\mathbf{x}_0 \in \partial\Omega$ if there exists an open ball $B \subset \Omega$ such that $\partial B \cap \partial\Omega = \{\mathbf{x}_0\}$.*

Fact: If $\partial\Omega$ is C^2-smooth, then Ω satisfies interior sphere condition.

Definition 1.5. *We say $\partial\Omega$ is C^m-smooth $m \geq 0$ if for all $p \in \partial\Omega$, there is a neighborhood N of p and shift & rotation of the coordinate system $(x_1, \cdots, x_n)$ such that*

i. there is a C^m-smooth function ϕ defined in a nbhd $Q \subset \mathbb{R}^{n-1}$ such that $\partial\Omega \cap N$ is exactly the graph of $\phi(x_2, \cdots, x_n) = x_1$;

ii. $\Omega \cap N = \{(x_1, \mathbf{x}') \in N \,; x_1 > \phi(\mathbf{x}')\}$.

证明. To prove the fact, we may without loss of generality assume that $p = \mathbf{0}$ and x_1-axis is the direction of inner normal of $\partial\Omega$ at p. Under this assumption, we know that $\phi(\mathbf{0}') = 0$ and $\nabla\phi(\mathbf{0}') = \mathbf{0}$. By Taylor expansion at the origin, we have

$$\phi(\mathbf{x}') = \frac{1}{2}(\mathbf{x}')^T D^2\phi(\mathbf{0}')\mathbf{x}' + o(|\mathbf{x}'|^2), \ \mathbf{x}' \approx \mathbf{0}'.$$

Let $C' > C$ be constants that are greater than the maximum eigenvalue of $D^2\phi(\mathbf{0}')$, then we further have that

$$\phi(\mathbf{x}') \leq C|\mathbf{x}'|/2 + o(|\mathbf{x}'|^2) \leq C'|\mathbf{x}'|/2, \ \mathbf{x}' \approx \mathbf{0}'.$$

We construct a ball B centered at $(R, \mathbf{0}')$ with radius $R > 0$. By definition we have for points $\mathbf{x} \in B$, $x_1 \geq \frac{1}{2R}|\mathbf{x}'|^2$ with equality holds only when $\mathbf{x} = \mathbf{0}$, provided R is tiny. Thus we've obtained a ball B above the graph of ϕ that intersects the boundary exactly at p. $\square$

Lemma 1.3.1. Hopf Boundary Point Lemma *Assume Ω bounded, L strictly elliptic on Ω, $a_{ij}(\mathbf{x})$, $b_i(\mathbf{x})$ and $c(\mathbf{x})$ are bounded on Ω. Let $u \in C^2(\Omega)$ satisfies*

- *$Lu \geq 0$ in Ω;*
- *u is continuous at $\mathbf{x}_0 \in \partial\Omega$ where the interior sphere condition is satisfied;*
- *$\mathbf{x}_0$ is a strictly local maximum point of u.*

Then, for any outward pointing vector $\vec{v}$ at $\mathbf{x}_0$, i.e. $\vec{v} \cdot (\mathbf{x}_0 - \boldsymbol{y}) > 0$ with $\boldsymbol{y}$ the center of the interior ball, we have $\frac{\partial u}{\partial \vec{v}}(\mathbf{x}_0) > 0$ if it exists, provided one of the following statements holds true:

1. *$c \equiv 0$ in Ω.*
2. *$c \leq 0$ in Ω and $u(\mathbf{x}_0) \geq 0$.*
3. *$u(\mathbf{x}_0) = 0$, regardless of the sign of c.*

证明. Let $B_R(\mathbf{y})$ be the interior ball that intersects the boundary at $\mathbf{x}_0$, $R > \rho > 0$, and $A = B_R(\mathbf{y}) \backslash \overline{B_\rho(\mathbf{y})}$. We will construct a function v, such that

a. $v \in C^\infty(\mathbb{R}^n)$, $Lv > 0$ in A if $c \leq 0$ in Ω;

b. $v\big|_{\partial B_R(\mathbf{y})} \equiv 0$;

c. $\frac{\partial v}{\partial \vec{v}}(\mathbf{x}_0) < 0$.

Now, define $\omega(\mathbf{x}) = u(\mathbf{x}) - u(\mathbf{x}_0) + \epsilon v(\mathbf{x})$, $\epsilon > 0$ small. Then, we have by assumptions

$$\begin{aligned} L\omega(\mathbf{x}) &= Lu(\mathbf{x}) - c(\mathbf{x})u(\mathbf{x}_0) + \epsilon Lv(\mathbf{x}) \\ &> Lu(\mathbf{x}) - c(\mathbf{x})u(\mathbf{x}_0) \\ &\geq 0, \end{aligned}$$

and on the boundaries

$$\omega\big|_{\partial B_R(\mathbf{y})} \leq 0, \text{ if } R \text{ is small enough,}$$

and

$$\omega\big|_{\partial B_\rho(\mathbf{y})} < -\delta + \epsilon v \leq 0.$$

WMP($c \le 0$) implies that $\omega \le 0$ in A. But $\omega(\mathbf{x}_0) = 0$, this implies that $\frac{\partial \omega}{\partial \vec{v}}(\mathbf{x}_0) \ge 0$, and so $\frac{\partial u}{\partial \vec{v}}(\mathbf{x}_0) > 0$.

The above arguments only holds for the cases 1. and 2., as for case 3., we define

$$L_0 u = a_{ij}u_{ij} + b_i u_i + c^- u = Lu - c^+ u \ge 0, \text{ in } \Omega \text{ \& close to } \mathbf{x}_0.$$

Now, applying case 2. to L_0 we obtain the same result.

We take $v(\mathbf{x}) = e^{-\alpha|\mathbf{x}-\mathbf{y}|^2} - e^{-\alpha R^2}$, for all $\mathbf{x} \in \mathbb{R}^n$, where $\alpha > 0$ is an undetermined constant. We may observe that

$$\nabla v(\mathbf{x}) = -2\alpha(\mathbf{x} - \mathbf{y})e^{-\alpha|\mathbf{x}-\mathbf{y}|^2},$$

which shows that $\frac{\partial v}{\partial \vec{v}}(\mathbf{x}_0) = \nabla v(\mathbf{x}_0) \cdot \vec{v} < 0$. Meanwhile,

$$v_{ij} = -2\alpha\delta_{ij}e^{-\alpha|\mathbf{x}-\mathbf{y}|^2} + 4\alpha^2(x_i - y_i)(x_j - y_j)e^{-\alpha|\mathbf{x}-\mathbf{y}|^2},$$

and so

$$\begin{aligned}
Lv &= a_{ij}v_{ij} + b_i v_i + cv \\
&= -2a_{ij}\alpha\delta_{ij}e^{-\alpha|\mathbf{x}-\mathbf{y}|^2} + 4a_{ij}\alpha^2(x_i - y_i)(x_j - y_j)e^{-\alpha|\mathbf{x}-\mathbf{y}|^2} \\
&\quad - \alpha b_i(x_i - y_i)e^{-\alpha|\mathbf{x}-\mathbf{y}|^2} + c\left[e^{-\alpha|\mathbf{x}-\mathbf{y}|^2} - e^{-\alpha R^2}\right] \\
&\ge e^{-\alpha|\mathbf{x}-\mathbf{y}|^2}\left[4\alpha^2\lambda_0|\mathbf{x}-\mathbf{y}|^2 - 2\alpha nM - 2\alpha nMR - M\right] \\
&\ge e^{-\alpha|\mathbf{x}-\mathbf{y}|^2}\left[4\alpha^2\lambda_0\rho^2 - 2\alpha[nM - nMR] - M\right] \\
&> 0,
\end{aligned}$$

if $\alpha > 0$ is taken large. □

1.3.2 Strong Maximum Principle

Theorem 1.3.1. Strong Maximum Principle *Let Ω be bounded and L strictly elliptic in Ω with a_{ij}, b_i and c bounded. Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \ge 0$ in Ω, and $\max_{\bar{\Omega}} u$ is achieved at some $\mathbf{x}_0 \in \Omega$. Then $u(\mathbf{x}) \equiv u(\mathbf{x}_0)$ in Ω, provided one of the following holds:*

1. $c \equiv 0$.
2. $c(\mathbf{x}) \le 0$ *in* Ω, $u(\mathbf{x}_0) \ge 0$.
3. $u(\mathbf{x}_0) = 0$.

证明. Define $\Omega^- = \{u(\mathbf{x}) < u(\mathbf{x}_0)\}$, and assume $\Omega^- \ne \emptyset$. Then, we see that Ω^- is open and claim that $\partial\Omega^- \cap \Omega \ne \emptyset$ by connectedness of Ω. Let $\mathbf{x}_1 \in \partial\Omega^- \cap \Omega$ and $\mathbf{x}_2 \in \Omega^-$ satisfying $dist(\mathbf{x}_1, \mathbf{x}_2) < dist(\mathbf{x}_2, \partial\Omega)$. Increasing radius of a ball centered at $\mathbf{x}_2$, the ball will first touch a point in $\partial\Omega^-$. Let $\mathbf{x}_3$ be the point, then Ω^- satisfies interior sphere condition there, and an simple application of Hopf's boundary point lemma finishes the proof. □

Examples:

1. **Separation of Solutions**

   Baby Case:

$$\begin{cases} u'' = f(x,u,u'), & x \in (a,b), \\ v'' = f(x,v,v'), & f \text{ "nice"}, \\ u \geq v, & \text{on } (a,b). \end{cases}$$

   Then $u \equiv v$, provided $u(c) = v(c)$ and $u'(c) = v'(c)$.

   PDE Case: Let Ω be bounded in $\mathbb{R}^n$, $Lu = a_{ij}u_{ij} + b_i u_i$ strictly elliptic, and a_{ij}, b_i bounded on Ω. Suppose $u, v \in C^2(\Omega)$ satisfying

   - $Lz = f(\mathbf{x}, z)$, $\mathbf{x} \in \Omega$, where for all $M > 0$, $f_z(x,z)$ bounded for all $\mathbf{x} \in \Omega$, $z \in [-M, M]$;
   - $u \geq v$ in Ω;
   - $u(\mathbf{x}_0) = v(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \Omega$.

   Then $u \equiv v$ in Ω.

   证明. We will first restrict our consideration on $\Omega_\epsilon = \{dist(\mathbf{x}, \partial\Omega) > \epsilon\}$, with $\epsilon > 0$. Since $\overline{\Omega_\epsilon}$ is compact, and u, v continuous on it, we may find some $M > 0$, by which the two functions are bounded. Observe that

$$\begin{aligned} L(u-v) &= f(\mathbf{x}, u) - f(\mathbf{x}, v) \\ &= \int_0^1 f_z(\mathbf{x}, tu + (1-t)v)(u-v)dt \\ &= c(x)(u-v), \end{aligned}$$

   which is an equation of the form

$$\begin{cases} Lw - cw = 0, & \text{in } \Omega \\ w \geq 0, & \text{in } \partial\Omega, \end{cases}$$

   where at $\mathbf{x}_0$ w reaches its local minimum 0. According to SMP(case 3.), we see $w \equiv 0$ in Ω. □

2. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, where Ω satisfies interior sphere condition at each $p \in \partial\Omega$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{v}}\Big|_{\partial\Omega} = 0, & \vec{v} \text{ outward pointing.} \end{cases}$$

   Then u is a constant.

   证明. (can also use energy method) Let $\mathbf{x}_0 \in \bar{\Omega}$ such that $\max_{\bar{\Omega}} u = u(\mathbf{x}_0)$.

case 1. $\mathbf{x}_0 \in \Omega$, SMP implies that $u \equiv u(\mathbf{x}_0)$.

case 2. $\mathbf{x}_0 \in \partial\Omega$, then $\mathbf{x}_0$ is a strict local maximum point, and so by Hopf, $\frac{\partial u}{\partial \vec{v}}(\mathbf{x}_0) > 0$, which is impossible.

□

Theorem 1.3.2. Comparison Principle (Robin Boundary Condition) *Suppose Ω bounded, L strictly elliptic on Ω and a_{ij}, b_i, c bounded on Ω, $c \leq 0$ on Ω, $\partial\Omega \in C^2$. $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy*

$$\begin{cases} Lu \geq Lv, & \textit{in } \Omega, \\ \frac{\partial u}{\partial \vec{v}} + \beta(\mathbf{x})u \leq \frac{\partial v}{\partial \vec{v}} + \beta(\mathbf{x})v, & \vec{v} \textit{ outward pointing,} \end{cases}$$

where $\beta \geq 0$. Then $u \leq v$ in Ω.

证明. Let $w = u - v$, then

$$\begin{cases} Lw \geq 0, & \text{in } \Omega, \\ \frac{\partial w}{\partial \vec{v}} + \beta(\mathbf{x})w \leq 0, & \text{on } \partial\Omega. \end{cases}$$

Let $M = \max_{\bar{\Omega}} w$, then if $M \leq 0$, we are done. If $M > 0$, then let $u(\mathbf{x}_0) = M$, we have

Case 1. $\mathbf{x}_0 \in \Omega$, then SMP(case 2.) implies that $w \equiv M$ all over the domain, which contradicts the boundary condition.

Case 2. max of w achieves only at boundary points, and so Hopf(case 2.) will also contradict the boundary condition.

□

1.4 Weak Maximum Principle for Second Order Parabolic Differential Equations

Baby Example: Let $\Omega \subset \mathbb{R}^3$ be an "oven", and u the temperature function in the oven. We suppose u satisfies an equation of the following form

$$u_t - a_{ij}u_{ij} + b_i u_i = f(\mathbf{x}, t),$$

where $f \geq 0$, which means that the oven is producing heat. Physically, one would find that the oven should reach its minimum temperature near its boundary, because it's producing heat in the interior. We define $\Gamma = \Omega \times \{0\} \cup \partial\Omega \times [0, T)$ and it can be shown that

$$\min_{\bar{\Omega} \times [0,T]} u = \min_{\Gamma} u.$$

Parabolic Boundary and Interior For a more general time-space domain $D \subset \mathbb{R}^n \times [0, T]$ such that $\bar{D} \cap \{t = 0\} \neq \emptyset$ and $\bar{D} \cap \{t = T\} \neq \emptyset$, we define its **parabolic boundary** to be $\overline{\partial D \cap \{0 \leq t < T\}}$. At the meantime, $\bar{D} \backslash \Gamma$ will be called the **parabolic interior** of D. It can be shown that every slice set in time variable of a parabolic interior is open in $\mathbb{R}^n$.

We now consider operators of the form

$$Lu = u_t - a_{ij}u_{ij} + b_i u_i + cu,$$

and we would say L strictly parabolic in D if there is a positive constant $\lambda_0 > 0$ such that

$$(a_{ij}) \succcurlyeq \lambda_0 I,$$

all over the parabolic interior.

Theorem 1.4.1. **WMP**$(c \equiv 0)$ *Assume L is strictly parabolic in D with $c \equiv 0$, and let $u \in C^0(\bar{D}) \cap C^{2,1}(\bar{D}\backslash\Gamma)$ satisfies $Lu \leq 0,\ (\mathbf{x}, t) \in \bar{D}\backslash\Gamma$. Then*

$$\max_{\bar{D}} u = \max_{\Gamma} u.$$

证明. **Special Case**: $Lu < 0$ in $\bar{D}\backslash\Gamma$. Let $(\mathbf{x}_0, t_0) \in \bar{D}$ such that $u(\mathbf{x}_0, t_0) = \max_{\bar{D}} u$.

Case 1. $(\mathbf{x}_0, t) \in \Gamma$, trivial;

Case 2. Otherwise, we have $u_t(\mathbf{x}_0, t_0) = 0$ and $\nabla_{\mathbf{x}} u(\mathbf{x}_0, t_0) = \mathbf{0}$, and $(u_{ij}(\mathbf{x}_0, t_0) \preccurlyeq 0)$. These calculations force $Lu(\mathbf{x}_0, t_0) \geq 0$, which contradicts the assumption.

General Case: $Lu \leq 0$ in $\bar{D}\backslash\Gamma$. Let $v = u - \epsilon t$, then $Lv = Lu - \epsilon < 0$ in $\bar{D}\backslash\Gamma$. With the special case applied to v, we see that $\max_{\bar{D}} v = \max_{\Gamma} v$. Letting $\epsilon \to 0$, we are done.

□

Remark:The conclusion above holds even if L is degenerate parabolic, i.e. $a_{ij} \succcurlyeq 0$.

Theorem 1.4.2. **WMP**$(c \geq 0)$ *Suppose $u \in C^0(\bar{D}) \cap C^{2,1}(\bar{D}\backslash\Gamma)$ satisfies $Lu \leq 0$ in $\bar{D}\backslash\Gamma$. Assume that L is degenerate parabolic on $\bar{D}\backslash\Gamma$. Then*

$$\max_{\bar{D}} u \leq \max_{\Gamma} u^+.$$

Theorem 1.4.3. *Let L be degenerate parabolic on $\bar{D}\backslash\Gamma$ with c bounded from below on $\bar{D}\backslash\Gamma$. Suppose $u \in C^0\bar{D} \cap C^{2,1}(\bar{D}\backslash\Gamma)$ satisfies*

$$Lu \leq 0 \ \textit{in} \ \bar{D}\backslash\Gamma,\ u\big|_{\Gamma} \leq 0,$$

then

$$u \leq 0 \ \textit{in} \ \bar{D}.$$

证明. Let M be the lower bound of c, and multiply Lu by e^{Mt}. Let $v = e^{Mt}u$, then

$$e^{Mt}Lu = v_t - a_{ij}v_{ij} + b_i v_i + (c - M)v = L^* v.$$

Here v satisfies conditions in WMP$(c \geq 0)$, and so

$$\max_{\bar{D}} v \leq \max_{\Gamma} v^+,$$

which forces $\max_{\bar{D}} u \leq 0$. □

Corollary 1.4.1. Comparison Principle *Assume L is degenerate parabolic on $\bar{D}\backslash\Gamma$, c bounded from below on $\bar{D}\backslash\Gamma$. $u, v \in C^0(\bar{D}) \cap C^{2,1}(\bar{D}\backslash\Gamma)$ satisfies*

$$\begin{cases} Lu \leq Lv, & \text{on } \bar{D}\backslash\Gamma, \\ u \leq v, & \text{on } \Gamma. \end{cases}$$

Then $u \leq v$ all over the domain.

1.5 Strong Maximum Principle for Second Order Parabolic Differential Equations

Baby model: $Lu \leq 0$ in a rigid domain (the corresponding time-space domain is a cylinder-type domain). Then if u takes its maximum value at $(\mathbf{x}_0, t_0)$ in the parabolic interior, u will be a constant before time t_0.

1.5.1 Baby Hopf Boundary Point Lemmas

Lemma 1.5.1. Baby Hopf 1 *Let $B_R^- = \{(\mathbf{x}, t) \in \mathbb{R}^{n+1};\ |\mathbf{x}|^2 + (t-R)^2 < R^2,\ 0 < t \leq R\}$ be a half ball. Suppose L is strictly parabolic on B_R^-, $a_{ij}, b_i \& c$ are bounded on B_R^-. Let $u \in C^{2,1}(B_R^-) \cap C^0(\overline{B_R^-})$ satisfy $Lu \leq 0$ on B_R^-. Assume there is a $P_0 = (\mathbf{x}_0, t_0) \in \Gamma$ such that $\mathbf{x}_0 \neq 0$, $u(\mathbf{x}_0, t_0) > u(\mathbf{x}, t)$ for all $(\mathbf{x}, t) \in \overline{B_R^-}\backslash\{P_0\}$. Then for any outward pointing vector $\vec{v} \in \mathbb{R}^{n+1}$ at P_0, we have*

$$\frac{\partial u}{\partial \vec{v}}(P_0) > 0,$$

provided one of the following is satisfied:

1. *$c \equiv 0$ in B_R^-.*
2. *$c \geq 0$ in B_R^- and $u(P_0) > 0$.*
3. *$u(P_0) = 0$ regardless of c.*

证明. Define $\Sigma = \{(\mathbf{x}, t) \in B_\delta^-;\ |\mathbf{x}| > |\mathbf{x}_0|/2\}$. We will construct $v \in C^\infty(\mathbb{R}^{n+1})$ such that

a. $Lv < 0$ in Σ if $c \geq 0$ on B_R^-;

b. $v\big|_{\Gamma_{B_R^-}} \equiv 0$;

c. $\frac{\partial v}{\partial \vec{v}}(P_0) < 0$.

With this, we define $w(\mathbf{x}, t) = u(\mathbf{x}, t) - u(P_0) + \epsilon v(\mathbf{x}, t)$. Then

- $Lw = Lu - cu(P_0) + \epsilon Lv \leq \epsilon Lv < 0$ on Σ;
- $w\big|_{\Gamma_\Sigma} \leq 0$ if ϵ is tiny.

With WMP($c \geq 0$) applied to w on Σ, we obtain that $w \leq 0$ on Σ. But $w(P_0) = 0$ and so $\frac{\partial w}{\partial \vec{v}}(P_0) \geq 0$ implies that $\frac{\partial u}{\partial \vec{v}}(P_0) \geq -\epsilon \frac{\partial v}{\partial \vec{v}}(P_0) > 0$, which leads to the conclusion in case of 1. and 2.. If 3. holds, we see

$$0 \geq Lu = u_t - a_{ij}u_{ij} + b_i u_i + c^+ u + c^- u \geq u_t - a_{ij}u_{ij} + b_i u_i + c^+ u =: L^* u,$$

from where we return to case 2..
To construct such a v, we define $v(\mathbf{x}, t) = e^{-\alpha\left(|\mathbf{x}|^2 + (t-R)^2\right)} - e^{-\alpha R^2}$, with $\alpha > 0$ to be determined. Thus b. is immediately satisfied. Observe that

$$\nabla_{(\mathbf{x},t)} v = e^{(\cdots)} \left((-2\alpha \mathbf{x}), -2\alpha(t-R)\right)^T,$$

and so

$$\frac{\partial v}{\partial \vec{v}}(P_0) = -2\alpha e^{(\cdots)}(\mathbf{x}_0, t_0 - R) \cdot \vec{v} < 0,$$

which satisfies condition c.. Finally, we have

$$v_{ij} = -2\alpha e^{(\cdots)}\delta_{ij} + 4\alpha^2 x_i x_j e^{(\cdots)},$$

which gives

$$\begin{aligned} Lv &= -2\alpha e^{(\cdots)}(t-R) + 2\alpha e^{(\cdots)} a_{ij}\delta_{ij} - 4\alpha^2 x_i x_j a_{ij} e^{(\cdots)} \\ &\quad + b_i\left(-2\alpha_i e^{(\cdots)}\right) + c\left[e^{(\cdots)} - e^{-\alpha R^2}\right] \\ &\leq e^{(\cdots)}\left[-4\alpha^2\lambda_0|\mathbf{x}_0|^2/4 + 2\alpha(R + Mn + MnR) + M\right] \quad \text{in } \Sigma \\ &< 0, \quad \text{if } \alpha \text{ is huge.} \end{aligned}$$

□

Remark:

1) This lemma is true if B_R^- is replaced by B_R, provided P_0 is neither the south pole nor the north.

2) True if B_R^- is shifted but not rotated.

Corollary 1.5.1. *This lemma can be naturally extended to any time-space domain with boundary that satisfy interior sphere condition, in which the interior sphere does not touch the "maximum point" at either the north or the south pole.*

Lemma 1.5.2. Baby Hopf 2 *let* $Q(R,h) = \{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R};\ |\mathbf{x}| < R,\ 0 < t \leq h\}$. *Suppose* L *is strictly parabolic on* $Q(R,h)$ *and* a_{ij}, b_i, c *bounded. Let* $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ *satisfy* $Lu \leq 0$ *in* Q. *If* $u(\boldsymbol{0}, h) > u(\mathbf{x}, t)$ *for all* $|\mathbf{x}| < R$, $0 < t < h$, *then* $u_t(\boldsymbol{0}, h) > 0$, *provided one of the following holds:*

1. $c \equiv 0$ *in* Q.

2. $c \geq 0$ *in* Q, $u(\boldsymbol{0}, h) \geq 0$.

3. $u(\boldsymbol{0}, h) = 0$.

证明. Take a large $\rho > 0$ and small $\delta > 0$, such that $N := B_\rho(\mathbf{0}, h-\rho) \cap \{t > h-\delta\} \subset Q$. We shall construct $v \in C^\infty(\mathbb{R}^{n+1})$ such that

a. $Lv < 0$ in N;

b. $v\big|_{\partial B_\rho(\mathbf{0},h-\rho)} \equiv 0$;

c. $v_t(\mathbf{0}, h) < 0$.

Now, let $w(\mathbf{x}, t) = u(\mathbf{x}, t) - u(\mathbf{0}, h) + \epsilon v(\mathbf{x}, t)$. Then $Lw = Lu - cu(\mathbf{0}, h) + \epsilon Lv < 0$ in N if $c \geq 0$ in Q. Meanwhile, $w\big|_{\Gamma_N} \leq 0$. Applying WMP($c \geq 0$) to w we obtain $\max_{\bar{N}} w \leq \max_{\Gamma_N} w^+ = 0$. Then $w \leq 0$ all over N. But $w(\mathbf{0}, h) = 0$, then $\frac{\partial w}{\partial t}(\mathbf{0}, h) \geq 0$, and so

$$u_t(\mathbf{0}, h) > 0.$$

To construct such a v, we define

$$v(\mathbf{x}, t) = \rho^2 - |\mathbf{x}|^2 - (t - h + \rho)^2.$$

Then it's clear that v satisfies b.. And $v_t(\mathbf{0}, h) = -2\rho < 0$ fulfills c.. Finally, we have

$$\begin{aligned}
Lv &= -2(t - h + \rho) + 2a_{ij}\delta_{ij} + b_i(-2x_i) + cv \\
&\leq -2(h - \delta - h + \rho) + 2nM + 2nMR + cv \\
&\leq -2(\rho - \delta) + 2nM + 2nMR + 2\delta\rho M \\
&< 0, \quad \text{if } \rho \text{ large, and } \delta \text{ small.}
\end{aligned}$$

Now, we've shown $u_t(\mathbf{0}, h) > 0$ if either 1. and 2. holds. As for case 3., we see $u \leq 0$ all over Q, and as before neglect $c^- u$ to return to case 2.. □

Corollary 1.5.2. *The conditions that appear in Baby Hopf 2 cannot be satisfied at the same time.*

证明.

$$0 \geq Lu(\mathbf{0}, h) = u_t(\mathbf{0}, h) - a_{ij}u_{ij}(\mathbf{0}, h) + b_i u_i(\mathbf{0}, h) + cu(\mathbf{0}, h) > 0.$$

□

This corollary leads us to prove that the maximum point can only appear on the boundary.

1.5.2 Strong Maximum Principle

Let D be a time-space domain, and $p \in D$. We collect all the points q in D such that there is a continuous path γ in D connecting them, which is non-decreasing in time t and denote this set by $S(p)$.

Theorem 1.5.1. Strong Maximum Principle *Suppose L is strictly parabolic on D, a_{ij}, b_i, c bounded. Let u be a nice function that satisfy $Lu \le 0$ in $\bar{D}$. If there is a $p_0 = (\mathbf{x}_0, t_0) \in \bar{D} \backslash \Gamma$ such that $u(p_0) = \max_{\bar{D}} u =: M$, then $u \equiv u(p_0)$ all over $S(p_0)$, provided one of the following holds:*

1. *$c \equiv 0$ in D;*
2. *$c \ge 0$ in D and $u(p_0) \ge 0$;*
3. *$u(p_0) = 0$, regardless of the sign of c.*

证明. Let $F = \{(\mathbf{x}, t) \in \bar{D};\ u(\mathbf{x}, t) = M\}$, $d_{p_0} = dist(p_0, \Gamma) > 0$.

Claim 1. $B_{d_{p_0}/3}(p_0) :== \{(\mathbf{x}, t_0);\ |\mathbf{x} - \mathbf{x}_0| \le d_{p_0}/3\} \subset F$. Otherwise there is a point $\bar{p} = (\bar{\mathbf{x}}, t)$ in the ball such that $u(\bar{p}) < M$. Let $\delta = dist(\bar{p}, F) > 0$ and define a semi-ellipsoid

$$E_\sigma = \left\{ (\mathbf{x}, t);\ \frac{|\mathbf{x} - \bar{\mathbf{x}}|^2}{(\sigma\delta)^2} + \frac{(t - t_0)^2}{\delta^2} < 1,\ t \le t_0 \right\},\ \sigma > 0.$$

If $0 < \sigma < 1$, $E_\sigma \cap F = \emptyset$, by definition of δ. If $\sigma\delta \ge d_{p_0}/3$, $p_0 \in E_\sigma$ implies that $E_\sigma \cap F \ne \emptyset$. Therefore we see that:

Sub-claim 1. Increasing σ we have that E_σ touches F before touching Γ.

Sub-claim 2. The touching point Q_0 is at neither the south nor the north pole. Otherwise $|\bar{p}Q_0| = \delta \ge dist(\bar{p}, F) = 2\delta$, which is impossible.

Now we can construct a ball B inscribed in E_σ "tangent" to F at Q_0, with Q_0 not the south or north pole of B. Baby Hopf 1 implies that for an outward pointing vector $\vec{v}$, $\frac{\partial u}{\partial \vec{v}}(Q_0) > 0$. However, by sub-claim 1., $Q_0 \in \bar{D} \backslash \Gamma$ and $Q_0 \in F$, we see that $\nabla_{(\mathbf{x},t)} u(Q_0) = 0$.

Claim 2. Define $C(p_0) = S(p_0) \cap \{t = t_0\}$, then $C(p_0) \subset F$. This is because $C(p_0)$ is connected, and by claim 1 and definition of F, $C(p_0) \cap F$ is both open and closed in $C(p_0)$. This also tells us that once a point p is in F, then $C(p)$ will be contained in F.

Claim 3. $u \equiv M$ on $S(p_0)$. Otherwise there is a $Q = (\mathbf{x}_2, t_2) \in S(p_0)$ such that $u(Q) < M$. Let $p_1 = (\mathbf{x}_1, t_1)$ be a point on the arc $\tilde{Qp_0}$ such that $u < M$ on $\tilde{Qp_1}$ and $u(p_1) = M$. By claim 2., $C(p_1) \subset F$, and $S(p_0)$ is already split by $C(p_1)$. Now, we construct a cylinder $Q_{p_1}(R, h) = \{(\mathbf{x}, t);\ |\mathbf{x} - \mathbf{x}_1| < R,\ t_1 - h < t \le t_1\}$ with R, h tiny, then with a simple application of Baby Hopf 2, we are done.

□

Application: Let Ω be bounded in $\mathbb{R}^n$ with $\partial\Omega \in C^2$, $D = \Omega \times (0, T)$, $S = \partial\Omega \times [0, T]$. Also let $Lu = u_t - a_{ij}(\mathbf{x}, t)u_{ij} + b_i(\mathbf{x}, t)u_i + f(\mathbf{x}, t, u)$ and define boundary operator to be

$$Bu = \frac{\partial u}{\partial \vec{v}} + \beta(\mathbf{x}, t)u,\ \beta \ge 0,\ \text{on } S.$$

Theorem 1.5.2. Comparison Principle *Assume* $u, v \in C^{2,1}(\bar{D})$ *satisfy*

$$\begin{cases} Lu \geq Lv, & \text{in } \bar{D} \backslash \Gamma, \\ Bu \geq Bv, & \text{on } S, \\ u\big|_{t=0} \geq v\big|_{t=0}, & \text{on } \bar{\Omega}. \end{cases}$$

Suppose L *is strictly parabolic on* D, a_{ij}, b_i *bounded on* $\bar{D}\backslash\Gamma$. *For all* $(\mathbf{x}, t)$ *in the parabolic interior,* $f_u(\mathbf{x}, t, u)$ *exists for any* $u \in (-\infty, +\infty)$. *For any* $R > 0$, *there is* $M > 0$ *such that*

$$|f_u(\mathbf{x}, t, u)| \leq M, \ \forall (\mathbf{x}, t) \in \bar{D}\backslash\Gamma, \ |u| \leq R.$$

Then $u \geq v$ *on* $\bar{D}$. *If* $u\big|_{t=0} \neq v\big|_{t=0}$, *then* $u(\mathbf{x}, t) > v(\mathbf{x}, t), \ t > 0, \ \mathbf{x} \in \bar{\Omega}$.

Remark: This also holds if $Bu = u$ on S, in which case "$\mathbf{x} \in \bar{\Omega}$" will be replaced by "$\mathbf{x} \in \Omega$".

证明. Let $w = u - v$. Then $f(\mathbf{x}, t, u) - f(\mathbf{x}, t, v) = c(\mathbf{x}, t)w$, where c is bounded in the parabolic interior. If $m := \min_{\bar{D}} w < 0$ is achieved inside the parabolic interior, then by SMP we see $w\big|_{t=0} = m$ which can only be 0 or does not occur. If m is achieved on the parabolic boundary, then we only have worry about S. Applying Baby Hopf 1, we may see it's not the case. □

Chapter 2

Sobolev Space Theory

2.1 Distributions and Fundamental Solution

2.1.1 Distributions

Preparations: Let $\Omega \subset \mathbb{R}^n$ be a domain and $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ the space of all definitely differentiable functions that have compact support in Ω. The space $\mathcal{D}(\Omega)$ is not empty:

$$j(\mathbf{x}) = \begin{cases} ce^{\frac{1}{|\mathbf{x}|^2-1}}, & \mathbf{x} \in \mathbb{R}^n,\ |\mathbf{x}| < 1; \\ 0, & \text{else}, \end{cases}$$

where c is chosen that the integral of j over the whole space is 1, then one can show that j is in $C_0^\infty(\mathbb{R}^n)$, with support the unit ball centered at the origin. Now, if we set

$$j_{\epsilon,\mathbf{y}}(\mathbf{x}) = \frac{1}{\epsilon^n} j\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon}\right),$$

we see the integral of $j_{\epsilon,\mathbf{y}}$ is still 1 and it has support the ball of radius ϵ centered at $\mathbf{y}$. Let $\mathbf{y}$ be a point in Ω, then when ϵ is small enough, $j_{\epsilon,\mathbf{y}}$ is in $\mathcal{D}(\Omega)$.

Definition 2.1. *A **distribution** on $\mathcal{D}(\Omega)$ is a linear functional*

$$f : \mathcal{D}(\Omega) \to \mathbb{R}$$
$$\phi \mapsto < f, \phi >$$

such that f is continuous in the sense that for any sequence $\{\phi_k\} \subset \mathcal{D}(\Omega)$:

i. Supp$\{\phi_k\} \subset\subset \Omega,\ \forall k \geq 1$;

ii. If for all $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$,

$$\|\partial^{\boldsymbol{\alpha}} \phi_k\|_{L^\infty(\Omega)} \to 0, \text{ as } k \to \infty,$$

we have $< f, \phi_k > \longrightarrow 0$ as $k \to \infty$.

We shall denote by $\mathcal{D}'(\Omega)$ the set of all distributions.

Examples:

i. Let $f \in L^1_{loc}(\Omega)$ the space of all locally integrable functions. With this function we may define a functional on $\mathcal{D}(\Omega)$ as follows:

$$\phi \in \mathcal{D}(\Omega) \xrightarrow{F} \int_\Omega f(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \in \mathbb{R}.$$

One may find that the functional $F \in \mathcal{D}'(\Omega)$, because given test functions $\{\phi_k\}$, we only have to check that

$$\begin{aligned}| < F, \phi_k > | &= \left| \int_\Omega f(\mathbf{x})\phi_k(\mathbf{x})d\mathbf{x} \right| \\ &\leq \int_\Omega |f||\phi_k| \\ &\leq \|\phi_k\|_{L^\infty} \int_\Omega |f| \\ &\overset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

Here, we would like to call F the functional *induced* by f, and we shall not distinguish them in the following contexts

ii. Suppose $f \in \mathcal{D}'(\Omega)$, and g definitely differentiable, then $f \cdot g$ defined by

$$\phi \in \mathcal{D}(\Omega) \to < f, g\phi >$$

is still a distribution.

iii. Fix $\mathbf{y} \in \Omega$, we define

$$\phi \in \mathcal{D}(\Omega) \xrightarrow{\delta_{\mathbf{y}}} \phi(\mathbf{y}),$$

and it is indeed a distribution. This kind of distribution is called δ-function (or Dirac function).

Definition 2.2. *Let f_k be a sequence of distributions, and f another one. We say $f_k \to f$ as $k \to \infty$ if for all fixed test function ϕ, $< f_k, \phi > \overset{k\to\infty}{\longrightarrow} < f, \phi >$.*

Theorem 2.1.1. *Let f_k be a sequence of integrable functions such that*

i. f_k concentrates at a point in the following sense:

$$\forall \delta > 0, \int_{\{|\mathbf{x}-\mathbf{y}|\geq\delta\}\cap\Omega} |f_k(\mathbf{x})|d\mathbf{x} \overset{k\to\infty}{\longrightarrow} 0,$$

ii. $\int_\Omega f_k \overset{k\to\infty}{\longrightarrow} A \in \mathbb{R}$ and f_k uniformly bounded by $M > 0$ in L^1.

Then f_k converges to $A\delta_{\mathbf{y}}$ in the sense of distributions.

证明. Given a test function ϕ, we see $A\phi(\mathbf{y}) = \lim_{k\to\infty} \int_\Omega f_k \phi(\mathbf{y})$, and so it suffices to show

$$\int_\Omega |f_k(\mathbf{x})||\phi(\mathbf{x}) - \phi(\mathbf{y})| d\mathbf{x} \stackrel{k\to\infty}{\longrightarrow} 0.$$

For $\epsilon > 0$, we choose $\delta > 0$ such that $|\phi(\mathbf{x}) - \phi(\mathbf{y})| < \epsilon$ for all $|\mathbf{x} - \mathbf{y}| < \delta$. Then the integral splits into two parts: one inside the ball $B_\delta(\mathbf{y})$ and another outside it, which finally gives the bound $M\epsilon + 2\|\phi\|_{L^\infty} \int_{\Omega \cap B_\delta(\mathbf{y})^c} |f_k|$ and hence the convergence. □

Examples:

i. Fix $\mathbf{y} \in \Omega$, then $j_{\epsilon,\mathbf{y}}$ converges to $\delta_\mathbf{y}$ in the sense of distributions;

ii. Suppose $f \in \mathcal{D}(\Omega)$ is Riemann integrable on $[0,1]$. Let P_k be a sequence of partitions of $[0,1]$ such that $||P_k|| \longrightarrow 0$ as $k \to \infty$. Then

$$\sum_{i=1}^{n_k} f(x_i^k)\phi(x_i^k)\Delta x_i^k \stackrel{k\to\infty}{\longrightarrow} \int_0^1 f(x)\phi(x)dx,$$

for any test function ϕ. Now, the left hand side can be written as $< \sum_{i=1}^{n_k} f(x_i^k)\delta_{x_i^k}\Delta x_i^k, \phi >$, which gives the fact that

$$\sum_{i=1}^{n_k} f(x_i^k)\delta_{x_i^k}\Delta x_i^k \longrightarrow f,\ k \to \infty,$$

in the sense of distribution;

iii. Heat kernal $\Gamma(\mathbf{x}, t) = \dfrac{1}{4\pi kt} e^{-\frac{|\mathbf{x}|^2}{4kt}}$. We have

$$\Gamma(\mathbf{x}, t) \stackrel{t\to 0^+}{\longrightarrow} \delta_\mathbf{0},$$

in the sense of distributions.

2.1.2 Derivatives of Distributions

Motivation: Let $f \in C^1(\Omega)$, which along with its partial derivatives is clearly locally integrable. Observe that

$$\begin{aligned} < \frac{\partial f}{\partial x_i}, \phi > &= \int_\Omega \frac{\partial f}{\partial x_i}(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \\ &= \int_\Omega \left[\frac{\partial}{\partial x_i}(f\phi) - f\frac{\partial \phi}{\partial x_i} \right] d\mathbf{x} \\ &= -\int_\Omega f\frac{\partial \phi}{\partial x_i} d\mathbf{x} + \int_\Omega \nabla \cdot \vec{F} d\mathbf{x} \\ &= -\int_\Omega f\frac{\partial \phi}{\partial x_i} d\mathbf{x} + \int_{\partial\Omega} \vec{F} \cdot \vec{n} ds \\ &= -\int_\Omega f\frac{\partial \phi}{\partial x_i} d\mathbf{x}, \end{aligned}$$

where $\vec{F} = (0, \cdots, 0, f\phi, 0, \cdots, 0)^T$, with $f\phi$ at the i-th position.

Definition 2.3. *For every $f \in \mathcal{D}'(\Omega)$, $\phi \in \mathcal{D}(\Omega)$, $i = 1, \cdots, n$, we define*

$$< \frac{\partial f}{\partial x_i}, \phi >= - < f, \frac{\partial \phi}{\partial x_i} > .$$

More generally, let $\boldsymbol{\alpha} \in \mathbb{N}^n$ and we define

$$< \partial^{\boldsymbol{\alpha}} f, \phi >= (-1)^{|\boldsymbol{\alpha}|} < f, \partial^{\boldsymbol{\alpha}} \phi > .$$

Remark: If the distribution has continuous partial derivatives, then its corresponding distributional derivatives coincide with them in the sense of distribution.

Example: Let

$$H(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \geq 0. \end{cases}$$

H is called the *Heaviside function*. It can be shown that $H \in \mathcal{D}'(\Omega)$, and for any $\phi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} < H', \phi > &= - < H, \phi' > \\ &= - \int_{-\infty}^{\infty} H(x)\phi(x)dx \\ &= - \int_{0}^{\infty} \phi'(x)dx \\ &= -\phi(\infty) + \phi(0) \\ &= \phi(0) \\ &=< \delta_0, \phi > . \end{aligned}$$

2.1.3 Distributional Solutions to PDEs

Consider a partial differential operator on Ω

$$Lu = \sum_{|\boldsymbol{\alpha}|=0}^{m} A_{\boldsymbol{\alpha}}(\mathbf{x}) \partial^{\boldsymbol{\alpha}} u,$$

with all $A_{\boldsymbol{\alpha}}$ smooth. If u is a distribution, then Lu is still a distribution with

$$\begin{aligned} < Lu, \phi > &= \sum_{|\boldsymbol{\alpha}|=0}^{m} < A_{\alpha} \partial^{\boldsymbol{\alpha}} u, \phi > \\ &= \sum_{|\boldsymbol{\alpha}|=0}^{m} (-1)^{|\boldsymbol{\alpha}|} < u, \partial^{\boldsymbol{\alpha}}(A_{\boldsymbol{\alpha}}\phi) > \\ &=< u, \sum_{|\boldsymbol{\alpha}|=0}^{m} (-1)^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}}(A_{\boldsymbol{\alpha}}\phi) > \\ &=:< u, L^* \phi >, \end{aligned}$$

where L^* is called the *formal adjoint* of L. The formal adjoint of the famous operator Laplcian "$-\Delta$" is itself.

Definition 2.4. *Consider PDE $Lu = f$ ☻ on Ω, with f a distribution. We say u is a **distributional solution** of ☻ if $Lu = f$ in the sense of distribution.*

Fundamental Solution: Now, we focus on a PDE $-\Delta u = \delta_0$ in $\mathbb{R}^n$. A solution of this equation is called the *fundamental solution* of $-\Delta u = 0$. Because δ_0 is radial, we expect the solution is also radial, i.e. $u = u(r)$. Recall that $-\Delta u = -u_r r - (n-1)u_r/r = \delta_0$, we see that

$$u_r r + (n-1)u_r/r = 0, \quad \text{if } r > 0.$$

Now, we have $(r^{n-1}u_r)_r = 0$, and so $u_r = c/r^{n-1}$. When $n = 2$, $u = c\ln r$, and when $n \geq 3$, $u = c/r^{n-2}$. What is the constant c?

"Tanglang Street" Method: When $n = 2$, we have

$$\begin{aligned}
&-\Delta u = \delta_0 \\
&\implies -(u_r r + u_r/r) = \delta_0 \\
&\implies -(ru_r)r = r\delta_0 \\
&\implies \int_0^r -(ru_r)rdr = \int_0^r r\delta_0 dr \\
&\implies -ru_r = \int_0^r r\delta_0 dr \\
&\implies -ru_r \cdot 2\pi = \int_{|\mathbf{x}|<r} \delta_0(\mathbf{x})d\mathbf{x} \\
&\implies u_r = -\frac{1}{2\pi}\ln r.
\end{aligned}$$

"Ivory Tower" Method: We regularize $c\ln r$ by $\frac{c}{2}\ln(r^2+\epsilon^2) =: u_\epsilon(\mathbf{x})$, with $\epsilon > 0$ small. Observe that u_ϵ is smooth for each index, and we may compute

$$-\Delta u_\epsilon = \frac{2c\epsilon^2}{(r^2+\epsilon^2)^2}.$$

Now it's easy to show that this is an approximation to the identity multiplied by $-2\pi c$, i.e.

$$-\Delta u_\epsilon \longrightarrow -2\pi c\delta_0$$

as $\epsilon \to 0$ in the sense of distributions. Now we have

$$\begin{aligned}
&\lim_{\epsilon\to 0} < -\Delta u_\epsilon, \phi > = < 2\pi c\delta_0, \phi > \\
&\implies \lim_{\epsilon\to 0} < u_\epsilon, -\Delta\phi > = RHS \\
&\implies \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} u_\epsilon(\mathbf{x})(-\Delta\phi(\mathbf{x}))d\mathbf{x} = RHS \\
&|\ln r| \in L^1\left(B_{1/2}(\mathbf{0})\right), \text{ and we may apply LDCT} \\
&\implies LHS = \int_{\mathbb{R}^2} u\,(-\Delta\phi)\,d\mathbf{x} = < -\Delta(c\ln r), \phi >,
\end{aligned}$$

which implies that $-\Delta(c\ln r)=-2\pi c\delta_0$ in the sense of distributions and that it suffices to choose $c=-\frac{1}{2\pi}$.

"Tanglang Street" Method: When $n\geq 3$, we see similarly

$$\begin{aligned}
&-\Delta u=\delta_0\\
&\Longrightarrow\ -(u_{rr}+(n-1)u_r/r)=\delta_0\\
&\Longrightarrow\ -(r^{n-1}u_r)r=r^{n-1}\delta_0\\
&\Longrightarrow\ \int_0^r -(r^{n-1}u_r)rdr=\int_0^r r^{n-1}\delta_0 dr\\
&\Longrightarrow\ -r^{n-1}u_r=\int_0^r r^{n-1}\delta_0 dr\\
&\Longrightarrow\ -r^{n-1}u_r\int_{|\sigma|=1,\,\sigma\in\mathbb{R}^n} d\sigma=\int_{|\mathbf{x}|<r}\delta_0(\mathbf{x})d\mathbf{x}\\
&\Longrightarrow\ u_r=-\frac{1}{|\mathbb{S}^{n-1}|r^{n-1}},
\end{aligned}$$

which gives $u(r)=\frac{1}{(n-2)n|B_1(\mathbf{0})|r^{n-2}}=:\Gamma(\mathbf{x})$.

"Ivory Tower" Method: To show $-\Delta\Gamma=\delta_0$ in the sense of distribution, i.e.

$$<\Gamma,-\Delta\phi>=<\delta_0,\phi>,$$

for all test function ϕ. To this end, we again regularize $\Gamma(\mathbf{x})$ as

$$\Gamma_\epsilon(\mathbf{x})=\frac{1}{(n-2)n|B_1(\mathbf{0})|(r^2+\epsilon^2)^{(n-2)/2}},\ \epsilon>0.$$

Question:

$$\int_{\mathbb{R}^n}\Gamma_\epsilon(-\Delta\phi)\longrightarrow\int_{\mathbb{R}^n}\Gamma(-\Delta\phi)?\ ☺$$

Answer: Yes! Observe that ϕ is smooth and compactly supported, then there exist big R and $M>0$ such that $Supp\{-\phi\}\subset B_R(\mathbf{0})$, $|-\Delta\phi|\leq M$ and so

$$|\Gamma_\epsilon(-\Delta\phi)|\leq c_n M\chi_{B_R(\mathbf{0})}\cdot\frac{1}{r^{n-2}}\in L^1(\mathbb{R}^n).$$

where $c_n=\frac{1}{(n-2)n|B_1(\mathbf{0})|}$. Applying LDCT, one can prove the answer. On the other hand, we check that $-\Delta\Gamma_\epsilon$ converges to δ_0

i.
$$\frac{\partial\Gamma_\epsilon}{\partial r}=(2-n)c_n(r^2+\epsilon^2)^{-n/2}r,$$

and

$$\begin{aligned}
\frac{\partial^2\Gamma_\epsilon}{\partial^2 r}&=c_n(2-n)\left[-\frac{n}{2}(r^2+\epsilon^2)^{-(n+2)/2}\cdot 2r^2+(r^2+\epsilon^2)^{-n/2}\right]\\
&=\frac{c_n(n-2)\left[(1-n)r^2+\epsilon^2\right]}{(r^2+\epsilon^2)^{(n+2)/2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
-\Delta\Gamma_\epsilon &= -\left(\frac{\partial^2\Gamma_\epsilon}{\partial^2 r} + \frac{n-1}{r}\frac{\partial\Gamma_\epsilon}{\partial r}\right) \\
&= c_n(n-2)\frac{n\epsilon^2}{(r^2+\epsilon^2)^{(n+2)/2}} \\
&= \frac{\epsilon^2}{|B_1(\mathbf{0})|(r^2+\epsilon^2)^{(n+2)/2}},
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^n} |-\Delta\Gamma_\epsilon(\mathbf{x})|d\mathbf{x} = \int_{|\sigma|=1}\int_0^\infty \frac{\epsilon^2}{|B_1(\mathbf{0})|(r^2+\epsilon^2)^{(n+2)/2}} r^{n-1} dr d\sigma.$$

Let $r = \epsilon\tan(\theta)$, then $dr = \epsilon\sec^2(\theta)d\theta$, and then

$$\begin{aligned}
RHS &= \frac{|\mathbb{S}^{n-1}|}{|B_1(\mathbf{0})|}\epsilon^2\int_0^{\pi/2}\frac{(\tan(\theta))^{n-1}\epsilon^n\sec^2(\theta)}{\epsilon^{n+2}(\sec(\theta))^{n+2}}d\theta \\
&= n\int_0^{\pi/2}\frac{(\sin(\theta))^{n-1}}{(\cos(\theta))^{n-1}}(\cos(\theta))^n d\theta \\
&= n\int_0^{\pi/2}(\sin(\theta))^{n-1} d\sin(\theta) \\
&= 1.
\end{aligned}$$

ii. $\int_{\mathbb{R}^n}\Gamma_\epsilon(\mathbf{x})d\mathbf{x} = 1$;

iii. For a fixed $\delta > 0$, we have

$$\begin{aligned}
\int_{|\mathbf{x}|>\delta} |-\Delta\Gamma_\epsilon(\mathbf{x})|d\mathbf{x} &= n\int_{\arctan\frac{\delta}{\epsilon}}^{\pi/2}(\sin(\theta))^{n-1}\cos(\theta)d\theta \\
&\longrightarrow 0,
\end{aligned}$$

as $\epsilon \to 0$.

Remark: For a fixed $\mathbf{y} \in \mathbb{R}^n$ we have in the sense of distribution $\Gamma_{\mathbf{y}}(\mathbf{x}) := \Gamma(\mathbf{x}-\mathbf{y})$ satisfies the equation $-\Delta_{\mathbf{x}} u = \delta_{\mathbf{y}}$.

Poisson Equation:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n,$$

where f denotes the density of charges and u the corresponding electric potential. Let $f(\mathbf{y})d\mathbf{y}$ be the total charges of a small piece of electric material near $\mathbf{y}$, then its electric potential at $\mathbf{x}$ is approximately given by $\Gamma(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y}$, and so it is reasonable to guess the observed potential should be

$$u(\mathbf{x}) = \int_{\mathbb{R}^n}\Gamma(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y}.$$

"Tanglang Street" Method: Check

$$\begin{aligned}
-\Delta_{\mathbf{x}} \int_{\mathbb{R}^n} \Gamma(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} &= \int_{\mathbb{R}^n} -\Delta_{\mathbf{x}}(\Gamma(\mathbf{x}-\mathbf{y})) f(\mathbf{y}) d\mathbf{y} \\
&= \int_{\mathbb{R}^n} \delta(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} \\
&= f(\mathbf{x}).
\end{aligned}$$

"Ivory Tower" Method:

Theorem 2.1.2. *Suppose $f \in C_0^2(\mathbb{R}^n)$, and let*

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Gamma(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y},$$

then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

证明. Since f is C^2-smooth and compactly supported, $u(\mathbf{x})$ is well-defined. Let $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$, with 1 at the i-th position, then we have

$$\begin{aligned}
\frac{u(\mathbf{x}+he_i)-u(\mathbf{x})}{h} &= \int_{\mathbb{R}^n} \Gamma(\mathbf{z}) \frac{f(\mathbf{x}+he_i-\mathbf{z})-f(\mathbf{x}-\mathbf{z})}{h} d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \Gamma(\mathbf{z}) \frac{\partial f}{\partial x_i}(\mathbf{x}+s(h)e_i-\mathbf{z}) d\mathbf{z},
\end{aligned}$$

where $f_{\mathbf{x}_i}$ is bounded and compactly supported and Γ is locally integrable. According to LDCT, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{u(\mathbf{x}+he_i)-u(\mathbf{x})}{h} &= \int_{\mathbb{R}^n} \Gamma(\mathbf{z}) \lim_{h\to 0} \frac{f(\mathbf{x}+he_i-\mathbf{z})-f(\mathbf{x}-\mathbf{z})}{h} d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \Gamma(\mathbf{z}) \frac{\partial f}{\partial x_i}(\mathbf{x}-\mathbf{z}) d\mathbf{z},
\end{aligned}$$

which exists and continuous (shown in a similar way.) In the same manner, one can handle the second derivatives, which immediately show that the integral function satisfies Poisson equation.

□

Remark: This integral function is called *Newtonian Potential.*

2.2 Weak Derivatives

Definition 2.5. *Suppose $u \in L^1_{loc}(\Omega)$ and its distributional derivative $\partial^{\boldsymbol{\alpha}} u$ can be realised by a locally integrable function v, then we say v is the **weak $\boldsymbol{\alpha}$-th** partial derivative of u, and simply write $v = \partial^{\boldsymbol{\alpha}} u$.*

Remark: If $u \in C^{|\boldsymbol{\alpha}|}(\Omega)$, then its weak derivative is exactly the classical one.

Definition 2.6. *We say $u \in L^1_{loc}(\Omega)$ is k-times **weakly differentiable** if all weak derivatives $\partial^{\boldsymbol{\alpha}} u$ with $|\boldsymbol{\alpha}| \leq k$ exist. The set of all such u is denoted by $W^k(\Omega)$.*

Example: Let $u(x) = |x|$ defined on the real line, then the weak derivative of it is

$$u'(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ 0, & \text{else.} \end{cases}$$

What about u''? The answer is that it does not exist: If weak u'' exists and equals $v(x)$, then v is locally integrable and for all test function ϕ we have

$$\begin{aligned} \int_{-\infty}^{\infty} v(x)\phi(x)dx &= \int_{-\infty}^{\infty} u(x)\phi''(x)dx \\ &= \int_{0}^{\infty} u(x)\phi''(x)dx + \int_{-\infty}^{0} u(x)\phi''(x)dx \\ &= u\phi'\Big|_{x=0}^{x=\infty} - \int_{0}^{\infty} u'(x)\phi'(x)dx + u\phi'\Big|_{x=-\infty}^{x=0} - \int_{-\infty}^{0} u'(x)\phi'(x)dx \\ &= 2\phi(0). \end{aligned}$$

Recall that

$$j(x) = j(\mathbf{x}) = \begin{cases} ce^{\frac{1}{|\mathbf{x}|^2-1}}, & \mathbf{x} \in \mathbb{R}^n,\ |\mathbf{x}| < 1; \\ 0, & \text{else,} \end{cases}$$

and we set $\phi_\epsilon = j(x/\epsilon)$. Thus we see $|v(x)\phi_\epsilon(x)| \le j(0)|v(x)|\chi_{B_1}(x)$ if $\epsilon < 1$, which is integrable. According to LDCT, $RHS \equiv 2j(0)$ while LHS converges to 0, as $\epsilon \to 0$.

2.2.1 Approximate Bad Functions by Good Ones

Definition 2.7. *For all $u \in L^1_{loc}(\Omega)$, the regularization of u is*

$$u_\epsilon(\mathbf{x}) = \int_\Omega j_\epsilon(\mathbf{x} - \mathbf{y})u(\mathbf{y})d\mathbf{y},$$

where $0 < \epsilon < dist(\mathbf{x}, \partial\Omega)$, and $j_\epsilon(\mathbf{z}) = \frac{1}{\epsilon^n} j(\frac{\mathbf{z}}{\epsilon})$.

Remark:

1. u_ϵ is well-defined because

   $$|j_\epsilon(\mathbf{x} - \mathbf{y})u(\mathbf{y})| \le \|j_\epsilon\|_\infty |u(\mathbf{y})|\chi_{B_\epsilon(\mathbf{x})}(\mathbf{y}),$$

   which is integrable.

2. If u is integrable on Ω, then for all $\mathbf{x} \in \mathbb{R}^n$ and $\epsilon > 0$ we define

   $$u_\epsilon(\mathbf{x}) = \int_{\mathbb{R}^n} j_\epsilon(\mathbf{x} - \mathbf{y})u(\mathbf{y})d\mathbf{y},$$

   where we extend u by letting it to be 0 outside Ω. Then $u_\epsilon(\mathbf{x}) = j_\epsilon * u(\mathbf{x})$.

Lemma 2.2.1. *For any fixed $\epsilon > 0$, $u_\epsilon \in C^\infty(\Omega_\epsilon)$, where $\Omega_\epsilon = \{\mathbf{x} \in \Omega;\ dist(\mathbf{x}, \partial\Omega) > \epsilon\}$. Moreover, if $u \in L^1(\Omega)$, then $u_\epsilon \in C^\infty(\mathbb{R}^n)$. Also, if Ω is bounded, then $Supp\{u_\epsilon\}$ is bounded and $u_\epsilon \in C_0^\infty(\mathbb{R}^n)$.*

证明. Let $e_i = (0, \cdot, 0, 1, 0, \cdots, 0)^T$, with 1 at the i-th position. For any fixed $\mathbf{x} \in \Omega_\epsilon$, we consider

$$\begin{aligned}\frac{u_\epsilon(\mathbf{x}+he_i) - u_\epsilon(\mathbf{x})}{h} &= \int_\Omega \frac{j_\epsilon(\mathbf{x}+he_i-\mathbf{y}) - j_\epsilon(\mathbf{x}-\mathbf{y})}{h} u(\mathbf{y}) d\mathbf{y} \\ &= \int_\Omega \frac{\partial}{\partial x_i} j_\epsilon(\mathbf{x}+\delta(h)e_i - \mathbf{y}) u(\mathbf{y}) d\mathbf{y},\end{aligned}$$

where $0 \le \delta(h) \le h$, and

$$\begin{aligned}\left|\frac{\partial}{\partial x_i} j_\epsilon(\mathbf{x}+\delta(h)e_i-\mathbf{y})u(\mathbf{y})\right| &\le \left\|\frac{\partial j_\epsilon}{\partial x_i}\right\|_\infty |u(\mathbf{y})| \chi_{B_{\epsilon+h}(\mathbf{x})}(\mathbf{y}) \\ &\le \left\|\frac{\partial j_\epsilon}{\partial x_i}\right\|_\infty |u(\mathbf{y})| \chi_{B_l(\mathbf{x})}(\mathbf{y}),\end{aligned}$$

with $\epsilon < l < dist(\mathbf{x}, \partial\Omega)$ and h small. Now, applying LDCT we see

$$\lim_{h\to 0} \frac{u_\epsilon(\mathbf{x}+he_i) - u_\epsilon(\mathbf{x})}{h} = \int_\Omega \frac{\partial}{\partial x_i} j_\epsilon(\mathbf{x}-\mathbf{y}) u(\mathbf{y}) d\mathbf{y}.$$

Now, suppose $\mathbf{x}_k$ is a sequence of points in Ω_ϵ such that $\mathbf{x}_k \overset{k\to\infty}{\longrightarrow} \mathbf{x}$ in Ω_ϵ. Then we have

$$\begin{aligned}\left|\frac{\partial u_\epsilon}{\partial x_i}(\mathbf{x}_k) - \frac{\partial u_\epsilon}{\partial x_i}(\mathbf{x})\right| &\le \int_\Omega \left|\frac{\partial j_\epsilon}{\partial x_i}(\mathbf{x}-\mathbf{y}) - \frac{\partial j_\epsilon}{\partial x_i}(\mathbf{x}_k - \mathbf{y})\right| |u(\mathbf{y})| d\mathbf{y} \\ &\quad \text{the integrand is dominated by an integrable function } 2\left\|\frac{\partial j_\epsilon}{\partial x_i}\right\|_\infty |u(\mathbf{y})| \chi_{B_\delta(\mathbf{x})}(\mathbf{y}) \\ &\quad \longrightarrow 0, \text{ as } k \to \infty.\end{aligned}$$

Similar proofs show that $u_\epsilon \in C^\infty(\Omega_\epsilon)$. When u is integrable, then u_ϵ is smooth all over the space. If further Ω is bounded, then we take $\mathbf{x}$ such that $dist(\mathbf{x}, \bar{\Omega}) > \epsilon$, and observe that $j_\epsilon(\mathbf{x}-\mathbf{y}) \equiv 0$ as a function in $\mathbf{y}$. □

Lemma 2.2.2. *If $u \in C^0(\Omega)$, then for any $\Omega' \subset\subset \Omega$ $(\overline{\Omega'} \subset \Omega)$*

$$u_\epsilon \overset{\epsilon\to 0}{\longrightarrow} u,$$

on $\overline{\Omega'}$ uniformly.

证明. Let $\epsilon \in (0, dist(\overline{\Omega'}, \partial\Omega))$, then

$$u_\epsilon(\mathbf{x}) = \int_\Omega j_\epsilon(\mathbf{x}-\mathbf{y}) u(\mathbf{y}) d\mathbf{y}$$

is well-defined on $\overline{\Omega'}$, and we may observe that the integrand supports on a small ball $B_\epsilon(\mathbf{x})$. If we let $z = \frac{\mathbf{x}-\mathbf{y}}{\epsilon}$, we see $d\mathbf{z} = \frac{1}{\epsilon^n} d\mathbf{y}$ and

$$u_\epsilon(\mathbf{x}) = \int_{B_1(\mathbf{0})} j(\mathbf{z}) u(\mathbf{x} - \epsilon\mathbf{z}) d\mathbf{z}.$$

Now, we rewrite $u(\mathbf{x}) = \int_{B_1(\mathbf{0})} j(\mathbf{z})u(\mathbf{x})d\mathbf{z}$ and observe that

$$|u_\epsilon(\mathbf{x}) - u(\mathbf{x})| \leq \int_{B_1(\mathbf{0})} j(\mathbf{z})|u(\mathbf{x} - \epsilon\mathbf{z}) - u(\mathbf{x})|d\mathbf{z}.$$

On a slightly larger domain Ω'' ($\Omega' \subset\subset \Omega'' \subset\subset \Omega$), we know u is uniformly continuous, and so we are done. □

Lemma 2.2.3. *Let $1 \leq p < \infty$, $u \in L^p_{loc}(\Omega)$ (L^p), then $u_\epsilon \xrightarrow{\epsilon\to 0} u$ in $L^p_{loc}(\Omega)$.*

证明. We want to show for any $\Omega' \subset\subset \Omega$

$$\int_{\Omega'} |u_\epsilon(\mathbf{x}) - u(\mathbf{x})|^p d\mathbf{x} \xrightarrow{\epsilon\to 0} 0.$$

Recall that

$$u_\epsilon(\mathbf{x}) = \int_{B_1(\mathbf{0})} j(\mathbf{z})u(\mathbf{x} - \epsilon\mathbf{z})d\mathbf{z},$$

we have

$$\begin{aligned}
\|u_\epsilon - u\|_{L^p(\Omega')} &= \left\| \int_{B_1(\mathbf{0})} j(\mathbf{z}) \left(u(\cdot - \epsilon\mathbf{z}) - u(\cdot)\right) d\mathbf{z} \right\|_{L^p(\Omega')} \\
&\leq \int_{B_1(\mathbf{0})} j(\mathbf{z}) \left\|u(\cdot - \epsilon\mathbf{z}) - u(\cdot)\right\|_{L^p(\Omega')} d\mathbf{z}.
\end{aligned}$$

As before, we consider an intermediate domain Ω'' and redefine u by setting it to be 0 outside Ω''. Now,

$$\|u_\epsilon - u\|_{L^p(\Omega')} \leq \int_{B_1(\mathbf{0})} j(\mathbf{z}) \left\|u(\cdot - \epsilon\mathbf{z}) - u(\cdot)\right\|_{L^p(\mathbb{R}^n)} d\mathbf{z},$$

and by continuity of $L^p(\mathbb{R}^n)$-norm in spacial translation, we see

$$\|u(\cdot - \epsilon\mathbf{z}) - u(\cdot)\|_{L^p(\mathbb{R}^n)} \xrightarrow{\epsilon\to 0} 0$$

uniformly for $\mathbf{z} \in B_1(\mathbf{0})$. Thus we obtain the $L^p(\Omega')$ convergence. □

Remark:

1. In the case of $u \in L^p(\Omega)$, $u_\epsilon \in C^\infty(\mathbb{R}^n)$, and

$$|u_\epsilon(\mathbf{x})| = \left| \int_\Omega j_\epsilon(\mathbf{x} - \mathbf{y})u(\mathbf{y})d\mathbf{y} \right| \leq \left(\int_\Omega |u(\mathbf{y})|^p d\mathbf{y} \right)^{1/p} \cdot \left(\int_\Omega |j_\epsilon(\mathbf{x} - \mathbf{y})|^{p'} d\mathbf{y} \right)^{1/p'},$$

   which implies that u_ϵ is well-defined.

2. $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ with $1 \leq p < \infty$. For $k \geq 1$, we define $\Omega_k = \{\mathbf{x} \in \Omega;\ dist(\mathbf{x}, \partial\Omega) > 1/k, |\mathbf{x}| < k\}$. Letting $u_k(\mathbf{x}) = u(\mathbf{x})\chi_{\Omega_k}(\mathbf{x}) \in L^p(\Omega)$, we see u_k converges to u in $L^p(\Omega)$ according to LDCT. For fixed $k \geq 1$, we apply Lemma (2.2.3) and find that

$$(u_k)_\epsilon \xrightarrow{\epsilon\to 0} u_k, \text{ in } L^p(\Omega),$$

   where ϵ is taken so small that $Supp\{(u_k)_\epsilon\} \subset\subset \Omega$.

3. Lemma (2.2.3) is not true when $p = \infty$.

Corollary 2.2.1. Well-definedness of Weak Derivatives *Let $u \in L^1_{loc}(\Omega)$, and suppose that v_1, $v_2 \in L^1_{loc}(\Omega)$ are $\boldsymbol{\alpha}$-th derivatives of u, then $v_1 = v_2$ a.e. on Ω.*

证明. By definition, we have

$$\int_\Omega v_1\phi = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega u\partial^{\boldsymbol{\alpha}}\phi = \int_\Omega v_2\phi,\ \phi \in \mathcal{D}(\Omega)$$

and so

$$\int_\Omega (v_1 - v_2)\phi = 0.$$

Replacing ϕ by $j_\epsilon(\mathbf{x}-\mathbf{y})$ and applying Lemma (2.2.3) we are done. □

Lemma 2.2.4. *If $u \in L^1_{loc}(\Omega)$ and weak $\partial^{\boldsymbol{\alpha}}u$ exists. Then for any $\mathbf{x} \in \Omega_\epsilon$*

$$\textit{Classical } \partial^{\boldsymbol{\alpha}}u_\epsilon(\mathbf{x}) = \left(\partial^{\boldsymbol{\alpha}}u\right)_\epsilon(\mathbf{x}).$$

证明. By Lemma (2.2.1), we have

$$\begin{aligned}
\partial^{\boldsymbol{\alpha}}u_\epsilon(\mathbf{x}) &= \int_\Omega \partial^{\boldsymbol{\alpha}}j_\epsilon(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} \\
&= \int_\Omega (-1)^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}}_{\mathbf{y}}\left(j_\epsilon(\mathbf{x}-\mathbf{y})\right)u(\mathbf{y})d\mathbf{y} \\
&= \int_\Omega j_\epsilon(\mathbf{x}-\mathbf{y})\partial^{\boldsymbol{\alpha}}u(\mathbf{y})d\mathbf{y} \\
&= \left(\partial^{\boldsymbol{\alpha}}u\right)_\epsilon(\mathbf{x}).
\end{aligned}$$

□

2.2.2 Properties of Weak Derivatives

Proposition 2.2.1. *Suppose $u \in L^1_{loc}(\Omega)$, and its first order weak derivatives exist and $\nabla u = 0$ a.e.. Then $u = Const.$ a.e. on Ω.*

证明. By Lemma (2.2.4), $\nabla u_\epsilon = (\nabla u)_\epsilon = \mathbf{0}$. Thus $u_\epsilon \equiv Const.C_\epsilon$ in Ω_ϵ. By Lemma (2.2.3), u_ϵ converges to u in $L^1_{loc}(\Omega)$ as $\epsilon \to 0$. Since Ω_ϵ is growing as $\epsilon \to 0$, we observe that, when restricted to a compactly embedded domain Ω' and after passage to a subsequence, $u_\epsilon \equiv C_\epsilon$ converges pointwise to u. This forces u to be a constant on Ω', and by arbitrariness of Ω', u is constant all over the domain. □

Theorem 2.2.1. *Let $u, v \in L^1_{loc}(\Omega)$, then $v =$ weak $\partial^{\boldsymbol{\alpha}}u$ if and only if there is a sequence of smooth functions u_k such that*

- *$u_k \longrightarrow u$ in $L^1_{loc}(\Omega)$ as $k \to \infty$;*
- *Classical $\partial^{\boldsymbol{\alpha}}u_k \longrightarrow v$ in $L^1_{loc}(\Omega)$ as $k \to \infty$.*

证明. “ $\Longleftarrow$ ”: For all test function ϕ, we have

$$\begin{aligned}\int_\Omega v\phi &= \lim_{k\to\infty}\int_\Omega \partial^{\boldsymbol{\alpha}} u_k\phi \\ &= \lim_{k\to\infty}(-1)^{|\boldsymbol{\alpha}|}\int_\Omega u_k\partial^{\boldsymbol{\alpha}}\phi,\end{aligned}$$

and then $v = $ weak $\partial^{\boldsymbol{\alpha}} u$.
“ $\Longrightarrow$ ”: For $k \geq 1$, let $\Omega_k = \{\mathbf{x}\in\Omega;\ |\mathbf{x}| < k,\ dist(\mathbf{x},\partial\Omega) > 1/k\}$. Then for large k, Ω_k is nonempty and compactly embedded in Ω. Define

$$\begin{aligned}u_k(\mathbf{x}) &= \left(u\big|_{\Omega_k}\right)_{1/k}(\mathbf{x}) \\ &= \int_{\Omega_k} j_{1/k}(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y},\ \mathbf{x}\in\Omega_k.\end{aligned}$$

By Lemma (2.2.1), $u_k \in C^\infty(\mathbb{R}^n)$. To show the convergence, one should observe that for a fixed $\mathbf{x}\in\Omega' \subset\subset \Omega_{k_0}$, k_0 large, and $k > k_0$, we have

$$\begin{aligned}u_k(\mathbf{x}) &= \int_{\Omega_k} j_{1/k}(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} \\ &= \int_{\Omega_k\cap B_{1/k}(\mathbf{x})} j_{1/k}(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} \\ &= \int_{\Omega_{k_0}} j_{1/k}(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y} \\ &= \left(u\big|_{\Omega_{k_0}}\right)_{1/k}(\mathbf{x}).\end{aligned}$$

According to Lemma (2.2.3), $\left(u\big|_{\Omega_{k_0}}\right)_{1/k}$ converges to $u\big|_{\Omega_{k_0}}$ in $L^1(\Omega_{k_0})$ and hence

$$u_k \overset{k\to\infty}{\longrightarrow} u \text{ in } L^1(\Omega').$$

On the other hand, by Lemma (2.2.4), we have

$$\begin{aligned}\partial^{\boldsymbol{\alpha}} u_k &= \partial^{\boldsymbol{\alpha}}\left(u\big|_{\Omega_{k_0}}\right)_{1/k}(\mathbf{x}) \\ &= \left(\partial^{\boldsymbol{\alpha}} u\big|_{\Omega_{k_0}}\right)_{1/k}(\mathbf{x}) \\ &= \left(v\big|_{\Omega_{k_0}}\right)_{1/k}(\mathbf{x}),\ \mathbf{x}\in\Omega' \\ &\overset{k\to\infty}{\longrightarrow} v,\ \text{in } L^1(\Omega').\end{aligned}$$

□

Remark: In “ $\Longleftarrow$ ”, we only need u_k to be $|\boldsymbol{\alpha}|$-times differentiable.

Theorem 2.2.2. Chain Rule *If $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$, and $u \in L^1_{loc}(\Omega)$. Then,*

- *$f(u) \in L^1_{loc}(\Omega)$;*
- *Weak $\nabla\left(f(u(\mathbf{x}))\right)$ exists and equals $f'(u(\mathbf{x}))\cdot\nabla u(\mathbf{x})$.*

证明. By previous theorem, there is a sequence u_k such that

$$\begin{cases} u_k \longrightarrow u, \\ \nabla u_k \longrightarrow \nabla u, \end{cases} \quad ☹$$

in $L^1_{loc}(\Omega)$. Now, consider $f(u_k(\mathbf{x})) \in C^1(\Omega)$, for any compactly embedded domain Ω',

$$\begin{aligned} \int_{\Omega'} |f(u_k(\mathbf{x})) - f(u(\mathbf{x}))| \, d\mathbf{x} &= \int_{\Omega'} |f'(\xi)||u_k(\mathbf{x}) - u(\mathbf{x})| d\mathbf{x} \\ &\leq \|f'\|_\infty \int_{\Omega'} \int_{\Omega'} |u_k(\mathbf{x}) - u(\mathbf{x})| d\mathbf{x} \\ &\longrightarrow 0, \end{aligned}$$

as $k \to \infty$. This imply that $f(u_k)$ converges to $f(u)$ in $L^1_{loc}(\Omega)$.

Now, we consider the following quantity

$$\begin{aligned} \int_{\Omega'} |\nabla f(u_k(\mathbf{x})) - f'(u(\mathbf{x}))\nabla u(\mathbf{x})| \, d\mathbf{x} &= \int_{\Omega'} |\nabla f(u_k(\mathbf{x})) - f'(u_k(\mathbf{x}))\nabla u(\mathbf{x}) + f'(u_k(\mathbf{x}))\nabla u(\mathbf{x}) - f'(u(\mathbf{x}))\nabla u(\mathbf{x})| \, d\mathbf{x} \\ &\leq \|f'\|_\infty \int_{\Omega'} |\nabla u_k - \nabla u| + \int_{\Omega'} |f'(u_k(\mathbf{x})) - f'(u(\mathbf{x}))||\nabla u(\mathbf{x})| d\mathbf{x} \\ &\longrightarrow 0, \end{aligned}$$

as $k \to \infty$. The first term converges because of ☹ and the second converges after passage to a subsequence according to LDCT. □

Corollary 2.2.2. *Suppose $u = u^+ + u^- \in W^1(\Omega)$, then $u^+, u^-, |u| \in W^1(\Omega)$, and*

$$\nabla u^+(\mathbf{x}) = \begin{cases} \nabla u(\mathbf{x}), & u(\mathbf{x}) > 0, \\ \boldsymbol{0}, & u(\mathbf{x}) \leq 0, \end{cases}$$

$$\nabla u^-(\mathbf{x}) = \begin{cases} \nabla u(\mathbf{x}), & u(\mathbf{x}) < 0, \\ \boldsymbol{0}, & u(\mathbf{x}) \geq 0, \end{cases}$$

$$\nabla |u|(\mathbf{x}) = \begin{cases} \nabla u(\mathbf{x}), & u(\mathbf{x}) > 0, \\ \boldsymbol{0}, & u(\mathbf{x}) = 0, \\ -\nabla u(\mathbf{x}), & u(\mathbf{x}) < 0. \end{cases}$$

证明. For all $\epsilon > 0$, let

$$f_\epsilon(u) = \begin{cases} \sqrt{u^2 + \epsilon^2} - \epsilon, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0. \end{cases}$$

Then we have

- $f(u) = 0$ if $u < 0$;
- $f'_\epsilon(0+) = \left. \dfrac{u}{\sqrt{u^2 + \epsilon^2}} \right|_{u=0} = 0$, and $f'_\epsilon(0-) = 0$, which implies that f_ϵ is in $C^1(\mathbb{R})$;
- $f_\epsilon(u) \longrightarrow u^+$ pointwise as $\epsilon \to 0$;

- $0 \le f_\epsilon(u) \le u^+$.

By Chain Rule, for any $\epsilon > 0$, we have $f_\epsilon(u(\mathbf{x})) \in W^1(\Omega)$ and

$$\text{Weak } \nabla\left(f_\epsilon(u(\mathbf{x}))\right) = f_\epsilon'(u(\mathbf{x}))\nabla u(\mathbf{x}) = \frac{u^+(\mathbf{x})}{\sqrt{u^2(\mathbf{x})+\epsilon^2}}\nabla u(\mathbf{x}).$$

To find the weak derivative of u^+, we observe that for arbitrary test function ϕ

$$\int_\Omega f_\epsilon(u(\mathbf{x}))\nabla\phi(\mathbf{x})d\mathbf{x} = -\int_\Omega \frac{u^+(\mathbf{x})}{\sqrt{u^2(\mathbf{x})+\epsilon^2}}\nabla u(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}.$$

Sending ϵ, along with the following facts

- $\left|\frac{u^+(\mathbf{x})}{\sqrt{u^2(\mathbf{x})+\epsilon^2}}\nabla u(\mathbf{x})\phi(\mathbf{x})\right| \le |\nabla u(\mathbf{x})||\phi(\mathbf{x})| \in L^1(\Omega)$;
- $|f_\epsilon(u(\mathbf{x}))\nabla\phi(\mathbf{x})| \le |\nabla\phi(\mathbf{x})||u(\mathbf{x})| \in L^1(\Omega)$,

we finally reach the equation

$$\int_\Omega u^+(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = -\int_\Omega \chi_{\{u(\mathbf{x})>0\}}(\mathbf{x})\nabla u(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}.$$

The rest are obtained in the same way. □

Corollary 2.2.3. *Suppose $u \in W^1(\Omega)$, and for some real number c we define*

$$\Omega_c = \{\mathbf{x} \in \Omega;\ u(\mathbf{x}) = c\}.$$

Then weak $\nabla u = 0$ a.e. on Ω_c.

证明.

$$\begin{aligned}\nabla u(\mathbf{x}) &= \nabla(u(\mathbf{x}) - c)\\ &= \nabla\left[(u(\mathbf{x}) - c)^+ + (u(\mathbf{x}) - c)^-\right]\\ &= \mathbf{0} + \mathbf{0} = \mathbf{0}.\end{aligned}$$

□

2.3 Sobolev Spaces

Definition 2.8. *Let $k \ge 0$ be an integer, $1 \le p < \infty$, we define*

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega);\ \partial^{\boldsymbol{\alpha}} u \in L^p(\Omega),\, |\boldsymbol{\alpha}| \le k\},$$

endowed with the following norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\boldsymbol{\alpha}|=0}^{k} |\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|^p d\mathbf{x}\right)^{1/p},$$

or equivalently,

$$|||u|||_{W^{k,p}(\Omega)} = \sum_{|\boldsymbol{\alpha}|=0}^{k} \|\partial^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)}.$$

Theorem 2.3.1. *$W^{k,p}(\Omega)$ is Banach.*

证明. Just consider $1 \le p < \infty$ ($p = \infty$ easier). $\|\cdot\|_{W^{k,p}(\Omega)}$ is a norm:

- $\|cu\|_{W^{k,p}(\Omega)} = |c| \, \|u\|_{W^{k,p}(\Omega)}$;
- $$\begin{aligned}
\|u+v\|_{W^{k,p}(\Omega)} &= \left(\int_\Omega \sum_{|\boldsymbol{\alpha}|=0}^{k} |\partial^{\boldsymbol{\alpha}} u(\mathbf{x}) + \partial^{\boldsymbol{\alpha}} v(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} \\
&\le \left\{ \int_\Omega \left[\left(\sum_{|\boldsymbol{\alpha}|=0}^{k} |\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|^p \right)^{1/p} + \left(\sum_{|\boldsymbol{\alpha}|=0}^{k} |\partial^{\boldsymbol{\alpha}} v(\mathbf{x})|^p \right)^{1/p} \right]^p d\mathbf{x} \right\}^{1/p} \\
&\le \left(\int_\Omega \sum_{|\boldsymbol{\alpha}|=0}^{k} |\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|^p d\mathbf{x} + \int_\Omega \sum_{|\boldsymbol{\alpha}|=0}^{k} |\partial^{\boldsymbol{\alpha}} v(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} \\
&\le \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)};
\end{aligned}$$
- If $\|u\|_{W^{k,p}(\Omega)} = 0$, then $\|u\|_{L^p(\Omega)} = 0$ and so $u = 0$.

How about completeness of $\|\cdot\|_{W^{k,p}(\Omega)}$? Suppose u_m is a Cauchy sequence in $W^{k,p}(\Omega)$, then it is Cauchy in $L^p(\Omega)$, and so are their partial derivatives $\partial^{\boldsymbol{\alpha}} u_m$. Recall that $L^p(\Omega)$ is Banach, we may find u_∞ and $v_{\boldsymbol{\alpha}}$ to be the corresponding limits. We claim that $v_{\boldsymbol{\alpha}} =$ weak $\partial^{\boldsymbol{\alpha}} u_\infty$ for all $|\boldsymbol{\alpha}| \le k$. Let ϕ be a test function and write down the integral equation we see

$$\int_\Omega \partial^{\boldsymbol{\alpha}} u_m \phi = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u_m \partial^{\boldsymbol{\alpha}} \phi.$$

By Hölder inequality, $LHS \longrightarrow \int_\Omega v_{\boldsymbol{\alpha}} \phi$ and $RHS \longrightarrow (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u_\infty \partial^{\boldsymbol{\alpha}} \phi$ as $m \to \infty$. Now, using $|||\cdot|||_{W^{k,p}(\Omega)}$, we see $u_m \longrightarrow u_\infty$ in $W^{k,p}(\Omega)$ as $m \to \infty$, which shows that $W^{k,p}(\Omega)$ is complete. $\square$

Theorem 2.3.2. *For $1 < p < \infty$, $W^{k,p}(\Omega)$ is separable and reflexive.*

证明. Define mapping

$$\begin{aligned}
T : W^{k,p}(\Omega) &\to \prod_{|\boldsymbol{\alpha}| \le k} L^p(\Omega) \\
u &\mapsto Tu = (\partial^{\boldsymbol{\alpha}} u)_{|\boldsymbol{\alpha}| \le k}.
\end{aligned}$$

Then T is linear and isometric from $W^{k,p}(\Omega)$ to its image, i.e. $\|Tu\| = |||u|||$. According to basic facts in Functional Analysis, we are done. $\square$

2.3.1 Density Result

$C^\infty(\Omega)$ may not be seen as subspace of $W^{k,p}(\Omega)$, but $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. To show this, we need partition of unity.

Theorem 2.3.3. Partition of Unity *Let* $\{U_i\}_{i=1}^{\infty}$ *be bounded open subsets of* Ω *such that*

- $\overline{U_i} \subset \Omega$, $\forall i \geq 1$;
- *Every compact* $K \subset \Omega$ *intersects only finitely many* U_i*'s;*
- $\cup_{i=1}^{\infty} U_i = \Omega$.

A partition of unity, subordinate to the open covering $\{U_i\}_{i=1}^{\infty}$ *is a sequence of* $C_0^{\infty}(\Omega)$ *functions* ϕ_i *such that*

1. *All* $\phi_i \geq 0$;
2. *Supp*$\{\phi_i\} \subset U_i$;
3. $\sum_{i=1}^{\infty} \phi_i(\mathbf{x}) = 1$, $\forall \mathbf{x} \in \Omega$.

证明. **Step 1**: Construct a new open covering V_i of Ω such that $\overline{V_i} \subset U_i$. Let $F_1 = \overline{U_1} \backslash \cup_{i=2}^{\infty} U_i$, then F_1 is closed and bounded, and $F_1 \bigcup \cup_{i=2}^{\infty} U_i = \Omega$. Since $\overline{U_1} \subset \Omega$, $\partial U_1 \subset \Omega$, and so $\partial U_1 \subset \cup_{i=2}^{\infty} U_i$. Thus $F_1 \subset U_1$. Take $0 < \epsilon < dist(F_1, \partial U_1)/2$, and let $V_1 = \{\mathbf{x} \in U_1;\ dist(\mathbf{x}, F_1) < \epsilon\}$, then V_1 is open, and $F_1 \subset V_1 \subset\subset U_1$. We also have $V_1 \bigcup \cup_{i=2}^{\infty} U_i = \Omega$.

Now, let $F_2 = \overline{U_2} \backslash (V_1 \bigcup \cup_{i=3}^{\infty} U_i)$, we see F_2 is compact, $F_2 \bigcup (V_1 \bigcup \cup_{i=3}^{\infty} U_i) = \Omega$, and $F_2 \subset U_2$. As before, we obtain V_2 and hence inductively V_i's for each $i \geq 1$.

Step 2: Construct $\psi_i \in C_0^{\infty}(U_i)$, $\psi_i \geq 0$ and $\psi_i > 0$ on $\overline{V_i}$ for $i \geq 1$. Let

$$\begin{aligned}\psi_i(\mathbf{x}) &= (\chi_{V_i})_\epsilon \\ &= \int_{U_i} \chi_{V_i}(\mathbf{y}) j_\epsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &= \int_{V_i} j_\epsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y},\end{aligned}$$

and according to Lemma (2.2.1), $\psi_i \in C_0^{\infty}(\mathbb{R}^n)$. Moreover, for all $\mathbf{x} \in \overline{V_i}$,

$$\begin{aligned}\psi_i(\mathbf{x}) &\geq \int_{B_\epsilon(\mathbf{x}) \cap V_i} \chi_{V_i}(\mathbf{y}) j_\epsilon(\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &> 0,\end{aligned}$$

and $Supp\{\phi_i\} \subset \{\mathbf{x} \in U_i;\ dist(\mathbf{x}, V_i) < \epsilon\} \subset\subset U_i$, if $\epsilon < dist(V_i, \partial U_i)/2$.

Step 3: Let $\psi(\mathbf{x}) = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$, then for all fixed $\mathbf{x}_0 \in \Omega$, there is some $i \geq 1$ such that $\mathbf{x} \in V_i$, and by Step 2, $\psi(\mathbf{x}_0) \geq \psi_i(\mathbf{x}_0) > 0$. Take a small $\delta > 0$ such that $\overline{B_\delta(\mathbf{x})} \subset \Omega$, then only finitely many U_i's intersects $\overline{B_\delta(\mathbf{x})}$, and recall that $Supp\{\psi_i\} \subset\subset U_i$, we see on the ball ψ is only a finite sum of smooth functions, and so by arbitrariness of $\mathbf{x}$, we know ψ is smooth all over Ω. Finally, letting $\phi_(\mathbf{x}) := \psi_i(\mathbf{x})/\psi(\mathbf{x})$, we are done.

□

Remark: How to evaluate integrals on surfaces? Suppose Ω is bounded and there are bounded open sets $\{(U_i, g_i)\}_{i=1}^{I}$ such that $\bar{\Omega} \subset \cup_{i=1}^{I} U_i$ and g_i's are the coordinate function of $\partial\Omega \cap U_i$. Then there exist $\phi_i \in C_0^{\infty}(U_i)$ such that

- $\phi_i \geq 0$;
- $\sum_{i=1}^{I} \phi_i \equiv 1$.

With this, we have for some proper function f on $\partial\Omega$,

$$\begin{aligned}\int_{\partial\Omega} f(\mathbf{x})dS &:= \sum_{i=1}^{I} \int_{\partial\Omega} f(\mathbf{x})\phi_i(\mathbf{x})dS \\ &= \sum_{i=1}^{I} \int_{U_i} (f\phi_i)(x_1, \cdots, x_{n-1}, g_i(x_1, \cdots, x_{n-1}))dS \\ &= \sum_{i=1}^{I} \int_{\mathbb{R}^{n-1}} (f\phi_i)(\mathbf{x}', g_i(\mathbf{x}'))\sqrt{1+|\nabla g_i|^2(\mathbf{x}')}d\mathbf{x}'.\end{aligned}$$

Theorem 2.3.4. Density Theorem for Sobolev Space *Let* $1 \leq p < \infty$, $k \geq 1$. *Then* $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$.

证明. Want to show, for all $u \in W^{k,p}(\Omega)$, $\epsilon > 0$, there is a $v \in C_0^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that

$$\|u - v\|_{W^{k,p}(\Omega)} < \epsilon.$$

In this situation, mollifiers are not enough, because by Lemma (2.2.4), $(\partial^{\boldsymbol{\alpha}} u)_\epsilon = \partial^{\boldsymbol{\alpha}} u_\epsilon$ only on a subset $\Omega_\epsilon \subset\subset \Omega$, and so $\partial^{\boldsymbol{\alpha}} u_\epsilon$ may not converge to $\partial^{\boldsymbol{\alpha}} u$.
Take large $R > 0$ such that $B_R(\mathbf{0}) \cap \Omega \neq \emptyset$, and define for each $j \geq 1$, $\Omega_j = \{\mathbf{x} \in \Omega;\ \mathbf{x} \in B_{R+j}(\mathbf{0}),\ dist(\mathbf{x}, \partial\Omega) > 1/j\}$. Then $\Omega_j \subset\subset \Omega_{j+1}$, and $\Omega_j \uparrow \Omega$. Let $U_j = \Omega_{j+1} \backslash \overline{\Omega_{j-1}}$, $j \geq 0$, with $\Omega_0 = \Omega_{-1} = \emptyset$. Then U_i's satisfy the conditions in previous theorem, and there is a partition of unity $\{\phi_j\}_{j=0}^\infty$ subordinate to $\{U_j\}_{j=0}^\infty$. Because $u \in W^{k,p}(\Omega)$, we have $\phi_j u \in W^{k,p}(\Omega)$, and $Supp\{\phi_j u\} \subset\subset \Omega_{j+1} \backslash \overline{\Omega_{j-1}}$.
We will consider $(\phi_j u)_h$. By Lemma (2.2.4), for small $h > 0$

$$\partial^{\boldsymbol{\alpha}}(\phi_j u)_h(\mathbf{x}) = \left(\partial^{\boldsymbol{\alpha}}(\phi_j u)\right)_h(\mathbf{x}),\ \forall \mathbf{x} \in U_j,\ |\boldsymbol{\alpha}| \leq k,$$

because when $h \approx 0$, its support must be compactly embedded in U_j, which also implies that the above equality holds on the whole $\mathbb{R}^n$. By Lemma (2.2.3), $\left(\partial^{\boldsymbol{\alpha}}(\phi_j u)\right)_h \stackrel{h\to 0}{\longrightarrow} \partial^{\boldsymbol{\alpha}}(\phi_j u)$ in $L^p(\Omega)$, and then

$$(\phi_j u)_h \stackrel{h\to 0}{\longrightarrow} \phi_j u, \quad \text{in } W^{k,p}(\Omega).$$

For $\epsilon > 0$, we take $h_j > 0$ so small that

$$\left\|(\phi_j u)_{h_j} - \phi_j u\right\|_{W^{k,p}(\Omega)} < \epsilon/2^j. ☺$$

We now set $v(\mathbf{x}) = \sum_{j=0}^\infty (\phi_j u)_{h_j}(\mathbf{x})$, and because it's locally a finite sum, we see v is well-defined and definitely differentiable. By ☺, we see v approximates u well in $W^{k,p}(\Omega)$. □

Remark: If $\partial\Omega$ is C^1 and Ω bounded, then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$. (see Gilbarg Trudinger *Sobolev Spaces*.)

2.3.2 Sobolev Imbedding Theorem

It is not hard to find that $C_0^\infty(\Omega) \subset W^{k,p}(\Omega)$. We take the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$, and denote it by $W_0^{k,p}(\Omega)$.

Theorem 2.3.5. Sobolev Inequality

i. If $1 \le p < n$, then

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \lesssim_{n,p} \|\nabla u\|_{L^p(\Omega)}, \ \forall u \in W_0^{k,p}(\Omega);$$

ii. If $p > n$, then for $u \in W_0^{k,p}(\Omega)$, we have $u \in C^0(\bar{\Omega})$ (there is $\tilde{u} \in C^0(\bar{\Omega})$ such that $\tilde{u} = u$ a.e. in Ω), and

$$\|u\|_{C^0(\bar{\Omega})} \lesssim_{n,p} |\Omega|^{\frac{1}{n}-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}.$$

证明. **Proof of *i.***: First assume $u \in C_0^1(\Omega)$, and extend $u = 0$ outside Ω, so that $u \in C_0^1(\mathbb{R}^n)$. By FTC, we have

$$\begin{aligned} |u(x_1, \cdots, x_n)| &= \left| \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \cdots, x_i, \cdots, x_i) dx_i \right| \\ &\le \int_{-\infty}^{x_i} |u_{x_i}| dx_i \\ &\le \int_{-\infty}^{\infty} |u_{x_i}| dx_i, \end{aligned}$$

and then

$$\begin{aligned} &|u|^n \le \prod_{i=1}^n \int_{-\infty}^{\infty} |u_{x_i}| dx_i \\ &\implies |u|^{\frac{n}{n-1}} \le \left[\prod_{i=1}^n \int_{-\infty}^{\infty} |u_{x_i}| dx_i \right]^{\frac{1}{n-1}} \\ &\implies \int_{-\infty}^{\infty} |u(x_1, \cdots, x_n)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{-\infty}^{\infty} |u_{x_1}(x_1, \cdots, x_n)| dx_1 \right)^{\frac{1}{n-1}} \cdot \int_{-\infty}^{\infty} \left[\prod_{i=2}^n \int_{-\infty}^{\infty} |u_{x_i}| dx_i \right]^{\frac{1}{n-1}} dx_1 \\ &\qquad \le \left(\int_{-\infty}^{\infty} |u_{x_1}| dx_1 \right)^{\frac{1}{n-1}} \times \prod_{i=2}^n \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |u_{x_i}| dx_i \right)^{\frac{1}{n-1} \cdot (n-1)} dx_1 \right]^{\frac{1}{n-1}} \\ &\implies \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x_1, \cdots, x_n)|^{\frac{n}{n-1}} dx_1 dx_2 \le \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |u_{x_2}| dx_2 \right) dx_1 \right]^{\frac{1}{n-1}} \\ &\qquad \times \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |u_{x_1}| dx_1 \right) dx_2 \right]^{\frac{1}{n-1}} \times \prod_{i=3}^n \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |u_{x_i}| dx_i \right) dx_1 dx_2 \right]^{\frac{1}{n-1}}, \end{aligned}$$

which, after induction, gives that

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}(\mathbf{x}) d\mathbf{x} \le \prod_{i=1}^n \left(\int_{\mathbb{R}^n} |u_{x_i}|(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{n-1}}$$

Taking the $\frac{n}{n-1}$-th root of both sides, wo get

$$\begin{aligned} LHS^{\frac{n-1}{n}} &\le \prod(\cdots)^{\frac{1}{n}} \\ &\le \frac{1}{n}\sum_{i=1}^{n}(\cdots) \\ &\le \frac{1}{\sqrt{n}}\int_{\mathbb{R}^n} |\nabla u|(\mathbf{x})d\mathbf{x}. \end{aligned}$$

Thus, we obtain

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \frac{1}{\sqrt{n}}\|\nabla u\|_{L^1(\mathbb{R})}\,,\ u\in C_0^1(\Omega),$$ 🔔

which is the desired inequality when $p=1$. Now, in 🔔, we replace u by $|u|^r$ ($r>1$ to be determined). We then have

$$\begin{aligned} \||u|^r\|_{L^{\frac{n}{n-1}}(\Omega)} &\le \frac{1}{\sqrt{n}}\left\|r|u|^{r-1}\nabla u\cdot sign(u)\right\|_{L^1(\Omega)} \\ &\le \frac{r}{\sqrt{n}}\|\nabla u\|_{L^p}\left\||u|^{r-1}\right\|_{L^{p'}(\Omega)}. \end{aligned}$$

Take r such that

$$rn' := \frac{rn}{n-1} = (r-1)p',$$

and we find that $r=\frac{p'}{p'-n'}>1$ and $\frac{rn}{n-1}=\frac{p'n'}{p'-n'}=\frac{pn}{n-p}$, and according to 🔔, we have

$$\left(\int_\Omega |u|^{\frac{pn}{n-p}}(\mathbf{x})d\mathbf{x}\right)^{\frac{n-p}{p}} \le \frac{r}{\sqrt{n}}\|\nabla u\|_{L^p(\Omega)}\left(\int_\Omega |u|^{\frac{pn}{n-p}}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p'}},$$

and hence

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \lesssim_{n,r} \|\nabla u\|_{L^p(\Omega)}\,,\ \forall u\in C_0^1(\Omega).$$ ☕

For general $u\in W_0^{1,p}(\Omega)$, by definition of it, there is a sequence $u_k\in C_0^\infty(\Omega)$ such that $u_k\longrightarrow u$ in $W^{1,p}(\Omega)$. Applying ☕ to u_k we have

$$\|u_k\|_{L^{\frac{np}{n-p}}(\Omega)} \lesssim_{n,r} \|\nabla u_k\|_{L^p(\Omega)}\,,$$

and by the convergence $u_k\longrightarrow u$ in $L^p(\Omega)$, we may find a subsequence such that $u_k\longrightarrow u$ pointwise. A simple application of Fatou's Lemma finishes the proof:

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \le \liminf_{k\to\infty}\|u_k\|_{L^{\frac{np}{n-p}}(\Omega)} \lesssim_{n,r} \lim_{k\to\infty}\|\nabla u_k\|_{L^p(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

Proof of *ii.*: Given $u\in C_0^1(\Omega)$, let $\tilde{u}=\frac{\sqrt{n}u}{\|\nabla u\|_{L^p(\Omega)}}$, and assume $|\Omega|=1$. Recall from proof of *i.*, we have

$$\||u|^r\|_{L^{\frac{n}{n-1}}(\Omega)} \le \frac{r}{\sqrt{n}}\|\nabla u\|_{L^p}\left\||u|^{r-1}\right\|_{L^{p'}(\Omega)},$$

and thus

$$\begin{aligned} \||\tilde{u}|^r\|_{n'} &\le \frac{r}{\sqrt{n}}\left\||u|^{r-1}\right\|_{L^{p'}(\Omega)}\|\nabla\tilde{u}\|_{L^p(\Omega)} \\ &= \frac{r}{\sqrt{n}}\left\||u|^{r-1}\right\|_{L^{p'}(\Omega)}\frac{\sqrt{n}}{\|\nabla u\|_{L^p}}\|\nabla u\|_{L^p} \\ &= r\left\||u|^{r-1}\right\|_{L^{p'}(\Omega)}, \end{aligned}$$

or equivalently,

$$\|\tilde{u}\|_{rn'}^{r} \leq r \|\tilde{u}\|_{p'(r-1)}^{r-1},$$

and hence

$$\begin{aligned}\|\tilde{u}\|_{rn'} &\leq r^{1/r} \|\tilde{u}\|_{p'(r-1)}^{1-1/r} \\ &\leq r^{1/r} \|\tilde{u}\|_{p'r}^{1-1/r},\end{aligned}$$

where we used the assumption $|\Omega| = 1$. Now, we take $r = \delta^m,\ m = 1, 2, \ldots$ with $\delta = \frac{n'}{p'} > 1$. We then have

$$\begin{aligned}\|\tilde{u}\|_{n'\delta^m} &\leq (\delta^m)^{\delta^{-m}} \|\tilde{u}\|_{\delta^{m-1}n'}^{1-\delta^{-m}} \\ &\leq (\delta^m)^{\delta^{-m}} \left[(\delta^{m-1})^{\delta^{-(m-1)}} \|\tilde{u}\|_{\delta^{m-2}n'}^{1-\delta^{-(m-1)}}\right]^{1-\delta^{-m}}, \quad \text{(Reversed Hölder)} \\ &\leq \delta^{m\delta^{-m}+(m-1)\delta^{-(m-1)}} \|\tilde{u}\|_{n'\delta^{m-2}}^{(1-\delta^{-m})(1-\delta^{-(m-1)})} \\ &\leq \delta^{m\delta^{-m}+\cdots+\delta^{-1}} \|\tilde{u}\|_{n'}^{(1-\delta^{-m})(1-\delta^{-(m-1)})\cdots(1-\delta^{-1})} \\ &= \delta^{\sum_{k=1}^{m} k\delta^{-k}} \left\| \frac{\sqrt{n}u}{\|\nabla u\|_p} \right\|_{n'}^{\prod_{k=1}^{m}(1-\delta^{-k})}.\end{aligned}$$

Recall that $\|u\|_{n'} \leq \frac{1}{\sqrt{n}} \|\nabla u\|_1 \leq \frac{1}{\sqrt{n}} \|\nabla u\|_p$, we have

$$\|\tilde{u}\|_{n'\delta^m} \leq \delta^{\sum_{k=1}^{\infty} k\delta^{-k}} = \chi < \infty.$$

Sending $m \to \infty$ we have

$$\|\tilde{u}\|_\infty \leq \chi,$$

and

$$\|u\|_\infty \leq \frac{\chi}{\sqrt{n}} \|\nabla u\|_{L^p(\Omega)},\ \forall u \in C_0^1(\Omega), \quad \diamondsuit$$

The general case that $u \in W_0^{k,p}(\Omega)$ is obtained through density arguments.
What if $|\Omega| \neq 1$? Let $v(\mathbf{y}) = u(|\Omega|^{1/n}\mathbf{y})$, and apply $\diamondsuit$ to it, we see

$$\|u\|_{C^0(\bar{\Omega})} = \|v\|_{C^0(\bar{\Omega}/|\Omega|^{1/n})} \lesssim \|\nabla v\|_{L^p(\Omega/|\Omega|^{1/n})} = \|\nabla v\|_{L^p(\Omega)} |\Omega|^{\frac{1}{n}-\frac{1}{p}}.$$

□

Theorem 2.3.6. Sobolev Imbedding Theorem *Suppose* $1 \leq p < \infty$*, then*

$$W_0^{k,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(\Omega), & \text{if } 1 \leq p < n, \\ L^q(\Omega),\ \forall 1 \leq q < \infty, & \text{if } p = n\ \&\ \Omega \text{ bounded}, \\ C^0(\bar{\Omega}), & \text{if } p > n\ \&\ \Omega \text{ bounded}. \end{cases}$$

证明. When $p \neq n$, the conclusion follows from Sobolev inequalities. The case $p = n$ is obtained by Hölder Inequality. □

Remark:

1. When $n < p < \infty$, and Ω bounded, we have

$$W_0^{1,p}(\Omega) \hookrightarrow C^\alpha(\bar{\Omega}),$$

where $\alpha = 1 - n/p$, (see HW4);

2. When $p = n$, the best estimate is the bound for the BMO-norm, (see *Partial Differential Equations* by Evans);

3. *ii.* implies that $1 \notin W_0^{k,p}(\Omega)$.

Corollary 2.3.1. Poincaré Inequality *Suppose Ω bounded and $1 \le p < \infty$, then for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \lesssim_{n,p,\Omega} \|\nabla u\|_{L^p(\Omega)}.$$

证明. We argue case by case:

Case 1. $1 \le p < n$, we have

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\overset{\text{Hölder}}{\lesssim}_{p,n,\Omega} \|u\|_{L^{\frac{np}{n-p}}(\Omega)} \\ &\lesssim_{p,n,\Omega} \|\nabla u\|_{L^p(\Omega)};\end{aligned}$$

Case 2. $p = n$ similar proof;

Case 3. $p > n$, we have

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\le \|u\|_{L^\infty(\Omega)} |\Omega|^{1/p} \\ &\lesssim_{p,n} |\Omega|^{1/n} \|\nabla u\|_{L^p(\Omega)}.\end{aligned}$$

□

Corollary 2.3.2. *Let $u \in W_0^{1,p}(\Omega)$, with Ω bounded, we may define an equivalent norm $\|\nabla u\|_{L^p(\Omega)}$ on this space, provided $1 \le p < \infty$.*

2.3.3 Relich-Kondrakov Compact Imbedding Theorem

Terminology: Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be Banach, then we say X is compactly imbedded into Y if there is an injective bounded linear map $i : X \to Y$ that is compact in the sense that bounded sequence in X has convergent subsequence in Y.

Theorem 2.3.7. *Suppose Ω is bounded and $1 \le p < n$. Then*

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega),\ \forall 1 \le q < \frac{np}{n-p}.$$

Remark:

1. If $p = n$ and Ω bounded, then the above compact imbedding holds for all $q \ge 1$;

2. If $p > n$ and Ω bounded, recall that

$$W_0^{1,p}(\Omega) \hookrightarrow C^{\alpha}(\bar{\Omega}),\ \alpha = 1 - \frac{n}{p},$$

where

$C^{\alpha}(\bar{\Omega}) = \{f \in C^0(\bar{\Omega});\ \text{ there is a constant } C > 0,\ |f(\mathbf{x}) - f(\mathbf{y})| \le C|\mathbf{x}-\mathbf{y}|^{\alpha},\ \forall \mathbf{x}, \mathbf{y} \in \bar{\Omega}.\}$

We also claim that

$$C^{\alpha}(\bar{\Omega}) \hookrightarrow\hookrightarrow C^{\beta}(\bar{\Omega}),\ 0 < \beta < \alpha.$$

(see HW4.)

証明. **Special Case** $q = 1$: We wish to show any bounded set $A \subset W_0^{1,p}(\Omega)$ (suppose the bound is $M > 0$) is precompact in $L^1(\Omega)$, (that is, its closure is compact.) We first claim that by extending $u \in W_0^{1,p}(\Omega)$ to be 0 outside Ω, we obtain $u \in W_0^{1,p}(\mathbb{R}^n)$, which can be easily shown using Theorem (2.2.1). Now, we extend all functions in A to be in $W_0^{1,p}(\mathbb{R}^n)$.

For $\epsilon > 0$, define

$$A_{\epsilon} = \{u_{\epsilon};\ u \in A\},$$

where $u_{\epsilon}(\mathbf{x}) = \int_{\mathbb{R}^n} u(\mathbf{y}) j_{\epsilon}(\mathbf{x}-\mathbf{y}) d\mathbf{y}$. We claim that for any fixed $\epsilon > 0$, A_{ϵ} is precompact in $L^1(\Omega)$. To this end, we observe that

$$\begin{aligned} |u_{\epsilon}(\mathbf{x})| &\le \int_{\mathbb{R}^n} |u(\mathbf{y})| j_{\epsilon}(\mathbf{x}-\mathbf{y}) d\mathbf{y} \\ &\le \frac{\|j\|_{\infty}}{\epsilon^n} \|u\|_1 \\ &\lesssim_{\epsilon,n} M, \end{aligned}$$

and

$$\begin{aligned} |\nabla u_{\epsilon}(\mathbf{x})| &= |(\nabla u)_{\epsilon}(\mathbf{x})| \\ &= \left| \int_{\mathbb{R}^n} \nabla u(\mathbf{y}) j_{\epsilon}(\mathbf{x}-\mathbf{y}) d\mathbf{y} \right| \\ &\le \frac{\|j\|_{\infty}}{\epsilon^n} \|\nabla u\|_1 \\ &\lesssim_{\epsilon,n} M. \end{aligned}$$

Thus, $\{u_{\epsilon}\}_{u \in A}$ is uniformly bounded on $\bar{\Omega}$ and equi-continuous. By Arzela-Ascoli Theorem, A_{ϵ} is precompact in $C^0(\bar{\Omega}) \hookrightarrow L^1(\Omega)$, which implies the Claim.

If u is further smooth and compactly supported, we have

$$\begin{aligned} |u_{\epsilon}(\mathbf{x}) - u(\mathbf{x})| &= \left| \int_{\mathbb{R}^n} j_{\epsilon}(\mathbf{x}-\mathbf{y}) \left(u(\mathbf{y}) - u(\mathbf{x})\right) d\mathbf{y} \right| \\ &= \left| \int_{\mathbb{R}^n} j(\mathbf{z}) \left(u(\mathbf{x}-\epsilon\mathbf{z}) - u(\mathbf{x})\right) d\mathbf{z} \right| \\ &= \left| \int_{\mathbb{R}^n} j(\mathbf{z}) \left(\int_0^{\epsilon} \frac{d}{ds} u(\mathbf{x}-s\mathbf{z}) ds \right) d\mathbf{z} \right| \\ &\le \int_{\mathbb{R}^n} j(\mathbf{z}) \left(\int_0^{\epsilon} |\nabla u(\mathbf{x}-s\mathbf{z})| ds \right) |\mathbf{z}| d\mathbf{z}. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^n} |u_\epsilon(\mathbf{x}) - u(\mathbf{x})| d\mathbf{x} \le \|\nabla u\|_{L^1(\mathbb{R}^n)} \epsilon. \quad \text{✋}$$

What if $u \notin C_0^\infty(\mathbb{R}^n)$? In this case, we take a sequence of $u_k \in C_0^\infty(\Omega)$ that converges to u in $W_0^{1,p}(\Omega)$ and hence in $W_0^{1,p}(\mathbb{R}^n)$. Applying ✋ to u_k, we have

$$\|(u_k)_\epsilon(\mathbf{x}) - u_k(\mathbf{x})\|_{L^1(\mathbb{R}^n)} \le \|\nabla u_k\|_{L^1(\mathbb{R}^n)} \epsilon,$$

and letting $k \to \infty$, we see ✋ still holds true for u.

Now, according to ✋, we have

$$\|u_\epsilon(\mathbf{x}) - u(\mathbf{x})\|_{L^1(\Omega)} \le \epsilon C(p, n, \Omega) M, \ \forall u \in A.$$

Since A_ϵ is precompact in $L^1(\Omega)$, for all $\delta > 0$, we can cover A_ϵ by finitely many balls of radius $\delta/2$ in $L^1(\Omega)$. We now choose $\epsilon = \delta/(2CM)$, then $\|u - u_\epsilon\|_1 \le \delta/2$, and so A is covered by finitely many balls of radius δ, which implies that A is precompact in $L^1(\Omega)$.

General Case $1 < q < \frac{np}{n-p} =: p^*$: In this case, there should be a constant $\lambda \in (0,1)$ such that

$$\lambda + (1-\lambda)/p^* = 1/q$$

Recall that if $\lambda \in (0,1)$, $p_1, p_2, q \ge 1$ and

$$\frac{\lambda}{p_1} + \frac{1-\lambda}{p_2} = \frac{1}{q},$$

then

$$\|f\|_{L^q} \le \|f\|_{L^{p_1}}^{\lambda} \|f\|_{L^{p_2}}^{1-\lambda}.$$

Applying this to $u \in A$, we have

$$\|u\|_{L^q(\Omega)} \le \|u\|_{L^1(\Omega)}^{\lambda} \|u\|_{L^{p^*}(\Omega)}^{1-\lambda}.$$

By Sobolev inequality, we have

$$\|u\|_{L^{p^*}(\Omega)} \lesssim_{p,n} M,$$

and hence

$$\|u\|_{L^q(\Omega)} \lesssim_{p,n,\lambda} M^{1-\lambda} \|u\|_{L^1(\Omega)}^{\lambda}.$$

With the same proof, we have for $u, v \in A$,

$$\|u - v\|_{L^q(\Omega)} \lesssim_{p,n,\lambda} M^{1-\lambda} \|u - v\|_{L^1(\Omega)}^{\lambda},$$

which implies that A is precompact in $L^q(\Omega)$.

$\square$

Chapter 3

L^2 Theory for Second Order Elliptic Equations

Through this chapter, we assume $\Omega \subset\subset \mathbb{R}^n$, and the operator in *divergence form*:

$$Lu = -\left(a_{ij}(\mathbf{x})u_{x_i}\right)_{x_j} + b_i(\mathbf{x})u_{x_i} + c(\mathbf{x})u, \ \mathbf{x} \in \Omega,$$

where $a_{ij}, b_i, c \in L^\infty(\Omega)$. We say that L is *strictly elliptic* on Ω if $(a_{ij}(\mathbf{x}))_{n\times n}$ is symmetric *a.e.* on Ω, and there is a positive constant λ_0 such that $(a_{ij}(\mathbf{x}))_{n\times n} \preccurlyeq \lambda_0 I_{n\times n}$ *a.e.* on Ω.

3.1 Lax-Milgram Theorem

Our main goal is to show the existence and uniqueness of

$$(DBVP)\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

Before that we introduce a new notation:

$$H_0^k(\Omega) = W_0^{k,2}(\Omega),$$

endowed with inner product:

$$(u,v)_{H_0^k(\Omega)} = \int_\Omega \sum_{|\boldsymbol{\alpha}|=0}^{k} \partial^{\boldsymbol{\alpha}} u(\mathbf{x}) \partial^{\boldsymbol{\alpha}} v(\mathbf{x}) d\mathbf{x}.$$

Notions of Weak Solution to (DBVP):

1. We require the weak solution to be in $H_0^1(\Omega)$, where the "0" take care of the Dirichlet Boundary Condition;

2. The equation cannot be considered as one for measurable functions, because if we assume on a plane that $a_{ij}(\mathbf{x}) = k(\mathbf{x})\delta_{ij}$, $c, b_i = 0$, with

$$k(\mathbf{x}) = \begin{cases} 1, & \text{on upper half plane,} \\ 0.1, & \text{on lower half plane,} \end{cases}$$

and $f = 0$, we find

$$-\nabla k \cdot \nabla u = k\Delta u,$$

where, according to later discussions, RHS is an L^2 function, while on LHS, ∇k looks like a δ-function, which forces us to discuss the equation in a more intricate way. In effect, the operator should be understood as distributions:

$$Lu = f \text{ in } \Omega \iff \forall \phi \in \mathcal{D}(\Omega), < Lu, \phi >=< f, \phi >,$$

where $< Lu, \phi >= \int_\Omega a_{ij}u_i\phi_j + b_i\phi + cu\phi =: \mathcal{L}(u, \phi)$;

3. For general distribution f, the problem (DBVP) is too hard, so we focus on the case $f \in (H_0^1(\Omega))^* =: H^{-1}(\Omega)$.

Now, we see a reasonable way to define weak solution to (DBVP) should be the following.

Definition 3.1. *Suppose $f \in H^{-1}(\Omega)$, we will say $u \in H_0^1(\Omega)$ is a* ***weak solution*** *to (DBVP) if for all $v \in H_0^1(\Omega)$, we have*

$$\mathcal{L}(u, v) =< f, v >_{H^{-1}, H_0^1} .$$

Remark:

- A weak solution must be a distributional solution;
- What are the ingredients of $H^{-1}(\Omega)$? Well, if $f \in L^2(\Omega)$, then f naturally induces a bounded linear functional on $H_0^1(\Omega)$. What are the others? According to Riesz's Representation Theorem, any Hilbert space is equivalent to its dual, and so to any bounded linear functional g on $H_0^1(\Omega)$ there corresponds a unique element $p \in H_0^1(\Omega)$ such that $< g, v >_{H^{-1}, H_0^1} = (p, v)_{H_0^1}$ for all $v \in H_0^1(\Omega)$. By definition of RHS, we have a realization of g, the sum of an L^2 function and the distributional derivative of another L^2 function.

Motivational Example: We consider

$$(DBVP) \begin{cases} -\Delta u = f \in H^{-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, (DBVP) has one and only one solution. Recall that $W_0^{1,p}(\Omega)$ has an equivalent norm $\|\nabla \cdot\|_{L^p(\Omega)}$, and since here $p = 2$, we may apply parallelogram principle and define a new inner product on $H_0^1(\Omega)$ as follows:

$$((u, v))_{H_0^1} = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}.$$

Now, a simple application of Riesz's Representation Theorem will give us the result. To tackle the general (DBVP) however, we require a generalization of RRT.

Theorem 3.1.1. Lax-Milgram Theorem *Let H be Hilbert, and $a(u,v)$ a bilinear form on H satisfying*

- *There is some $N > 0$ such that*

$$|a(u,v)| \leq N \|u\| \|v\|, \; u,v \in H;$$

- *There is $\gamma > 0$ such that*

$$a(u,u) \geq \gamma \|u\|^2 .$$

Then, for any $f \in H^*$, there corresponds a unique $u \in H$ that realizes f in the sense that

$$a(u,v) = < f, v >_{H^*,H}, \; \forall v \in H.$$

Moreover, we have the bound

$$\|u\| \leq \frac{1}{\gamma} \|f\| .$$

证明. Observe that for a fixed $u \in H$, the mapping $v \mapsto a(u,v)$ is linear and bounded, and thus according to RRT, there is a unique $A_u \in H$ such that

$$(A_u, v) = a(u,v), \; \forall v \in H.$$

Claim 1. A_u induces an injective linear and bounded map from H to itself. Linearity is clear to find, and we focus on boundedness. To see this, we observe that $|(A_u,v)| = |a(u,v)| \leq M \|u\| \|v\|$. After dividing both sides by $\|v\|$, we obtain the boundedness. Since $a(u,u) \geq \gamma \|u\|^2$, we have $\|A_u\| \|u\| \geq (A_u, u) \geq \gamma \|u\|^2$ ☜, and thus A_u is injective.

Also by RRT, we have $< f, v >= (\tilde{f}, v)$, with $\|f\|_{H^*} = \left\|\tilde{f}\right\|_H$, and so we only need to solve $Au = A_u = \tilde{f}$. It suffices to show A is onto.
Claim 2. $R(A)$ is closed. This can be obtain through Cauchy-sequence arguments with the aid of ☜.
Claim 3. $R(A) = H$. If not, then there is a $0 \neq g \in H$ such that $g \perp R(A)$, and hence $A_g \perp g$. However, $0 = (A_g, g) = a(g,g) \geq \gamma \|g\|^2$.

Now, also because of ☜, we have $\|A^{-1}\| \leq \frac{1}{\gamma}$, and then we obtain the bound for u. □

Now, we return to the general problem

$$(DBVP) \begin{cases} Lu = f \in H^{-1}(\Omega), & \text{in } \Omega, \\ u(\mathbf{x}) = 0, & \text{on } \partial\Omega. \end{cases}$$

Before discussing it, we look at the following baby example

$$\begin{cases} -u'' - u = \sin(x), & x \in (0,\pi), \\ u(0) = 0 = u(\pi), \end{cases}$$

which has no solution. To see this, we multiply both sides by $\sin(x)$ and integrate over $(0, \pi)$, and obtain

$$\int_0^\pi -u'' \sin(x) - u \sin(x) dx = \int_0^\pi \sin^2(x) dx.$$

Do integration by parts on LHS, we have

$$LHS = u(\sin(x))' \Big|_{x=0}^{x=\pi} - \int_0^\pi u(\sin(x))'' + u \sin(x) dx = 0,$$

which is impossible. On the other hand, the following initial value problem must have solutions

$$\begin{cases} u'' = f(x, u, u'), \\ u(0) = u_0, \\ u'(0) = u_1. \end{cases}$$

From the above discussions, one should keep in mind that (DBVP) may not always have solutions and therefore, we need to consider the problem in a more thorough way.

Theorem 3.1.2. *There is a constant $\sigma_0 > 0$ such that if $\sigma \geq \sigma_0$, then for all $f \in H^{-1}(\Omega)$, the following revised problem*

$$(DBVP) \begin{cases} L_\sigma u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has one and only one weak solution, where $L_\sigma u = Lu + \sigma u$.

证明. We consider the bilinear form associated to L_σ:

$$\mathcal{L}_\sigma(u, v) = \int_\Omega [a_{ij} u_i v_j + b_i u_i v + (c + \sigma) uv] \, d\mathbf{x}, \; u, v \in H_0^1(\Omega).$$

Check:

- Boundedness of $\mathcal{L}_\sigma$:

$$\begin{aligned} |\mathcal{L}_\sigma(u, v)| &\leq \| |a_{ij}| \|_\infty \|\nabla u\|_2 \|\nabla v\|_2 + \left\| \vec{b} \right\|_\infty \|\nabla u\|_2 \|v\|_2 + (\|c\|_\infty + \sigma) \|u\|_2 \|v\|_2 \\ &\leq M \|u\|_{H_0^1} \|v\|_{H_0^1} ; \end{aligned}$$

- Coercivity of $\mathcal{L}_\sigma$:

$$\begin{aligned} L_\sigma(u, u) &\geq \lambda_0 \int_\Omega |\nabla u|^2 d\mathbf{x} - \left\| \vec{b} \right\|_\infty \|\nabla u\|_2 \|u\|_2 - \|c\|_\infty \|u\|_2^2 + \sigma \|u\|_2^2 \\ &\geq \lambda_0 \int_\Omega |\nabla u|^2 d\mathbf{x} - \epsilon \|\nabla u\|_2^2 - \frac{1}{4\epsilon} \left\| \vec{b} \right\|_\infty^2 \|u\|_2^2 + (c - \|c\|_\infty) \|u\|_2^2 \\ &\overset{\epsilon = \frac{\lambda_0}{2}}{=} \frac{\lambda_0}{2} \int_\Omega |\nabla u|^2 d\mathbf{x} + \left(\sigma - \|c\|_\infty - \frac{1}{2\lambda_0} \left\| \vec{b} \right\|_\infty^2 \right) \|u\|_2^2 , \end{aligned}$$

  and so we only have to take $\sigma_0 = \|c\|_\infty + \frac{1}{2\lambda_0} \left\| \vec{b} \right\|_\infty^2$, and norm induced by $((\cdot, \cdot))_{H_0^1}$.

Now, applying Lax-Milgram theorem, we are done. □

3.2 Fredholm Operator Theory

In order to handle the harder problem

$$(DBVP)\begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We need Fredholm Operator Theory.

Definition 3.2. *Let X and Y be Banach, $T : X \to Y$ linear and bounded. We say that T is **Fredholm** if*

- *$dim Ker(T) < \infty$;*
- *$Im(T)$ is closed;*
- *$codim Im(T) < \infty$.*

*For such operators, we define $ind(T) = dim\, Ker(T) - Codim\, Im(T)$, the **Fredholm index** of T.*

Theorem 3.2.1. Riesz-Fredholm Compact Perturbation Theorem *If T is Fredholm and*

$$K : X \to Y,$$

is compact. Then $T + K$ is Fredholm and

$$ind(T + K) = ind(T).$$

This theorem has been proved in the course *Functional Analysis (Graduate)*, and here we only talk about how to use it. Recall that given $u \in H_0^1$, $\mathcal{L}(u, \cdot)$ induces a bounded linear functional on H_0^1, and because $\mathcal{L}$ is a bounded bilinear form, we know L induces a bounded linear mapping from H_0^1 to its dual. Now, it is convenient to observe that to ask whether (DBVP) is solvable is equivalent to ask whether $Im(L) = H^{-1}$, and whether the solution is unique $Ker(L) = 0$.

Theorem 3.2.2. *The operator $L : H_0^1 \to H^{-1}$ is Fredholm with index 0.*

证明. According to RFCPT and previous result for L_σ, it suffices to show that the inclusion $I : H_0^1 \to H^{-1}$ is compact. Observe that $I : H_0^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$ is the composite of two canonical embeddings, where the first is compact embedding due to RKCIT, and thus we are done. □

Corollary 3.2.1. Fredholm Alternative *(DBVP) has a unique solution if and only if it is solvable for every $f \in H^{-1}$.*

Question: What if uniqueness of L fails?

Let u, v be smooth and compactly supported, then

$$\begin{aligned}\int_\Omega Luv &= \int_\Omega [a_{ij}u_iv_j + b_iu_iv + cuv] \\ &= \int_\Omega -\left(a_ijv_i\right)_j u - (b_iv)_iu + cvu \\ &=: \int_\Omega L^*vu.\end{aligned}$$

Here, we call L^* the formal adjoint of L, and L^*v is considered as an element in H^{-1}.

Theorem 3.2.3. Existence and Uniqueness Theorem for (DBVP)

1. *Fredholm Alternatives: (DBVP) has a unique solution if and only if it has a solution for all $f \in H^{-1}$. (Already proved);*

2. *The subspace of H_0^1 consisting of weak solutions of (DBVP) when $f = 0$ is of finite dimension, with dimension equal that of solution space of*

$$\begin{cases} L^*v = 0, & in\ \Omega, \\ v = 0, & on\ \partial\Omega. \end{cases}$$

Or we may write $dim\, Ker(L) = dim\, Ker(L^*)$;

3. *For general f, (DBVP) has a weak solution if and only if $< f, v >= 0$ for all $v \in Ker(L^*)$. Or we may write $Im(L) = {}^\perp\mathrm{Ker}(L^*)$.*

证明. Recall that we have shown $L : H_0^1 \to H^{-1}$ is Fredholm with $ind(L) = 0$. Similarly, one can show that L^* is also Fredholm with index 0. Now, we write

$$H_0^1 = Ker(L) \bigoplus X_1,$$

$$H^{-1} = Im(L) \bigoplus Y_1,$$

with X_1, Y_1 closed. Let $Ker(L^*) = span\{e_1, \cdots, e_k\} \subset H_0^1$, $k = dim\, Ker(L^*)$. By Hahn-Banach, there are linearly independent $f_1, \cdots, f_k \in H^{-1}$ such that $< f_i, e_j >= \delta_{ij},$.

Claim 1. $span\{f_1, \cdots, f_k\} \cap Im(L) = 0$. If not, there should be $u \in H_0^1$ such that $Lu \neq 0$ and $Lu = c_1f_1 + \cdots + c_kf_k$. Applying Lu to e_j, we have

$$c_i =< Lu, e_i >=< L^*e_i, u >= 0.$$

Claim 2. $dim\, Ker(L^*) \leq dim\, Ker(L)$. Because $f_i \in H^{-1} = Im(L) \bigoplus Y_1$, then there should be $u_i \in H_0^1$ and $y_i \in Y_1$ such that $f_i = Lu_i + y_i$. We claim that y_i's are linearly independent. If not, there are $d_1, \cdots, d_k$, not all 0, and $\sum_i d_iy_i = 0$, and hence

$$\sum_i d_if_i = \sum_i d_iLu_i,$$

where $LHS \in span\{f_1, \cdots, f_k\}$, while $RHS \in Im(L)$. By claim 1, $LHS = 0$, which contradicts that f_i's are linearly independent. Thus $k \leq dim\, Y_1 = dim\, Ker(L)$. Similarly, one can show the reverse.
Now, to show $Im(L) = {}^{\perp}\mathrm{Ker}(\mathrm{L}^*)$, we first observe that $Im(L) \subset {}^{\perp}\mathrm{Ker}(\mathrm{L}^*)$. Given $u \in H_0^1$, and for every $v \in Ker(L^*)$, we have $< Lu, v >=< L^*v, u >= 0$. To show the reverse, we take $f \in {}^{\perp}\mathrm{Ker}(\mathrm{L}^*)$. From Claim 1 and 2, we know $H^{-1} = \Im(L) \bigoplus span\{f_1, \cdots, f_k\}$. This implies that there is $r \in Im(L)$, $c_1, \cdots, c_k$ constants, such that

$$f = r + c_1 f_1 + \cdots + c_k f_k = Lu + \cdots,$$

for some $u \in H_0^1$. Applying f to e_i, we have $c_i =< f, e_i > - < Lu, e_i >= 0$, and hence $f = r \in Im(L)$.

$\square$

Example: Consider on $\Omega = (0, \pi)$, $Lu = -u'' - u$,

$$(BVP)\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

For what $f \in H^{-1}$ does (BVP) have a weak solution? Answer: $f \perp \sin(x)$.

Theorem 3.2.4. A user-friendly theorem *Suppose Ω bounded, $u \in H^1(\Omega) \cap C^0(\bar{\Omega})$ and $u = 0$ on the boundary. Then $u \in H_0^1(\Omega)$.*

证明. Just need to show $u^+ \in H_0^1(\Omega)$. By HW4, $(u^+ - \epsilon)^+ \in H_0^1(\Omega)$. On the other hand $(u^+ - \epsilon)^+$ converges to u^+ in $H^1(\Omega)$. $\square$

3.3 An Introduction to Homogenization

Suppose the thermal conductivity of a rod is given by $a(x/\epsilon)$, where $a(y)$ is a periodic measurable bounded function withe period $y_0 > 0$. We also assume that $\lambda_0 \leq a(y) \leq 1/\lambda_0$, for some $\lambda_0 > 0$. Suppose f is an L^2 function, and the temperature u of the rod is given by

$$\begin{cases} u_t = (a(x/\epsilon)u_x)_x + f(x), & x \in (0, b), \\ u = 0, & \text{in } \{0, b\}. \end{cases}$$

Its steady state is then given by

$$\begin{cases} -(a(x/\epsilon)u_x)_x = f(x), & x \in (0, b), \\ u = 0, & \text{in } \{0, b\}. \end{cases}$$

When f, we multiply both sides by u, and after using integration by parts, we know that the above equation has at most one solution. According to Fredholm Alternative, this equation has solution for every $f \in H^{-1}$. We denote this solution by u_ϵ, and wish to study $\lim_{\epsilon \to 0} u_\epsilon$.

Boundedness of u_ϵ in $H_0^1(\Omega)$:

$$\int_0^b a(x/\epsilon)u_x^2 dx = \int_0^b f(x)u(x)dx,$$

and by ellipticity of a and Poincaré inequality, we obtain the following A priori estimate

$$\|u_x\|_2 \leq \|f\|_2 C/\lambda_0. \text{✊}$$

Accrording to Banach-Eberlein Theorem, and $H_0^1 \hookrightarrow L^2$, we know after passage to a subsequence, there is an $u_0 \in H_0^1$ such that $u_\epsilon \to u_0$ weakly in H_0^1 and strongly in L^2.

Now, let $v_\epsilon(x) = a(x/\epsilon)u_\epsilon'(x)$, we claim that after passage to a subsequence, $v_\epsilon \to v_0$ weakly in H^1 and strongly in L^2, which can be proved also by the A priori estimate ✊. But why $v_\epsilon \in H^1$? According to the equation, we know its weak derivative is exactly $f \in L^2$.

Claim (Generalized Riemann-Lebesgue) For all $h \in L^\infty(\mathbb{R})$, periodic with period $y_0 > 0$, we have

$$h\left(\frac{x}{\epsilon}\right) \overset{*}{\rightharpoonup} < h > := \frac{\int_0^{y_0} h(y)dy}{y_0}.$$

To prove this claim, we first study the simplest case: $g(x) = \chi_I$, with $I = (c,d)$ some interval in (o,b). Then applying h on g, we have

$$\begin{aligned}
\int_c^d h(x/\epsilon)dx &= N\int_c^{c+\epsilon y_0} h(x/\epsilon)dx + O(\epsilon) \\
&= \epsilon N \int_{c/\epsilon}^{y_0 + c/\epsilon} h(z)dz + O(\epsilon) \\
&= \epsilon N \int_0^{y_0} h(z)dz + O(\epsilon),
\end{aligned}$$

with $N \geq 1$ such that $c + N\epsilon y_0 \leq d < c + (N+1)\epsilon y_0$. We further have

$$\begin{aligned}
LHS &= N\epsilon y_0 \frac{\int_0^{y_0} h(y)dy}{y_0} + O(\epsilon) \\
&= (d-c) < h > + O(\epsilon).
\end{aligned}$$

Thus, the case for χ_I is done, and so are step functions. For general $g \in L^1(0,b)$, we already know step functions are dense in it, that is, for all $\delta > 0$, there is a step function s such that $\|g - s\|_1 < \delta$. Now, we have

$$\begin{aligned}
\left|\int_0^b g(x)h(x/\epsilon)dx - < h > \int_0^b g(x)dx\right| &\leq \left|\int_0^b s(x)h(x/\epsilon)dx - < h > \int_0^b s(x)dx\right| \\
&\quad + (\|h\|_\infty + | < h > |)\|g - s\|_1.
\end{aligned}$$

Thus, we have proved the claim.

To see the convergence of v_ϵ, we observe that $u_\epsilon' = v_\epsilon / a(x/\epsilon)$. Recall that $LHS = u_\epsilon'$ is weakly convergent in L^2 to u_0', and so is RHS. According to the above claim and the fact that $v_\epsilon \to v_0$ strongly in L^2, we have $RHS \to v_0 < \frac{1}{a} >$ weakly in L^2, and hence

$$v_0(x) = \frac{1}{< \frac{1}{a} >} u_0'(x).$$

Recall in the weak sense, we have $-v_0' = f$, and hence u_0 must satisfy the equation

$$\begin{cases} -\frac{1}{<\frac{1}{a}>}u_0''(x) = f(x), & x \in (0,b), \\ u_0(0) = 0 = u_0(b). \end{cases}$$

3.4 Eigenvalue Problem

Baby Example: Recall in ODE, the equation

$$\frac{d\vec{X}}{dt} = A_{n\times n}\vec{X},$$

admits exactly one solution provided the initial value. A trial solution is a solution of the form $\vec{X} = e^{\lambda t}\vec{C}$. Inserting this formula into the equation, we obtain an algebraic equation

$$A\vec{C} = \lambda\vec{C}.$$

Recall if A is symmetric, then

$$\inf_{\mathbf{x}\neq\mathbf{0},\mathbf{x}\in\mathbb{R}^n} \frac{(A\mathbf{x},\mathbf{x})}{|\mathbf{x}|^2} =: \lambda_1,$$

will be the first eigenvalue of A. Reducing the space that we are taking infimum, we obtain a sequence of numbers $\lambda_1, \cdots, \lambda_n$, each corresponds a minimizer $\mathbf{x}_1, \cdots, \mathbf{x}_n$.

We wish to apply this idea to

$$(EP) \begin{cases} -(a_{ij}(\mathbf{x})u_i)_j + c(\mathbf{x})u = \lambda u, & \mathbf{x} \in \Omega \subset\subset \mathbb{R}^n, \\ u = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

We always assume a_{ij}, c bounded, and a_{ij} strictly elliptic with ellipticity coefficient λ_0.

Definition 3.3. *A* ***weak solution*** *of (EP) is $u \in H_0^1(\Omega)$ such that*

$$\mathcal{L}(u,v) =: \int_\Omega a_{ij}u_iv_j + cuvd\mathbf{x} = \lambda\int_\Omega uvd\mathbf{x},\ \forall v \in H_0^1(\Omega).$$

Let

$$\lambda_1 = \inf_{u\neq 0, u\in H_0^1(\Omega)} \frac{\mathcal{L}(u,u)}{\|u\|_2^2},$$

and for $u \in H_0^1(\Omega)$

$$\mathcal{R}(u) := \frac{\mathcal{L}(u,u)}{\|u\|_2^2}.$$

Lemma 3.4.1. *λ_1 is an eigenvalue of (EP).*

证明. It can be shown, for any u

$$R(u) \geq -\|c\|_\infty,$$

and hence $\lambda_1 > -\infty$. Taking minimizing sequence and applying Banach-Eberlein Theorem, we may find a minimizer $v \in H_0^1(\Omega)$. v is a minimizer of $R(\cdot)$, and so if we define $f_w(t) = R(v+tw)$, for $t \approx 0$, and $w \in H_0^1(\Omega)$, we have $f_w'(0) = 0$, and thus direct computation tells us that v is exactly a weak solution.

□

Remark:

- All eigenvalues of (EP) must be real;
- λ_1 is the smallest one.

Lemma 3.4.2. *Any eigenfunction that corresponds to λ_1 cannot change sign. λ_1 corresponds to only one eigenfunction.*

证明. If u minimizes $R(\cdot)$, then so is $|u|$. Thus $|u|$ is still an eigenfunction. Apply De Giorgi-Nash Theorem, we have $u \in C^{\alpha}_{loc}(\Omega)$, for some $\alpha \in (0,1)$. If u changes sign, then $|u|$ must be 0 somewhere in the domain. Now, using Harnack's inequality, we know it's impossible. (If $u \geq 0$ on Ω, then there is $C = C(\Omega', \Omega, a_{ij}, b_i, c) > 0$ such that $\sup_{\Omega'} u \leq C \inf_{\Omega'} u$ for all $\Omega' \subset\subset \Omega$.)

Suppose there are two eigenfunctions $u_1 \neq u_2$ that correspond to λ_1, we consider $g(c) = (u_1 + cu_2, u_2)_{L^2}$, and find that when $c_0 = -(u_1, u_2)_{L^2}/(u_2, u_2)_{L^2})$, $g(c_0) = 0$, and hence $u_1 + c_0 u_2$ must change sign, which is impossible. $\square$

Lemma 3.4.3. *Let*

$$\lambda_2 = \inf_{u \neq 0, u \in H_0^1, u \perp u_1} R(u),$$

then $\lambda_2 > \lambda_1$ is also an eigenvalue.

证明. As before, we may obtain u_2 as the eigenfunction corresponding λ_2. It can be shown that $u_2 \perp u_1$ both in L^2 and H_0^1. $\square$

For $k \geq 1$, define

$$\lambda_{k+1} = \inf_{u \in H_0^1, u \neq 0, u \perp_{L^2} u_1, \cdots, u_k} R(u),$$

and one can similarly prove (λ_k, u_k)'s are all the eigenpairs.

Lemma 3.4.4. *$\lambda_k \to \infty$, as $k \to \infty$.*

证明. It is easy to obtain

$$\lambda_0 \int_\Omega |\nabla u_k|^2 \leq (\|c\|_\infty + \lambda_k) \int_\Omega u_k^2.$$

Normalize $\int_\Omega u_k^2 = 1$, we see if there is a bounded subsequence $\lambda_{k_j} \leq M$, we see u_{k_j}'s must have weakly convergent subsequence in H_0^1 (strong convergence in L^2). But u_k's are mutually orthogonal, which makes the convergence impossible. $\square$

Induce a new inner product on $H_0^1(\Omega)$:

$$((u,v))^* = \mathcal{L}(u,v) + \|c\|_\infty (u,v).$$

It can be shown that $((u,v))^*$ is equivalent to the previous one, and we denote the induced norm $\|\cdot\|^*$.

Lemma 3.4.5. *Let c_k be constants such that $\|c_k u_k\|^* = 1$, then $c_k u_k$'s form an orthonormal basis for $H_0^1(\Omega)$.*

证明. If there is $u \in H_0^1(\Omega)$ such that

$$0 = ((u, u_k))^* = (\lambda_k + \|c\|_\infty)(u, u_k)_{L^2}.$$

Because $\lambda_k + \|c\|_\infty > 0$, we know

$$(u, u_k)_{L^2} = 0,$$

and by definition of λ_k's, $R(u) \geq \lambda_k$ for $k \geq 1$, which is impossible by Lemma (3.4.4). $\square$

Lemma 3.4.6. *u_k's form an orthonormal basis for $L^2(\Omega)$.*

证明. If $u \in H_0^1(\Omega)$, then we are done by the above lemma. (Need to use $((u, u_k))^* = (\lambda_k + \|c\|_\infty)(u, u_k)_{L^2}$.) Suppose otherwise, we recall $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we set v_m such a sequence. Then, by prior discussion, we may write

$$v_m = \sum_{k=1}^{\infty}(v_m, u_k)u_k.$$

Also, it can be shown $\sum_{k=1}^{\infty}(u, u_k)^2 \leq \|u\|_2^2$, thus $\sum_{k=1}^{\infty}(u_k, u)u_k$ is well-defined. We then have

$$\begin{aligned}\left\|u - \sum_{k=1}^{\infty}(u_k, u)u_k\right\|_2 &\leq \left\|u - \sum_{k=1}^{\infty}(u_k, v_m)u_k\right\|_2 \\ &\leq \|u - v_m\|_2 \\ &\longrightarrow 0,\end{aligned}$$

as $m \to \infty$.

$\square$

Lemma 3.4.7. *Let (λ, e) be an eigen pair, then there is some $k \geq 1$ such that $\lambda = \lambda_k$, and e is in the eigenspace of λ_k.*

证明. A corollary of previous lemma. $\square$

Chapter 4

Regularity Theory for Second Order Elliptic Equations

4.1 L^2 Regularity

Goal: Let Ω be bounded in $\mathbb{R}^n$, and

$$Lu = -\left(a_{ij}(\mathbf{x})u_i\right)_j + b_i(\mathbf{x})u_i(\mathbf{x}) + c(\mathbf{x})u, \ \mathbf{x} \in \Omega.$$

Assume L is strictly ellptic with ellipticity coefficient $\lambda_0 > 0$, $a_{ij} \in C^1(\Omega) \cap L^\infty(\Omega)$, $, b_i, c \in L^\infty(\Omega)$ and a_{ij} symmetric almost everywhere. We wish to study how smooth the solution to the equation $Lu = f$ is, for proper functions.

Definition 4.1. *Let Ω be a domain, and Ω' compactly supported in it. u is some real function on Ω. Then we define the i-th ($i = 1, \cdots, n$) **difference quotient** of size h is*

$$\nabla_i^h u(\mathbf{x}) = \frac{u(\mathbf{x} + he_i) - u(\mathbf{x})}{h}, \ \mathbf{x} \in \Omega',$$

where $0 < |h| < dist(\Omega', \partial\Omega)$ and e_i is the unit vector with i-th entry 1. For notational convenience, we set

$$\nabla^h u = (\nabla_1^h u, \cdots, \nabla_n^h u)^T.$$

Theorem 4.1.1. *i. Suppose $1 \le p < \infty$ and $u \in W^{1,p}(\Omega)$, then for any $1 < |h| < dist(\Omega', \partial\Omega)$, we have*

$$\left\|\nabla^h u\right\|_{L^p(\Omega')} \le n \left\|\nabla u\right\|_{L^p(\Omega)}.$$

ii. $1 < p < \infty$, and $u \in L^p(\Omega') \cap L^1(\Omega)$, and there is a constant $C > 0$ such that

$$\left\|\nabla^h u\right\|_{L^p(\Omega')} \le C, \ \forall 0 < |h| < dist(\Omega', \partial\Omega).$$

Then $u \in W^{1,p}(\Omega')$ with $\left\|\nabla u\right\|_{L^p(\Omega')} \le C$. (False if $p = 1$.)

证明. i. Suppose first that $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$. For all $\mathbf{x} \in \Omega'$ and $0 < |h| < dist(\Omega', \partial\Omega)$, we have

$$\begin{aligned} |u(\mathbf{x}+he_i) - u(\mathbf{x})| &= \left| \int_0^1 \frac{d(u(\mathbf{x}+the_i))}{dt} dt \right| \\ &= \left| \int_0^1 hu_i(\mathbf{x}+the_i) dt \right| \\ &\leq \int_0^1 |u_i(\mathbf{x}+the_i)| dt |h|, \end{aligned}$$

which forces

$$|\nabla_i^h u(\mathbf{x})| \leq \int_0^1 |u_i(\mathbf{x}+the_i)| dt.$$

Thus, we have

$$\begin{aligned} \left\|\nabla_i^h u\right\|_{L^p(\Omega')} &\leq \int_0^1 \|u_i(\cdot + the_i)\|_{L^p(\Omega')} dt \\ &= \int_0^1 \left(\int_{\Omega'+the_i} |u_i(\mathbf{y})|^p d\mathbf{y} \right)^{1/p} dt \\ &\leq \int_0^1 \left(\int_{\Omega} |u_i(\mathbf{y})|^p d\mathbf{y} \right)^{1/p} dt \\ &= \|u_i\|_{L^p(\Omega)}, \end{aligned}$$

and hence

$$\left\|\nabla^h u\right\|_{L^p(\Omega')} \leq \sum_i \left\|\nabla_i^h u\right\|_{L^p(\Omega')} \leq n \|\nabla u\|_{L^p(\Omega)}.$$

Now, for $u \in W^{1,p}(\Omega)$. By density theorem, there exists a sequence $u^k(\mathbf{x})$ in $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ such that $u^k \longrightarrow u$ in $W^{1,p}(\Omega)$ as $k \to \infty$. A simple application of Fatou's lemma gives us the result.

ii. Because $\nabla^h u$ is bounded in $L^p(\Omega')$, we see after passage to a subsequence, $\nabla^h u$ converges to some $\vec{v}$ weakly in $L^p(\Omega')$, as $h \to 0$. We claim that $\vec{v}$ is the weak derivative of u. It suffices to show for any test function ϕ in Ω', we have

$$\int_{\Omega'} v_i \phi = -\int_{\Omega'} u\phi_i,$$

which is obtained by taking $h \to 0$ on both sides of the equality

$$\int_{\Omega'} \frac{u(\mathbf{x}+he_i) - u(\mathbf{x})}{h} \phi(\mathbf{x}) = -\int_{\Omega'} u(\mathbf{x}) \frac{\phi(\mathbf{x}-he_i) - \phi(\mathbf{x})}{-h}.$$

Now, weak $\nabla u = \vec{v} \in L^p(\Omega')$, and the bound is immediately obtained using the weak convergence.

□

Remark: In case of part *i.*, we may consider a domain Ω in the upper half plane $H = \{x_n \geq 0\}$, and so $\Omega' \subset\subset \Omega$ may be chosen to touch the boundary $\partial H \cap \partial\Omega$. The bounds hold true for $1 \leq i \leq n-1$ in this situation.

Theorem 4.1.2. Interpolation Inequality *Let Ω be a domain with C^1-smooth boundary. For any $\epsilon_0 > 0$, there is $K = K(\epsilon_0, m, p, \Omega)$ such that for all $1 < \epsilon \le \epsilon_0$, integer $0 < j \le m-1$ and $u \in W^{m,p}(\Omega)$ $1 \le p < \infty$*

$$\sum_{|\boldsymbol{\alpha}|=j} \|\partial^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)} \le K \left(\epsilon \sum_{|\boldsymbol{\alpha}|=m} \|\partial^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)} + \epsilon^{-\frac{j}{m-j}} \|u\|_{L^p(\Omega)} \right).$$

Remark: This helps one to establish a new equivalent norm consisting of only the function itself and its highest order derivatives.

证明. We only give a proof for $n = 1$, $\Omega = (0,1)$, $m = 2$, $j = 1$. Suppose $u \in C^2[0,1]$. By mean value theorem, we have

$$\left| \frac{u(\eta) - u(\xi)}{\eta - \xi} \right| = |u'(\lambda)|,$$

and so

$$|u'(\lambda)| \le 3(|u(\eta)| + |u(\xi)|).$$

According to F.T.C., we have

$$u'(x) = u'(\lambda) + \int_\lambda^x u''(t)dt,$$

and hence

$$|u'(x)| \le 3(|u(\eta)| + |u(\xi)|) + \int_0^1 |u''(t)|dt.$$

Integrate w.r.t. ξ from 0 to 1/3 and η from 2/3 to 1, and see that

$$|u'(x)| \le 9 \int_0^1 |u(t)|dt + \int_0^1 |u''(t)|dt.$$

Thus,

$$\begin{aligned} |u'(x)|^p &\le 2^{p-1} 9^p \left(\int_0^1 |u| \right)^p + 2^{p-1} \left(\int_0^1 |u''| \right)^p \\ &\le 2^{p-1} 9^p \left(\int_0^1 |u|^p \right) + 2^{p-1} \left(\int_0^1 |u''|^p \right), \text{☺}. \end{aligned}$$

For $v \in C^2[a,b]$, if we define $u(t) = v(bt + (1-t)a)$, we obtain a similar estimate

$$\int_a^b |v'|^p \le \frac{2^{p-1} 9^p}{(b-a)^p} \int_a^b |v|^p + 2^{p-1}(b-a)^p \int_a^b |v''|^p.$$

W.T.S. for all $\epsilon_0 > 0$, there is such K such that if $0 < \epsilon \le \epsilon_0$,

$$\|u'\|_{L^p(0,1)} \le K \left(\epsilon \|u''\|_{L^p(0,1)} + \epsilon^{-1} \|u\|_{L^p(0,1)} \right).$$

For $0 < \epsilon \le 1$, let $K_p = 2^{p-1}9^p$, consider $2(K_p/\epsilon)^{1/p} \ge N \ge (K_p/\epsilon)^{1/p} > 1$, and observe

$$\begin{aligned}\int_0^1 |u'|^p &= \sum_{j=1}^N \int_{\frac{j-1}{N}}^{\frac{j}{N}} |u'|^p \\ &\overset{☺}{\le} K_p N^p \int_{\frac{j-1}{N}}^{\frac{j}{N}} |u|^p + \frac{K_p}{N^p}\int_{\frac{j-1}{N}}^{\frac{j}{N}} |u''|^p \\ &\le 2^p K_p^2 \left(\frac{1}{\epsilon^p}\int_0^1 |u|^p + \epsilon^p \int_0^1 |u''|^p\right);\end{aligned}$$

If $1 < \epsilon \le \epsilon_0$, then by ☺

$$\|u'\|_{L^p(0,1)} \le K_p^{1/p}\left(\|u''\|_{L^p} + \|u\|_{L^p}\right) \le K_p^{1/p}\left(\epsilon\|u''\|_{L^p} + \frac{\epsilon_0}{\epsilon}\|u\|_{L^p}\right).$$

Simply take $K = K_p^{1/p}\epsilon_0$. □

Theorem 4.1.3. Interior H^2-regularity Theorem *Suppose $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ with $f \in L^2(\Omega)$, then for any $\Omega' \subset\subset \Omega$, $u \in H^2(\Omega')$ and*

$$\|u\|_{H^2(\Omega')} \lesssim_{L,\Omega',\Omega} \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}. \tag{4.1.1}$$

证明. According to the interpolation inequality, it suffices to show

$$\sum_{|\boldsymbol{\alpha}|=2} \|\partial^{\boldsymbol{\alpha}} u\|_{L^2(\Omega')} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right). ☹$$

Why ☹ is enough? Take subdomain $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$, then ☹ holds true if Ω replaced by Ω_1. Further we take $\eta \in C_0^\infty(\Omega)$ a cut-off function such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on Ω_1. Recalling from the HW, $v = \eta^2 u \in H_0^1(\Omega)$, and by definition of weak solution, we have

$$\int_\Omega a_{ij}u_i(2\eta\eta_i u + \eta^2 u_i) + b_i u_i \eta^2 u + cu^2\eta^2 = \int_\Omega f\eta^2 u.$$

By strict ellipticity

$$\begin{aligned}LHS &\ge \lambda_0 \int_\Omega \eta^2|\nabla u|^2 - C\int_\Omega \left(\eta|u||\nabla u| + \eta^2|u||\nabla u| + \eta^2 u^2\right) \\ &\ge \lambda_0 \int_\Omega \eta^2|\nabla u|^2 - \frac{C}{2}\int_\Omega\left(\epsilon\eta^2|\nabla u|^2 + \frac{1}{\epsilon}u^2\right) - C\int_\Omega u^2 \\ &\ge \frac{\lambda_0}{2}\int_\Omega \eta^2|\nabla u|^2 - C\int_\Omega u^2,\end{aligned}$$

and at the mean time

$$RHS \le \frac{1}{2}\left(\int_\Omega |f|^2 + \int_\Omega u^2\right).$$

The two estimates combined can lead to (4.1.1).

Now, to show ☹, we want to show

$$\int_\Omega \eta^2|\nabla_k^h(\nabla u)| \le C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}\right).$$

Take $\eta \in C_0^\infty(\Omega_1)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on Ω', where $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$. Letting $v \in H_0^1(\Omega)$, and $\tilde{f} = f - b_i u_i - c$, we have

$$\int_\Omega a_{ij} u_i v_j = \int_\Omega \tilde{f} v.$$

We further take $v = \nabla_k^{-h}\left(\eta^2 \nabla_k^h u\right)$, then for small $|h|$, v is well defined in Ω_1 and is compactly supported. According to HW, we know $v \in H_0^1(\Omega)$. Inserting v into the above equality, we observe

$$\begin{aligned}
-\int_\Omega a_{ij} u_i \left(\nabla_k^{-h}\left(\eta^2 \nabla_k^h u\right)\right)_j &= -\int_\Omega a_{ij} u_i \nabla_k^{-h}\left(\eta^2 \nabla_k^h u\right)_j \\
&= \int_\Omega \nabla_k^h\left(a_{ij} u_i\right)\left(\eta^2 \nabla_k^h u\right)_j \\
&= \int_\Omega \left(\nabla_k^h(u_i) a_{ij}(\cdot + h e_k) + u_i \nabla_k^h a_{ij}\right)\left(2\eta\eta_j \nabla_k^h u + \eta^2 \nabla_k^h(u_j)\right) \\
&= \int_{\Omega_1} (\cdots) \\
&\ge \lambda_0 \int_{\Omega_1} \eta^2 |\nabla_k^h(\nabla u)|^2 - C\int_{\Omega_1} \eta\left(|\nabla_k^h(\nabla u)|^2 + |\nabla u||\nabla_k^h u| + |\nabla u||\nabla_k^h(\nabla u)|\right) \\
&\ge \lambda_0 \int_{\Omega_1} \eta^2 |\nabla_k^h(\nabla u)|^2 - C\int_{\Omega_1} \frac{1}{2}\left(\epsilon\eta^2 |\nabla_k^h(\nabla u)|^2 + \frac{|\nabla_k^h u|^2}{\epsilon}\right) \\
&\quad - C\int_{\Omega_1}\left(\frac{\eta^2|\nabla u|^2}{2} + \frac{|\nabla_k^h u|^2}{2}\right) - C\int_{\Omega_1} \frac{1}{2}\left(\epsilon\eta^2 |\nabla_k^h(\nabla u)|^2 + \frac{|\nabla u|^2}{\epsilon}\right) \\
&\ge \lambda_0 \int_{\Omega_1} \eta^2 |\nabla_k^h(\nabla u)|^2/2 - C\int_{\Omega_1} |\nabla_k^h u|^2 - C\int_{\Omega_1} |\nabla u|^2.
\end{aligned}$$

Recalling $\tilde{f} = f - b_i u_i - cu$, we have similarly

$$\begin{aligned}
-\int_\Omega \tilde{f} v &\le \int_\Omega \left(\epsilon v^2 + \frac{\tilde{f}^2}{4\epsilon}\right) \\
&\le \epsilon n \int_\Omega \eta^2 |\nabla_k^h(\nabla u)|^2 + C\int_{\Omega_1} (\nabla_k^h u)^2 + C_\epsilon \int_\Omega (f^2 + b_i^2 u_i^2 + c^2 u^2) \\
&\le \epsilon n \int_\Omega \eta^2 |\nabla_k^h(\nabla u)|^2 + C\left(\int_\Omega f^2 + \int_\Omega u^2 + \int_\Omega |\nabla u|^2\right).
\end{aligned}$$

When ϵ is small, we may apply *ii.* of Theorem (4.1.1), and then we are done because $0 \le \eta \le 1$ and $\eta \equiv 1$ on Ω'. □

Theorem 4.1.4. Higher Order Interior Regularity *Let Ω bounded in $\mathbb{R}^n$, and suppose*

$$Lu = -(a_{ij}(\mathbf{x}) u_i)_j + b_i(\mathbf{x}) u_i + c(\mathbf{x}) u$$

is strictly elliptic in Ω, $a_{ij} \in C^{m+1}(\Omega) \cap L^\infty(\Omega)$, $b_i, c \in C^m(\Omega) \cap L^\infty(\Omega)$, where $m \ge 0$ (when $m = 0$, no need to assume $b_i, c \in C^0(\Omega)$), and $f \in H^m(\Omega)$. Let $u \in H^1(\Omega)$ be a weak solution of $Lu = f$, then $u \in H_{loc}^{m+2}(\Omega)$, and for any compactly supported subdomain Ω', we have

$$\|u\|_{H^{m+2}(\Omega')} \lesssim_{L,\Omega',\Omega} \|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}.$$

证明. Formally apply ∂_i to both sides of $Lu = f$, and obtain a new equation $Lw = \tilde{f}$. Check that $\partial_i u$ is a weak solution to it. Iteratively applying the prior theorem, we are done. □

Corollary 4.1.1. *Suppose a_{ij}, b_i, c and f are smooth on Ω, then u is also smooth on Ω.*

证明. Recall that by Sobolev Imbedding Theorem, we have

$$W^{m+2,2}_{loc}(\Omega) \hookrightarrow C^{m+2-\frac{n}{2}}(\Omega),$$

for $m + 2 - \frac{n}{2}$ not an integer. □

Now, we wish to study the global regularity of the solution. Before that, we study the effect of the boundary geometry on functions.

Definition 4.2. *Given a domain Ω, we say its boundary $\partial\Omega$ is C^k-smooth for $k \geq 0$ if for all $\mathbf{x}_0 \in \partial\Omega$, there is $r > 0$ such that after rotating the domain in $\mathbb{R}^n$ w.r.t. $\mathbf{x}_0$*

- *$B_r(\mathbf{x}_0) \cap \Omega = \{\mathbf{x} \in B_r(\mathbf{x}_0);\ x_n > \phi(x_1, \cdots, x_{n-1}) =: \phi(\mathbf{x}')\}$, with ϕ some C^k-smooth function;*
- *$\partial\Omega \cap B_r(\mathbf{x}_0) = \{\mathbf{x} \in B_r(\mathbf{x}_0);\ x_n = \phi(\mathbf{x}')\}$.*

Write $\mathbf{x}_0 = (x^0_1, \cdots, x^0_n) = (\mathbf{x}'_0, x^0_n)$. With the above defintion, we may locally straighten $\partial\Omega$ near $\mathbf{x}_0$ by setting

$$\begin{aligned} \Phi : B_r(\mathbf{x}_0) &\to \mathbb{R}^n \\ \mathbf{x} &\mapsto (\mathbf{x}' - \mathbf{x}'_0, x_n - \phi(\mathbf{x}')). \end{aligned}$$

Thus, Φ is C^k-smooth on the ball, and it is clearly injective according to its defintion. (We denote its image by N^+.) Its Jacobian is

$$D\Phi_{\mathbf{x}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_1(\mathbf{x}) & -\phi_2(\mathbf{x}) & \cdots & 1 \end{pmatrix}.$$

Thus, the volume form $|\det(D\Phi^* D\Phi)| d\mathbf{x} = d\mathbf{x}$ and so the reparametrization Φ is equiareal. Moreover, for $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$, we may define $\tilde{u}(\mathbf{y}) = u(\Phi^{-1}(\mathbf{y}))$ for $\mathbf{y} \in N^+$, and obtain the bound

$$\begin{aligned} \|\nabla_{\mathbf{y}} \tilde{u}\|_{L^p(N^+)} &= \left\| \left(D\Phi^{-1}_{\bullet}\right)^* (\nabla_{\mathbf{x}} u)(\Phi^{-1}(\bullet)) \right\|_{L^p(N^+)} \\ &\leq \left\| D\Phi^{-1} \right\|_{L^\infty(N^+)} \|\nabla_{\mathbf{x}} u\|_{L^p(B_r(\mathbf{x}_0)\cap\Omega)} \\ &\leq C_1 \|\nabla_{\mathbf{x}} u\|_{L^p(B_r(\mathbf{x}_0)\cap\Omega)}, \end{aligned}$$

and reversely, we also have

$$\|\nabla_{\mathbf{x}} u\|_{L^p(B_r(\mathbf{x}_0)\cap\Omega)} \leq C_2 \|\nabla_{\mathbf{y}} \tilde{u}\|_{L^p(N^+)}.$$

After a density argument, we know $W^{1,p}(\Omega \cap B_r(\mathbf{x}_0))$ is equivalent to $W^{1,p}(N^+)$ by the map $u \mapsto \tilde{u}$. Similarly one can show $W_0^{1,p}(\Omega \cap B_r(\mathbf{x}_0))$ is equivalent to $W_0^{1,p}(N^+)$ by using the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ for general Ω.

Theorem 4.1.5. Trace Theorem *Let Ω be bounded with C^1-smooth boundary. Then there is a linear and bounded operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega),\ 1 \leq p < \infty$$

such that

1. *$Tu = u\big|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$;*
2. *Generally, we have the bound*

$$\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)},\ u \in W^{1,p}(\Omega).$$

*Tu will be called the **trace** of u on $\partial\Omega$.*

证明. Recalling that $\partial\Omega \in C^1$ implies that $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, we start with a function $u \in C^1(\bar{\Omega})$. Let $B_r(\mathbf{x}_0)$ be a small ball centered at $\mathbf{x}_0 \in \partial\Omega$ such that we have a straightening map $\Phi : B_r(\mathbf{x}_0) \cap \Omega \to N^+$ (the same meaning as before.) By prior arguments, we know Φ induces a canonical equivalence between $W^{1,p}(A)$ to $W^{1,p}(\Phi(A))$ for any subdomain $A \subset B_r(\mathbf{x}_0) \cap \Omega$, and hence only have to consider the problem on N^+. By compactness of $B_r(\mathbf{x}_0)$ and because Φ is a C^1-diffeomorphism, there should be some small $\epsilon > 0$ such that the set $\{y_n = \epsilon\} \cap N^+$ is simply connected and its projection to $\{y_n = 0\}$ covers $B_{r/2}(\mathbf{0}')$. According to F.T.C., we have

$$u(\mathbf{y}', 0) = u(\mathbf{y}', t) - \int_0^t u_{y_n}(\mathbf{y}', s) ds,\ 0 < t < \epsilon,$$

and so

$$|u(\mathbf{y}', 0)|^p \leq 2^p \left(|u(\mathbf{y}', t)|^p + \left| \int_0^\epsilon u_{y_n}(\mathbf{y}', s) ds \right|^p \right),\ 0 < t < \epsilon.$$

Therefore, we have

$$\begin{aligned} \int_{B_{r/2}(\mathbf{0}')} |u(\mathbf{y}', 0)|^p d\mathbf{y}' &\leq \frac{2^p}{\epsilon} \left(\int_0^\epsilon \int_{\{y_n = \epsilon\} \cap N^+} |u(\mathbf{y}', t)|^p d\mathbf{y}' dt + \epsilon^p \int_0^\epsilon \int_{\{y_n = \epsilon\} \cap N^+} |u_{y_n}(\mathbf{y}', s)|^p d\mathbf{y}' ds \right) \\ &\leq C_{p,\epsilon} \left(\int_\Omega |u|^p + \int_\Omega |\nabla u|^p \right). \end{aligned}$$

Because $\partial\Omega$ is compact, we obtain the bound all over the boundary. Now, for general $u \in W^{1,p}(\Omega)$, one may first find a sequence of $C^1(\bar{\Omega})$ functions $u_n \in C^1(\bar{\Omega})$ that approximates u in $W^{1,p}(\Omega)$. According to the above bound, we know $u_n\big|_{\partial\Omega}$ is a Cauchy sequence in $L^p(\partial\Omega)$, and hence one may define $Tu = \lim_{n\to\infty} u_n\big|_{\partial\Omega}$ with limit taken in $L^p(\partial\Omega)$. After a density argument, we know the operator T is well-defined and we also have the bound in item 2.. In item 1., for $u \in W^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, we simply use the uniform convergence. Notice this case happens when $p > n$. □

Theorem 4.1.6. *Suppose Ω is bounded with C^1-smooth $\partial\Omega$, $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if an only if $Tu = 0$.*

证明. By definition of $W_0^{1,p}(\Omega)$, u can be approximated by a sequence of $C_0^\infty(\Omega)$ functions, whose traces are 0. By trace theorem, we know that $Tu = 0$ because T is continuous. Reversely, if $Tu = 0$, then there is a sequence of $C^1(\bar{\Omega})$ functions u_m such that u_m converges to u in $W^{1,p}(\Omega)$ and Tu_m converges to 0 as $n \to \infty$. After choosing a proper parametrization, we reduce the problem to the case where $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}_+$ the upper half plane, and $\partial\Omega$ is the hyperplane $\{y_n = 0\}$. Consider a nondecreasing cut-off function on $\phi \in C_0^\infty(\mathbb{R}_+)$ such that $\phi \equiv 1$ on $(0, 1)$ and $\phi \equiv 0$ on $[2, \infty)$, and define

$$\hat{u}_{k,m}(\mathbf{y}) = u_m(\mathbf{y})(1 - \phi(ky_n)),\ k, m \geq 1.$$

By HW, each $\hat{u}_{k,m}(\mathbf{y})$ are in $W_0^{1,p}(\Omega)$. It then suffices to show $\hat{u}_{k,m} \longrightarrow u_m$ in $W^{1,p}(\Omega)$ as $k \to \infty$. Observe that

$$\nabla_{\mathbf{y}'}\hat{u}_{k,m}(\mathbf{y}) = \nabla_{\mathbf{y}'}u_m(\mathbf{y}),$$

and

$$\partial_{y_n}\hat{u}_{k,m}(\mathbf{y}) = \partial_{y_n}u_m(\mathbf{y})(1 - \phi(ky_n)) - k\phi'(ky_n)u_m(\mathbf{y}).$$

Therefore, we have

$$\begin{aligned}
\|\nabla\hat{u}_{k,m} - \nabla u_m\|_{L^p(\Omega)} &\leq \left(\int_\Omega |\partial_{y_n}u_m(\mathbf{y})\phi(ky_n) + k\phi'(ky_n)u_m(\mathbf{y})|^p d\mathbf{y}\right)^{1/p} \\
&\leq \left(\int_\Omega |\partial_{y_n}u_m(\mathbf{y})\phi(ky_n)|^p d\mathbf{y}\right)^{1/p} + \left(\int_\Omega |k\phi'(ky_n)u_m(\mathbf{y})|^p d\mathbf{y}\right)^{1/p} \\
&\leq o(1) + \left(\int_{1/k}^{2/k}\int_{\mathbb{R}^{n-1}} k^pC^p|u_m(\mathbf{y}', y_n)|^p d\mathbf{y}'dy_n\right)^{1/p} \\
&\overset{F.T.C.}{\leq} o(1) + \left(\int_{1/k}^{2/k}\int_{\mathbb{R}^{n-1}} k^pC^p\left|u_m(\mathbf{y}', 0) + \int_0^{y_n}\partial_{y_n}u_m(\mathbf{y}', s)ds\right|^p d\mathbf{y}'dy_n\right)^{1/p} \\
&\overset{\text{Minkowski+Hölder}}{\leq} o(1) + Ck\left(\int_{1/k}^{2/k} s^{p-1}ds\right)^{1/p} \|\partial_{y_n}u_m\|_{L^p(\mathbb{R}^{n-1}\times[0,2/k])} \\
&\leq C\|\partial_{y_n}u_m\|_{L^p(\mathbb{R}^{n-1}\times[0,2/k])} \\
&\overset{L.D.C.T.}{\longrightarrow} 0,
\end{aligned}$$

as $k \to \infty$. □

Now, we arrive at the gate to investigate the global estimate. Recall

$$Lu = -(a_{ij}(\mathbf{x})u_i)_j + b_i(\mathbf{x})u_i + c(\mathbf{x})u.$$

We assume that $a_{ij} \in C^1(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $\partial\Omega \in C^2$ and L is strictly elliptic. If the boundary condition is Dirichlet, it is reasonable to ask whether the solution is globally H^2.

Theorem 4.1.7. *Suppose $u \in H_0^1(\Omega)$ is a weak solution of*

$$(DBVP)\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$. Then $u \in H^2(\Omega)$, and we have

$$\|u\|_{H^2(\Omega)} \lesssim_{\Omega,L} \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} .$$

证明. Let $\mathbf{x}_i$, $i = 1, \cdots, N$ be finitely many points on $\partial\Omega$ and $\{(B_i := B_{r_i}(\mathbf{x}_i), \Phi_i)\}$ be an atlas of $\partial\Omega$ and $B_0 \subset\subset \Omega$ such that $\Omega \subset \cup_{i=0}^N B_i$. By interior regularity, the bound on B_0 is automatically obtained, and the central issue is to obtain the bound

$$\sum_{|\boldsymbol{\alpha}|=2} \|\partial^{\boldsymbol{\alpha}} u\|_{L^p(\Omega\cap B_i)} \lesssim \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} . 🚗$$

Here if the above estimate is true, then one may apply interpolation inequality to reduce the H^1-norm of u to its L^2-norm.

To show 🚗, we take η a test function on $B_{2r_i}(\mathbf{x}_i)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on B_i. Apply L on $\eta u =: \bar{u}$, we have

$$\begin{aligned} L(\bar{u}) &= -(a_{ij}(\eta u)_i)_j + b_i(\eta u)_i + c\eta u \\ &= \eta f + \text{trash} \\ &=: \bar{f} \in L^2(\Omega), \bigcirc \end{aligned}$$

where by computation we know "trash" has its L^2-norm bounded by $C\|u\|_{H^1}$. Moreover, we have $\bar{u} = 0$ on $\partial(\Omega \cap B_{2r_i}(\mathbf{x}_i))$. The idea is that for $k = 1, \cdots, n-1$, we can as before do difference quotient (we may reduce the problem to the case that the domain is exact the upper half plane)

$$\frac{u_i(\mathbf{x} + he_k) - u_i(\mathbf{x})}{h}, \ i = 1, \cdots, n.$$

The remaining term $\frac{\partial^2 u}{\partial x_n^2}$ will be obtained by using the equation

$$-a_{nn}\frac{\partial^2 u}{\partial x_n^2} = \text{blahblah},$$

where $a_{nn}(\mathbf{x}) > \lambda_0$ by strict ellipticity.

Let Φ be a C^2-smooth diffeomorphism that transforms $B_{2r_i}(\mathbf{x}_i) \cap \Omega$ into a subdomain N_2^+ of the upper half space $\{y_n > 0\}$, and $\overline{B_{2r_i}(\mathbf{x}_i)} \cap \partial\Omega = \{y_n = 0\} \cap \partial N_2^+$. We also denote $\Phi(B_i)$ by N_1^+. For any $v \in H_0^1(B_{2r_i}(\mathbf{x}_i) \cap \Omega)$, we have an induced map $v \mapsto \tilde{v} = v \circ \Phi^{-1}$. We also set $\tilde{u} = \bar{u} \circ \Phi^{-1}$, $\tilde{a}_{kl} = \frac{\partial y_k}{\partial x_i} a_{ij} \circ \Phi^{-1} \frac{\partial y_l}{\partial x_j}$, and $\tilde{b}_k = \frac{\partial y_k}{\partial x_i} b_i \circ \Phi^{-1}$, and so are $\tilde{c}$, $\tilde{f}$. Since $\bar{u}$ is a weak solution to $\bigcirc$, we have

$$\begin{aligned} &\int_{\Omega\cap B_{2r_i}} a_{ij}\bar{u}_i v_j + b_i \bar{u}_i v + c\bar{u}v = \int_{\Omega\cap B_{2r_i}} \bar{f}v \\ \Longrightarrow &\int_{N_2^+} \tilde{a}_{kl}\tilde{u}_k\tilde{v}_l + \tilde{b}_k\tilde{u}_k\tilde{v} + \tilde{c}\tilde{u}\tilde{v} = \int_{N_2^+} \tilde{v}\tilde{f}. \end{aligned}$$

Observe that $\tilde{a}_{kl} \in C^1(\overline{N_2^+})$, and $\tilde{b}_k, \tilde{c} \in L^\infty(\Omega)$. By linear algebra and that Φ is nondegenerate, we know $\tilde{u}_{kl}$ is still strictly elliptic.

Now, we have reduced the problem into the case that the boundary is part of the hyperlane $\{y_n = 0\}$. Notice that N_1^+ has positive distance from $\partial N_2^+ \cap \{y_n > 0\}$, and so in directions e_k for $k = 1, \cdots, n-1$, we may do as before difference quotients and obtain the $L^2(N_1^+)$-bounds for $\partial_l \partial_k u$ with $(i,k) \neq (n,n)$. By interior regularity, we may pointwise do differentiation on the equation (here we need Φ to be C^2), and obtain

$$\tilde{u}_{nn} = \frac{-(\tilde{a}_{kl})_l \tilde{u}_k - \tilde{a}_{kl}\tilde{u}_{kl} - \tilde{f} + \text{trash}}{\tilde{a}_{nn}}.$$

$\square$

Theorem 4.1.8. Higher Order Global Regularity *Suppose L is strictly elliptic, $a_{ij} \in C^{m+1}(\bar{\Omega})$, $m \geq 0$, $b_i, c \in C^m(\bar{\Omega})$, $f \in H^m(\Omega)$, $\partial\Omega \in C^{m+2}$, then any weak solution $u \in H_0^1(\Omega)$ of the (DBVP) must be in $H^{m+2}(\Omega)$, and*

$$\|u\|_{H^{m+2}(\Omega)} \lesssim_{L,\Omega} \|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}.$$

Corollary 4.1.2. *If $a_{ij}, b_i, c \in C^\infty(\bar{\Omega})$, $f \in C^\infty(\bar{\Omega})$, $\partial\Omega \in C^\infty$, then $u \in C^\infty(\bar{\Omega})$.*

Epilogue of L^2-theory: (De Giorgi-Nash-Moser theory) Suppose Ω bounded, a_{ij}, b_i, c bounded functions and L strictly elliptic.

Theorem 4.1.9. *Suppose u is a weak solution of $Lu = f$ in Ω with $f \in L^q(\Omega)$, $q > \frac{n}{2}$. Then*

1. *(interior regularity) For all $\Omega' \subset\subset \Omega$, there is some $\alpha = \alpha(n, L, \Omega', \Omega, q)$ such that $u \in C^\alpha(\bar{\Omega}')$, and*
$$\|u\|_{C^\alpha(\bar{\Omega}')} \lesssim_{L,\Omega',\Omega,n,q} \|u\|_{L^2(\bar{\Omega})} + \|f\|_{L^q(\Omega)};$$
2. *If $u \in H_0^1(\Omega)$, then $u \in L^\infty(\Omega)$, and*
$$\|u\|_{L^\infty(\Omega)} \lesssim_{L,\Omega',\Omega,n,q} \|u\|_{L^2(\bar{\Omega})} + \|f\|_{L^q(\Omega)};$$
3. *If $u \in H_0^1(\Omega)$, and $\partial\Omega \in C^2$, then there is some $\alpha = \alpha(n, L, \Omega, q)$ such that $u \in C^\alpha(\bar{\Omega})$, and*
$$\|u\|_{C^\alpha(\bar{\Omega})} \lesssim_{n,L,\Omega,q} \|u\|_{L^2(\bar{\Omega})} + \|f\|_{L^q(\Omega)}.$$

Neumann-Robin Boundary Value Problem: We consider

$$(RBVP)\begin{cases} Lu = f, & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}_A} + \beta(\mathbf{x})u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\vec{n}_A \coloneqq \nabla u \cdot (a_{ij})_{n\times n}\vec{n}$, with $\vec{n}$ the unit outer normal field on $\partial\Omega$. A question is that how can one define its weak solution? Formally, we suppose everything is smooth, and for $v \in C^1(\bar{\Omega})$, we have

$$\int_\Omega -(a_{ij}u_i)_j v + (b_i u_i + cu)v = \int_\Omega fv.$$

Integration by parts gives that

$$LHS = \int_\Omega a_{ij} u_i v_j - \int_{\partial\Omega} (a_{ij} u_i \vec{n}_j) v + \int_\Omega (b_i u_i + cu) v = \int_\Omega fv, \ \circledast.$$

Definition 4.3. *We say u is a weak solution of (RBVP), if $u \in H^1(\Omega)$ and $\circledast$ holds for all $v \in H^1(\Omega)$.*

From now on, we assume $\partial\Omega \in C^1$, $a_{ij}, b_i, c \in L^\infty(\Omega)$, $\beta \in L^\infty(\partial\Omega)$ and L strictly elliptic. Existence: Full version of Fredholm Alternatives hold; Uniqueness holds if $b_i = 0$, $c \geq 0$ and $\beta \geq 0$.

Theorem 4.1.10. Global H^2-regularity *Suppose $\partial\Omega \in C^2$, $a_{ij} \in C^1(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $\beta \in C^1(\partial\Omega)$, u is a weak solution of (RBVP). Then $u \in H^2(\Omega)$, and*

$$\|u\|_{H^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}.$$

For more information, see Tag der Prüfung's Thesis.

4.2 L^p-theory for Elliptic Equations $(1 < p < \infty)$

Let Ω be bounded, and

$$Lu = a_{ij} u_{ij} + b_i u_i + cu,$$

with $a_{ij} \in C^0(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$. We also assume that L is strictly elliptic on the domain.

Definition 4.4. *We say u is a **strong solution** of*

$$(BVP) \begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

if

- *$u \in W^{2,p} \cap W_0^{1,p}(\Omega)$;*
- *PDE holds pointwise on Ω.*

Fredholm Alternative: Suppose $\partial\Omega \in C^2$, then uniqueness of (BVP) is equivalent to the existence of strong solution for all $f \in L^p(\Omega)$. We will not prove this fact here.

We also have the following estimates (without proofs).

Theorem 4.2.1. Interior L^p-estimate *Let u be a strong solution of $Lu = f$, then for any $\Omega' \subset\subset \Omega$, we have*

$$\|u\|_{W^{2,p}(\Omega')} \lesssim_{n,p,L,\Omega',\Omega} \|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Theorem 4.2.2. Global L^p-estimate *Let u be a strong solution of $Lu = f$ and $\partial\Omega \in C^2$, then we have*

$$\|u\|_{W^{2,p}(\Omega)} \lesssim_{n,p,L,\Omega} \|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}.$$

L^p-regularity Theory

Theorem 4.2.3. *1. Suppose $\partial\Omega \in C^2$, $u \in W^{2,p}(\Omega)$ is a strong solution of (BVP), $f \in L^q(\Omega)$ for $p < q < \infty$, then $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$;*

2. If $u \in W^{2,p}(\Omega)$ is a strong solution of $Lu = f$, $f \in L^q(\Omega)$, with $p < q < \infty$, then $u \in W^{2,q}_{loc}(\Omega)$.

证明. 1. (**Bootstrap Method**) Consider

$$\begin{cases} a_{ij}v_{ij} + b_i v_i = f - cv, & \text{on } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2.1}$$

We want the right hand side of the equation to be in $L^q(\Omega)$. To see this, we recall by Sobolev Imbedding

$$W^{2,p}(\Omega) \hookrightarrow L^{p_1}(\Omega),$$

for

$$p_1 = \begin{cases} \frac{np}{n-2p}, & \text{if } n - 2p > 0, \\ \text{arbitrarily big}, & \text{if } n - 2p \leq 0. \end{cases}$$

Then $p_1 > p$ and hence $u \in L^{p_1}(\Omega)$. If $p_1 \geq q$, then RHS of the PDE is in $L^q(\Omega)$. Thus, the equation (4.2.1) has "$c = 0$", which implies that it has uniqueness of solutions both in L^p and L^q settings. According to Fredholm Alternatives, (4.2.1) has a unique strong solution $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Clearly, v is also a strong solution in L^p setting, and hence by uniqueness of solution $u \equiv v$, and thus $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. If $p < p_1 < q$, then RHS of PDE is in $L^{p_1}(\Omega)$. By above arguments, we have $u \in W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega)$. Because

$$W^{2,p_1}(\Omega) \hookrightarrow L^{p_2}(\Omega),$$

where p_n is defined similarly to p_1, with p replaced by p_{n-1} for $n \geq 2$. Thus RHS of PDE is in $L^{\min(q,p_2)}(\Omega)$. By repeating the above arguments, we have either $p_k \geq q$ for some $k \geq 2$, or p_k an increasing sequence bounded by q. It suffices to consider the latter case. Observe that both sides of

$$p_{k+1} = \frac{np_k}{n - 2p_k}$$

must have a limit. The limit of the sequence p_∞ then satisfies the equality

$$p_\infty = \frac{np_\infty}{n - 2p_\infty},$$

and so $p_\infty = 0$, which is impossible, because each $p_k \geq 1$;

2. For any $\Omega' \subset\subset \Omega$, we take η a cut-off function that is compactly supported on Ω, and equals 1 on Ω'. Let $v = \eta u$, then we have

$$Lv = a_{ij}v_{ij} + b_j v_j + cv = \eta Lu + a_{ij}\eta_{ij}u + 2a_{ij}\eta_i u_j + b_i \eta_i u,$$

and $v\big|_{\partial\Omega_1} = 0$ for $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$ and $\partial\Omega_1 \in C^\infty$. Observe $u \in W^{2,p}(\Omega) \hookrightarrow L^{p_1}(\Omega)$, and $u_i \in W^{1,p}(\Omega) \hookrightarrow L^{\overline{p_1}}(\Omega)$ for

$$\overline{p_1} = \begin{cases} \frac{np}{n-p}, & \text{if } n-p > 0, \\ \text{arbitrarily big}, & \text{if } n-p \leq 0. \end{cases}$$

Then $p < \overline{p_1} < p_1$, and thus in $\faStackOverflow$, $RHS \in L^{\min(q,\overline{p_1})}(\Omega)$. If $\overline{p_1} \geq q$, then we are done. Otherwise, we have, according to 1., $v \in W^{2,\overline{p_1}}(\Omega_1)$, and thus $u \in W^{2,\overline{p_1}}(\Omega')$. Similar to 1., we may do bootstrap, and obtain the result.

□

Theorem 4.2.4. Higher Order Regularity

1. *(Global Version) Suppose $\partial\Omega \in C^{m+2}$, $f \in W^{m,q}(\Omega)$ for $1 < q < \infty$ with $m \geq 1$. $a_{ij}, b_i, c \in C^m(\bar{\Omega})$. If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ is a strong solution of*

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

   Then $u \in W^{2+m,q}(\Omega)$;

2. *(Interior Version) Suppose $f \in W^{m,q}(\Omega)$, $a_{ij}, b_i, c \in C^m(\bar{\Omega})$, $u \in W^{2,p}(\Omega)$ is a strong solution of $Lu = f$, then $u \in W^{m+2,q}_{loc}(\Omega)$.*

Schauder Theory for Classical Solutions

We consider the operator $Lu = a_{ij}u_{ij} + b_iu_i + cu$ on bounded domain Ω, where $\partial\Omega \in C^{2+\alpha}$, and $a_{ij}, b_i, c \in C^\alpha(\bar{\Omega})$.

Fredholm Alternative: Consider

$$(BVP) \begin{cases} Lu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then uniqueness of $C^{2+\alpha}(\bar{\Omega})$ solution for (BVP) is equivalent to the existence of such a solution for every $f \in C^\alpha(\bar{\Omega})$.

Remark:

1. Uniqueness prevails if $c \leq 0$ on Ω (Maximum Principle);

2. Existence under weak condition (see Gilbarg Trudinger): Assume $a_{ij}, b_i, c \in C^\alpha_{loc}(\Omega) \cap L^\infty(\Omega)$ and strict ellipticity of L; $\partial\Omega$ satisfies exterior sphere condition at every point and $c \leq 0$ all over the domain. Then for every $\phi \in C^0(\partial\bar{\Omega})$ and $f \in C^\alpha_{loc} \cap L^\infty(\Omega)$, there is a unique $u \in C^0(\bar{\Omega}) \cap C^{2+\alpha}_{loc}(\Omega)$ solving the equation.

Theorem 4.2.5. Schauder Estimates

1. *(Interior Estimate) Suppose* $u \in C^{2+\alpha}_{loc}(\Omega)$ *is a solution of* $Lu = f \in C^\alpha(\bar{\Omega})$*. Then for any* $\Omega' \subset\subset \Omega$*, we have*

$$\|u\|_{C^{2+\alpha}(\overline{\Omega'})} \lesssim_{\alpha,n,L,\Omega',\Omega} \|f\|_{C^\alpha(\bar{\Omega})} + \|u\|_{C^\alpha(\bar{\Omega})};$$

2. *(Global Estimate) Suppose* $\partial\Omega \in C^{2+\alpha}$*,* $u \in C^{2+\alpha}(\bar{\Omega})$ *is a solution to the (BVP), then*

$$\|u\|_{C^{2+\alpha}(\bar{\Omega})} \lesssim_{\alpha,n,L,\Omega} \|f\|_{C^\alpha(\bar{\Omega})} + \|u\|_{C^\alpha(\bar{\Omega})}.$$

User-friendly Regularity Theorem

Theorem 4.2.6. 1. *(Global Version) Suppose* $a_{ij}, b_i, c, f \in C^\alpha(\bar{\Omega})$ *and* $\partial\Omega \in C^{2+\alpha}$*. If* $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ $(1 < p < \infty)$ *is a strong solution of (BVP), then* $u \in C^{2+\alpha}(\bar{\Omega})$*;*

2. *(Interior Version) Suppose* $a_{ij}, b_i, c, f \in C^\alpha_{loc}(\bar{\Omega})$*,* $u \in W^{2,p}_{loc}(\Omega)$ *is a strong solution of* $Lu = f$*, then* $u \in C^{2+\alpha}_{loc}(\Omega)$*;*

3. *Let* $Au = -(a_{ij}u_i)_j + b_iu_i + cu$*. Suppose* A *is strictly elliptic on the domain,* $a_{ij} \in C^1(\bar{\Omega})$*,* $b_i, c \in L^\infty(\Omega)$*. Assume* $u \in H^1_0(\Omega)$ *is a weak solution of*

$$\begin{cases} Au = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\partial\Omega \in C^2$*, and* $f \in L^p(\Omega)$*, with* $p \geq \frac{2n}{n+2}$ *for* $n \geq 3$ *(arbitrary if* $n = 1, 2$*). Then* $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$*;*

4. *Suppose conditions in 3.,* $a_{ij} \in C^{1+\alpha}(\bar{\Omega})$*,* $b_i, c, f \in C^\alpha(\bar{\Omega})$*,* $\partial\Omega \in C^{2+\alpha}$ $(0 < \alpha < 1)$*. Then* $u \in C^{2+\alpha}(\bar{\Omega})$*.*

证明. 1. For any $q \in (p, \infty)$, we have $f \in C^\alpha(\bar{\Omega}) \hookrightarrow L^q(\Omega)$. By Global L^q-regularity, we see

$$u \in W^{2,q}(\Omega) \hookrightarrow C^{2-\frac{n}{q}}(\bar{\Omega}).$$

Thus, $f - cu \in C^\alpha(\bar{\Omega})$. Now, consider

$$\begin{cases} a_{ij}v_{ij} + b_iv_i + cv = f - cu, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}.$$

Observe that the problem has uniqueness of solution because "$(c \equiv 0)$". According to Fredholm Alternative (Schauder Setting), it has a unique solution $v \in C^{1,\alpha}(\bar{\Omega})$. It is evident that $v \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ is also a strong solution of . Also by uniqueness of solution, we have $u \equiv v$;

2. For any $\Omega' \subset\subset \Omega$, we take a cut-off function η on Ω, such that $0 \leq \eta \leq 1$ and equals 1 on Ω'. By L^p-interior regularity, we have $u \in W^{2,q}_{loc}(\Omega) \hookrightarrow C^{2-\frac{n}{q}}_{loc}(\Omega)$ for any $q > 1$. If we

choose q to be large, we have $u \in C^{1+\alpha}_{loc}(\Omega)$. Let $\bar{u} = \eta u$, we then have a new equation

$$\begin{cases} L\bar{u} = \eta f + a_{ij}\eta_{ij}u + a_{ij}(\eta_i u_j + \eta_j u_i) + b_i \eta_i u \in C^{\alpha}(\bar{\Omega}), & \text{on } \Omega, \\ \bar{u} = 0, & \text{on } \partial\Omega. \end{cases}$$

Thus $\bar{u} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ is a strong solution of the above equation. By 1., we know $\bar{u} \in C^{2+\alpha}(\bar{\Omega})$, which implies that $u \in C^{2+\alpha}_{loc}(\Omega)$;

3. When $p \geq 2$, then $f \in L^2(\Omega)$, and so by H^2-global regularity, we have $u \in W^{2,2}(\Omega)$. We claim u is a strong solution in this setting: For any $v \in C^\infty_0(\Omega)$, because u is a weak solution, we have

$$\int_\Omega a_{ij}u_i v_j + b_i u_i v + cuv = \int_\Omega fv.$$

Recalling what we did in HW, we have

$$\begin{aligned} \text{First term of } LHS &= \int_\Omega -(a_{ij}u_i)_j v \\ &= \int_\Omega \left[-(a_{ij})_j u_i - a_{ij}u_{ij}\right] v. \end{aligned}$$

Replacing v by an approximation of identity, we see

$$-a_{ij}u_{ij} + (b_i - (a_{ij})_j)u_i + cu = f, \text{ a.e.}$$

Now, L^p-regularity theory gives the result. When $1 < p < 2$, we define

$$\tilde{A}v = -(a_{ij}v_{ij}) + (b_i - (a_{ij})_j)v_i,$$

and consider

$$\begin{cases} \tilde{A}v = f - cu \in L^p(\Omega) & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}.$$

By Fredholm Alternative, has one and only one strong solution $v \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. Is v an H^1_0-weak solution?

- $v \in L^2(\Omega)$? Recall that $W^{2,p(\Omega)} \hookrightarrow L^q(\Omega)$, where $q \geq \frac{np}{n-2p}$. But $p \geq \frac{2n}{n+2}$ will evidently imply that ;
- $\nabla v \in L^2(\Omega)$? Observe that $\nabla v \in W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ for $r \geq \frac{np}{n-p}$. But we also have $\frac{np}{n-p} \geq 2$;
- $Tv = 0$ in H^1-sense. Because $\partial\Omega \in C^1$, we have that $C^\infty(\bar{\Omega})$ is dense in $W^{2,p}(\Omega)$. Thus, there is a sequence $v_k \in C^\infty(\bar{\Omega})$ that converges to v in $W^{2,p}(\Omega) \hookrightarrow H^1(\Omega)$. Thus, by continuity of trace in $H^1(\Omega)$, we have $Tv_k = v_k\big|_{\partial\Omega} \longrightarrow Tv$ in $L^2(\partial\Omega)$ as $k \to \infty$. But $v \in W^{1,p}_0(\Omega)$, we have $Tv_k = v_k\big|_{\partial\Omega} \longrightarrow 0$ in $L^p(\Omega)$ as $k \to \infty$. After passage to a subsequence, we have $Tv = 0\, a.e.$, which implies that $v \in H^1_0(\Omega)$.

Because v is a $W^{2,p}$ strong solution of $\tilde{A}v = f - cu$ in Ω, we have for all $w \in H_0^1(\Omega)$

$$\int_\Omega \tilde{A}vw = \int_\Omega (f - cu)w,$$

where by old HW, we have $LHS = \int_\Omega a_{ij}u_iw_j + b_iu_iw$. Thus v is a weak solution of $\tilde{A}v = f - cv$. Because "$c \equiv 0$", we have uniqueness, and hence $u \equiv v$;

4. Apply 3. and 1..

□

Remark:

- $f \in L^p(\Omega)$ induces a linear and bounded functional on $H_0^1(\Omega)$. Define for $v \in H_0^1(\Omega) \mapsto \int_\Omega fv$, we have the following bound

$$\int_\Omega |fv| \le \|f\|_p \|v\|_{p'} .$$

Recalling $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ with $q \ge \frac{2n}{n-2}$, we only have to show $p' \le q$. Observe

$$\frac{p}{p-1} \le \frac{2n}{n-2}$$

is equivalent to

$$p \ge \frac{2n}{n+2},$$

which is exactly the assumption on p;

- If B.C. is Robin/Neumann $\frac{\partial u}{\partial \vec{n}_A} + \beta u = 0$ on $\partial\Omega$, then 3. holds if $\beta \in C^1(\partial\Omega)$ and 4. holds if $\beta \in C^{1+\alpha}(\partial\Omega)$.

Chapter 5

Function Space Theories for Second Order Parabolic Equations

Let Ω be bounded, and $0 < T < \infty$, we define $Q_T = \Omega \times (0,T)$, $S_T = \Omega \times (0,T)$ and $\Gamma_T = S_T \cup \bar{\Omega} \times \{0\}$. A parabolic operator defined on the domain is of the form

$$Mu = \frac{\partial u}{\partial t} + Au,$$

where

$$Au = -(a_{ij}(\mathbf{x},t)u_i)_j + b_i(\mathbf{x},t)u_i + c(\mathbf{x},t)u.$$

We say M is *strictly parabolic* on Q_T if there is $\lambda_0 > 0$ such that

$$(a_{ij}(\mathbf{x},t)) \succcurlyeq \lambda_0 I_{n\times n},\ a.e.\ \text{on}\ Q_T.$$

5.1 L^2-theory for Parabolic Equations

Definition 5.1. Anisotropic Sobolev Spaces

- *For $k \geq 1$, we define*

$$W_p^{2k,k}(Q_T) = \{u \in L^p(Q_T);\quad weak\ \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta}u \in L^p(Q_T)\ for\ |\boldsymbol{\alpha}| + 2|\beta| \leq 2k\},$$

  on which we may introduce a norm

$$\|u\|_{W_p^{2k,k}(Q_T)} = \left(\sum_{|\boldsymbol{\alpha}|+2|\beta|\leq 2k} \left\|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta}u\right\|_{L^p(Q_T)}^p\right)^{1/p};$$

- *For $l, k = 0$ or 1, we define*

$$W_p^{l,k}(Q_T) = \{u \in L^p(Q_T);\ \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}u, \partial_t^{\beta}u \in L^p(Q_T),\ |\boldsymbol{\alpha}| \leq l, \beta \leq k\}.$$

Remark:

1. $W_p^{2,1} = \{u, u_t, \nabla_{\mathbf{x}} u, \nabla^2_{\mathbf{x}} u \in L^p(Q_T)\}$;

2. When $l = k = 1$, then $W_p^{1,1}(Q_T) = W^{1,p}(Q_T)$;

3. We say

   $\overset{o}{W}{}_p^{1,k}(Q_T) =$ closure of $C^\infty(\overline{Q_T})$ functions with restriction 0 on $\overline{S_T}$ in the space $W_p^{1,k}(Q_T)$.

Theorem 5.1.1. Density Theorem *If $\partial\Omega \in C^1$, then all spaces above (except $\overset{o}{W}{}_p^{1,k}(Q_T)$) have dense subset $C^\infty(\overline{Q_T})$.*

Theorem 5.1.2. Imbedding Theorem

1. *Suppose $1 < P < \infty$. Then*

$$W_p^{0,1}(Q_T) \hookrightarrow C^0([o, T]; L^p(\Omega)),$$

   that is, for any $u \in W_p^{0,1}(Q_T)$, the function $t \in [0, T] \mapsto u(\cdot, t) \in L^p(\Omega)$ is continuous, and

$$\max_{t \in [0,T]} \|u(\cdot, t)\|_{L^p(\Omega)} \le (p + 1/T)^{1/p} \|u\|_{W_p^{0,1}(Q_T)}$$

2. *Let $k \ge 1$, $\partial\Omega \in C^2$ and $u \in W_2^{2k,k}(Q_T)$. Then*

$$\begin{aligned} & u \in C^0([0, T]; H^{2k-1}(\Omega)) \\ & u_t \in C^0([0, T]; H^{2k-3}(\Omega)) \\ & \vdots \\ & \frac{\partial^{k-1} u}{\partial t^{k-1}} \in C^0([0, T]; H^1(\Omega)). \end{aligned}$$

   Moreover, these inclusions are continuous.

证明. 1. Let $u \in C^\infty(\overline{Q_T})$, we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega |u(\mathbf{x}, t)|^p d\mathbf{x} &= \int_\Omega \frac{d}{dt} |u(\mathbf{x}, t)|^p d\mathbf{x} \\ &= \int_\Omega p |u(\mathbf{x}, t)|^{p-1} u_t \cdot sign(u) d\mathbf{x}, \end{aligned}$$

where $p > 1$ makes sure the above equality. Now, for any $0 \le s, t \le T$, we have

$$\begin{aligned} \int_\Omega |u(\mathbf{x}, t)|^p d\mathbf{x} - \int_\Omega |u(\mathbf{x}, s)|^p d\mathbf{x} &\le p \left| \int_s^t \int_\Omega |u(\mathbf{x}, \tau)|^{p-1} |u_t(\mathbf{x}, \tau)| d\mathbf{x} d\tau \right| \\ &\le p \left(\int_{Q_T} |u_t|^p \right)^{1/p} \left(\int_{Q_T} |u|^p \right)^{\frac{p-1}{p}} \\ &\le p \|u\|^p_{W_p^{0,1}(Q_T)}. \end{aligned}$$

Observe by continuity,

$$\begin{aligned}\int_{Q_T} |u(\mathbf{x},t)|^p d\mathbf{x}dt &= \int_0^T \left(\int_\Omega |u(\mathbf{x},t)|^p d\mathbf{x} \right) dt \\ &= T \int_\Omega |u(\mathbf{x},s)|^p d\mathbf{x},\end{aligned}$$

for some $s \in [0,T]$. Therefore, we have

$$\begin{aligned}\int_\Omega |u(\mathbf{x},t)|^p d\mathbf{x} &\leq \frac{1}{T} \int_{Q_T} |u(\mathbf{x},t)|^p d\mathbf{x}dt + p \left\| u \right\|^p_{W_p^{0,1}(Q_T)} \\ &\leq (p + 1/T) \left\| u \right\|^p_{W_p^{0,1}(Q_T)},\end{aligned}$$

and thus we have the estimate. A density argument show that this is true for all elements in $W_p^{0,1}(Q_T)$;

2. It suffices to show

$$W_2^{2k,k}(Q_T) \hookrightarrow C^0([0,T]; H^{2k-1}(\Omega)).$$

To see this, we need

Theorem 5.1.3. Extension Theorem *For all $\Omega \subset\subset \Omega'$, we set $Q'_T = \Omega' \times (0,T)$. Then there is an extension operator*

$$E : W_p^{2k,k}(Q_T) \to \overset{o}{W}{}_p^{2k,k}(Q'_T),$$

such that

- *E is linear and bounded;*
- *$Eu\big|_{Q_t} = u$;*
- *$Eu = 0$ near $\overline{S'_T}$.*

Suppose $u \in C^\infty(\overline{Q_T})$, and let $\bar{u} = Eu$, we have

$$\begin{aligned}\frac{d}{dt} \int_{\Omega'} |\nabla \bar{u}|^2 &= \int_{\Omega'} 2 \nabla \bar{u} \cdot \nabla \bar{u}_t \\ &= -2 \int_{\Omega'} \Delta \bar{u} \bar{u}_t.\end{aligned}$$

For all $0 \leq t, s \leq T$, we have

$$\begin{aligned}\int_{\Omega'} |\nabla \bar{u}(\mathbf{x},t)|^2 d\mathbf{x} - \int_{\Omega'} |\nabla \bar{u}(\mathbf{x},s)|^2 d\mathbf{x} &\leq 2 \int_{Q'_T} |\Delta \bar{u} \bar{u}_t| \\ &\leq \int_{Q'_T} |\Delta \bar{u}|^2 + \int_{Q'_T} |\bar{u}_t|^2 \\ &\leq \left\| \bar{u} \right\|_{\overset{o}{W}{}_2^{2,1}(Q'_T)} \\ &\leq C \left\| u \right\|_{W_2^{2,1}(Q_T)}.\end{aligned}$$

Taking $s \in [0,T]$ such that

$$\begin{aligned}\int_{\Omega'} |\nabla \bar{u}(\mathbf{x},s)|^2 d\mathbf{x} &= \frac{1}{T}\int_{Q'_T} |\nabla \bar{u}|^2 \\ &\leq \frac{1}{T}\|\bar{u}\|^2_{W_2^{2,1}(Q'_T)},\end{aligned}$$

we then have

$$\int_{\Omega} |\nabla u(\mathbf{x},t)|^2 d\mathbf{x} \leq (C/T + C)\|u\|^2_{W_2^{2,1}(Q_T)}, \; t \in [0,T].$$

Combining this estimate with the one from 1., we have

$$\|u(\cdot,t)\|_{C^0([0,T];H^1(\Omega))} \lesssim \|u\|_{W_2^{2,1}(Q_T)}, \; \forall u \in C^\infty(\overline{Q_T}).$$

A density argument shows this estimate also hold in $W_2^{2,1}(Q_T)$, which shows the case when $k=1$. Now, for $k \geq 2$, by what has been proven

$$\max_{t\in[0,T]} \|u(\cdot,t)\|_{H^1(\Omega)} \lesssim \|u\|_{W_2^{2,1}(Q_T)}.$$

But for $\partial_{\mathbf{x}} u \in W_2^{2,1}(Q_T)$, we still have the above estimate. What's more, we have

$$\max_{t\in[0,T]} \|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u(\cdot,t)\|_{H^1(\Omega)} \lesssim \|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u\|_{W_2^{2,1}(Q_T)}, \; |\boldsymbol{\alpha}| + 2 \leq 2k,$$

which exhibits that

$$\max_{t\in[0,T]} \|u\|_{H^{2k-1}(\Omega)} \lesssim \|u\|_{W_2^{2k,k}(Q_T)}.$$

□

Dirichlet Initial Boundary Value Problem

We consider the following equation

$$(DIBVP)\begin{cases} \frac{\partial u}{\partial t} + Au = f(\mathbf{x},t), & \text{on } Q_T, \\ u(\mathbf{x},0) = \phi(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u = 0, & \text{on } S_T, \end{cases}$$

where $M = \frac{\partial}{\partial t} + A$ is strictly parabolic.

Our central issue is the existence and uniqueness of the solution to (DIBVP). To this end, we further assume that $a_{ij}, b_i, c \in L^\infty(Q_T)$, $f \in L^2(Q_T)$ and $\phi \in L^2(\Omega)$.

Definition 5.2. *We say $u(\mathbf{x},t)$ is a weak solution to (DIBVP) if*

- $u \in \overset{o}{W}{}_2^{1,1}(Q_T)$ *and* $u(\cdot,t) \overset{t\to 0^+}{\longrightarrow} \phi(\cdot)$ *in* $L^2(\Omega)$*;*
- *For all* $v \in \overset{o}{W}{}_2^{1,1}(Q_T)$*, we have*

$$\int_{Q_T} [u_t v + a_{ij} u_i v_j + b_i u_u v + cuv] = \int_{Q_T} fv.$$

Remark: The above integral equation is equivalent to that, for any $\bar{t} \in [0,T]$

$$\int_{Q_{\bar{t}}} [u_t v + a_{ij} u_i v_j + b_i u_i v + cuv] = \int_{Q_{\bar{t}}} fv,\ v \in \overset{o}{W}{}_2^{1,1}(Q_T).$$

To see this, we take η_k a sequence of functions in $C^\infty([0,T])$ such that $\eta_k \equiv 0$ on $[\bar{t},T]$, $0 \le \eta_k \le 1$ and $\eta_k(t) \overset{k\to\infty}{\longrightarrow} 1$, for all $t \in [0,\bar{t})$. Now, replacing v by $v\eta_k$ and a simple application of L.D.C.T. will show this fact.

Energy Estimate (A priori estimate)

Taking $v = u$, we have

$$\int_{Q_T} u_t u + a_{ij} u_i u_j + b_i u_i u + cu^2 = \int_{Q_T} fu,$$

We claim that

$$\int_{Q_T} u_t u = \frac{1}{2}\left(\int_\Omega u^2(\cdot,\bar{t}) - \int_\Omega \phi^2\right),$$

which is clear to see when u is smooth. For general $u \in \overset{o}{W}{}_2^{1,1}(Q_T)$, recalling its definition, we may choose a sequence u_k in $C^\infty(\overline{Q_T})$ such that u_k vanishes near S_T and u_k converges to u in the ambient space. Using the norms, we know the integrals also converges.

Returning to , we have

$$\begin{aligned}\frac{1}{2}\int_\Omega u^2(\cdot,\bar{t}) - \frac{1}{2}\int_\Omega \phi^2 + \lambda_0 \int_{Q_{\bar{t}}} |\nabla_{\mathbf{x}} u|^2 &\le \int_{Q_T} \left(fu - b_i u_i u - cu^2\right) \\ &\le \int_{Q_{\bar{t}}} \frac{f^2+u^2}{2} + \int_{Q_{\bar{t}}} \epsilon |\nabla_{\mathbf{x}} u|^2 + \frac{1}{4\epsilon}\int_{Q_{\bar{t}}} u^2 |\vec{b}|^2 + \|c\|_{L^\infty(Q_T)} \int_{Q_T} u^2.\end{aligned}$$

Take $\epsilon = \lambda_0/2$, we have

$$\frac{1}{2}\int_\Omega u^2(\mathbf{x},\bar{t})d\mathbf{x} + \frac{\lambda_0}{2}\int_{Q_{\bar{t}}} |\nabla_{\mathbf{x}} u(\mathbf{x},t)|^2 d\mathbf{x}dt \le \frac{1}{2}\int_\Omega \phi^2(\mathbf{x})d\mathbf{x} + C\int_{Q_T} u^2(\mathbf{x},t) + f^2(\mathbf{x},t)d\mathbf{x}dt,$$

which shows that

$$\max_{t\in[0,T]} \int_\Omega u^2(\mathbf{x},t)d\mathbf{x} + \lambda_0 \int_{Q_T} |\nabla_{\mathbf{x}} u(\mathbf{x},t)|^2 d\mathbf{x}dt \lesssim_L \int_\Omega \phi^2(\mathbf{x})d\mathbf{x} + \int_{Q_T} u^2(\mathbf{x},t) + f^2(\mathbf{x},t)d\mathbf{x}dt,$$

We would like to call

$$\max_{t\in[0,T]} \int_\Omega u^2(\mathbf{x},t)d\mathbf{x}$$

mathematical energy, and

$$\lambda_0 \int_{Q_T} |\nabla_{\mathbf{x}} u(\mathbf{x},t)|^2 d\mathbf{x}dt$$

interfacial energy.

We have a further result: For $t \in [0,T]$, if we set $g(t) = \int_{Q_t} u^2(\mathbf{x},t)d\mathbf{x}dt$, then we have that $g \in AC([0,T])$, $g(0) = 0$ and

$$g'(t) \le Cg(t) + \int_\Omega \phi^2 + C\int_{Q_T} f^2 \le Cg(t) + R,\ t - a.e.$$

Therefore, we have

$$(e^{-Ct}g(t))' = Re^{-Ct},$$

which shows that

$$g(t) \le \frac{1}{C}(e^{Ct} - 1)R.$$

Inserting this into , we obtain

$$\max_{t\in[0,T]} \int_\Omega u^2(\mathbf{x},t)d\mathbf{x}dt + \int_{Q_T} |\nabla_{\mathbf{x}} u(\mathbf{x},t)|^2 d\mathbf{x}dt \le 2e^{CT}\left(\int_\Omega \phi^2(\mathbf{x})d\mathbf{x} + C\int_{Q_T} f^2(\mathbf{x},t)d\mathbf{x}dt\right).$$

Theorem 5.1.4. Uniqueness of Solutions *With the above energy estimate, we know that there is at most one weak solution to (DIBVP).*

Existence: Galerkin Method

Special Case: Eigen-expansion method. Consider

$$\begin{cases} u_t - \Delta u = 0, & \text{on } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega, \\ u(\mathbf{x},0) = \phi(\mathbf{x}). \end{cases}$$

To solve this problem, we set $u(\mathbf{x},t) = X(\mathbf{x})T(t)$, and obtain

$$\frac{T'(t)}{T(t)} = \frac{\Delta X(\mathbf{x})}{X(\mathbf{x})} = -\lambda,\ t > 0, \mathbf{x} \in \Omega,$$

which reduces the original problem to an eigenvalue problem.

Let L be autonomous (independent of time t), and b_i's are 0, that is, $Lu = -(a_{ij}(\mathbf{x})u_i)_j + c(\mathbf{x})u$. Suppose L is strictly elliptic on the domain, and a_{ij}, c are bounded. Now, we set $Mu = \frac{\partial u}{\partial t} + Lu$, with L defined above, in the parabolic domain $Q_T = \Omega \times (0,T)$. We consider the following problem

$$(IBVP)\begin{cases} Mu = f(\mathbf{x},t) \in L^2(Q_T), & (\mathbf{x},t) \in Q_T, \\ u = 0, & \text{on } S_T, \\ u(\mathbf{x},0) = \phi(\mathbf{x}) \in L^2(\Omega), & \mathbf{x} \in \Omega. \end{cases}$$

Definition 5.3. *We say u is a weak solution of (IBVP) if*

- $u \in \overset{o}{W}{}_2^{1,1}(Q_T)$;
- *For every $v \in \overset{o}{W}{}_2^{1,1}(Q_T)$, we have*

$$\int_0^T \int_\Omega [u_t v + a_{ij}u_i v_j + cuv] = \int_0^T \int_\Omega fv.$$

Consider

$$(EP)\begin{cases} Lu = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

According to previous results, we know it admits a sequence of eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \longrightarrow \infty,$$

with corresponding eigenfunctions

$$e_1(\mathbf{x}) > 0, e_2(\mathbf{x}), \cdots$$

where $\|e_k\|_{L^2(\Omega)} = 1$, and $e_k \perp_{L^2 \& H_0^1} e_l$ for $k \neq l$. The inner product on $H_0^1(\Omega)$ is redefined as

$$((u, v)) = \int_\Omega a_{ij} u_i v_j + \left(c(\mathbf{x}) + \|c\|_{L^\infty(\Omega)}\right) uv.$$

Fix $t \in [0, T]$, we write $u(\mathbf{x}, t) = \sum_{k=1}^\infty c_k(t) e_k(\mathbf{x})$, and similarly $f(\mathbf{x}, t) = \sum_{k=1}^\infty d_k(t) e_k(\mathbf{x})$. Now, by the equation, we can formally write

$$\sum_{k=1}^\infty \left(c_k'(t) + \lambda_k c_k(t)\right) e_k(\mathbf{x}) = \sum_{k=1}^\infty d_k(t) e_k(\mathbf{x}),$$

which gives

$$\begin{cases} c_k'(t) + \lambda_k c_k(t) = d_k(t), \quad t \in [0, T], \\ c_k(0) = (\phi, e_k)_{L^2(\Omega)}. \end{cases}$$

Observe that $\lambda_k > \lambda_1 > 0$, $k \geq 2$, we have

$$c_k(t) = e^{-\lambda_k t} (\phi, e_k)_{L^2(\Omega)} + \int_0^t e^{-\lambda_k (t-s)} d_k(s) ds.$$

Theorem 5.1.5. Existence of Solution *Suppose $f \in L^2(Q_T)$ and $\phi \in H_0^1(\Omega)$. Then*

$$\sum_{k=1}^\infty c_k(t) e_k(\mathbf{x}) =: u$$

converges in $\overset{o}{W}{}_2^{1,1}(Q_T)$. Then u is a weak solution of (IBVP). Moreover, $u \in C^0([0,T]; H_0^1(\Omega))$, and

$$u(\cdot, t) \xrightarrow{t \to 0} \phi(\cdot)$$

in $H_0^1(\Omega)$.

证明. We will show this result in three steps.

Step 1: Because $f \in L^2(Q_T)$, we have $f(\cdot, t) \in L^2(\Omega)$ for almost every $t \in [0, T]$. By expansion, we have

$$f(\cdot, t) = \sum_{k=1}^\infty d_k(t) e_k(\mathbf{x})$$

in $L^2(\Omega)$, where $d_k(t) = (f(\cdot, t), e_k)_{L^2(\Omega)}$. To each $m \geq 1$, we set

$$f_m(\mathbf{x}, t) = \sum_{k=1}^m d_k(t) e_k(\mathbf{x}), \ (\mathbf{x}, t) \in Q_T.$$

Then

$$\|f_m - f\|^2_{L^2(Q_T)} = \int_0^T \int_\Omega |f_m - f|^2,$$

but

$$\begin{aligned}\int_\Omega |f_m(\mathbf{x},t) - f(\mathbf{x},t)|^2 d\mathbf{x} &\leq \int_\Omega 2f_m^2(\mathbf{x},t) + 2f^2(\mathbf{x},t)d\mathbf{x} \\ &\leq \int_\Omega 4f^2(\mathbf{x},t)d\mathbf{x} \in L^1(0,T),\end{aligned}$$

we then may apply L.D.C.T., and find that f_m converges to f in $L^2(Q_T)$ as $m \to \infty$.

Step 2: Since $f \in L^2(Q_T)$, $d_k \in L^2(0,T)$, we know

$$c_k(t) = e^{-\lambda_k t}(\phi, e_k)_{L^2(\Omega)} + \int_0^t e^{-\lambda_k(t-s)} d_k(s)ds$$

is absolutely continuous on $[0,T]$, and $c_k' \in L^2(0,T)$ exists in both *a.e.* $-$ *classical* & weak cases.

Define $u_m(\mathbf{x},t) = \sum_{k=1}^m c_k(t)e_k(\mathbf{x}) \in \overset{o}{W}{}_2^{1,1}(Q_T)$ the truncated version of u. Then we claim that u_m is a weak solution of

$$\begin{cases} Mu_m = f_m(\mathbf{x},t), & (\mathbf{x},t) \in Q_T, \\ u_m = 0, & \text{on } S_T, \\ u_m(\mathbf{x},0) = \phi_m(\mathbf{x}) = \sum_{k=1}^m (\phi, e_k)_{L^2(\Omega)} e_k(\mathbf{x}). \end{cases}$$

Observe that for almost every $t \in (0,T)$, $u_m(\cdot,t) \in H_0^1(\Omega)$, we have

$$\begin{aligned} Lu_m &\overset{H^{-1}}{=} \sum_{k=1}^m c_k(t)\lambda_k e_k(\mathbf{x}) \, (Le_k \overset{H^{-1}}{=} \lambda_k e_k) \\ &= \sum_{k=1}^m (d_k(t) - c_k'(t))\, e_k(\mathbf{x}) \\ &= f_m(\mathbf{x},t) - \frac{\partial u_m(\mathbf{x},t)}{\partial t}, \ \mathbf{x} \in \Omega. \end{aligned}$$

Now, applying a general $v \in \overset{o}{W}{}_2^{1,1}(Q_T)$ on both sides, we have (because for almost all t, $v \in H_0^1(\Omega)$, we have I.B.P.)

$$\int_\Omega [a_{ij}(\mathbf{x})(u_m)_i v_j + c(\mathbf{x})u_m v]\,(\mathbf{x},t)d\mathbf{x} = \int_\Omega \left(f_m(\mathbf{x},t) - \frac{\partial u_m(\mathbf{x},t)}{\partial t} \right) v(\mathbf{x},t)d\mathbf{x}.$$

Thus, an integration over $(0,T)$ gives that u_m satisfies the PDE in the weak sense. As for the initial value, we have

$$\|u_m(\cdot,t) - \phi(\cdot)\|_{L^2(\Omega)} \leq \sum_{k=1}^m \left| c_k(t) - (\phi, e_k)_{L^2(\Omega)} \right|,$$

where RHS clearly converges to 0 as $t \to 0$, which proves the claim.

Step 3: By energy estimates applied to u_m, we have

$$\max_{t\in[0,T]} \int_\Omega u_m^2(\mathbf{x},t)d\mathbf{x} + \int_{Q_T} |\nabla u_m(\mathbf{x},t)| d\mathbf{x}dt \lesssim \|f_m\|^2_{L^2(Q_T)} + \|\phi_m\|^2_{L^2(Q_T)} .$$

How about $\frac{\partial u_m}{\partial t}$? Recalling for almost every $t \in (0,T)$, we have

$$\begin{cases} Lu_m \overset{H^{-1}}{=} f_m(\mathbf{x},t) - \frac{\partial u_m(\mathbf{x},t)}{\partial t}, \\ u_m\big|_{\partial\Omega} = 0. \end{cases}$$

Applying $v = \frac{\partial u_m}{\partial t} \in H_0^1(\Omega)$, we have

$$\int_\Omega \left[a_{ij}(u_m)_i \left(\frac{\partial u_m}{\partial t}\right)_j + cu_m \frac{\partial u_m}{\partial t} \right] = \int_\Omega f_m \frac{\partial u_m}{\partial t} - \int_\Omega \left(\frac{\partial u_m}{\partial t}\right)^2 .$$

Observing that $\left(\frac{\partial u_m}{\partial t}\right)_j = \sum_{k=1}^m c_k'(e_k)_j$, we then have

$$LHS = \frac{d}{dt}\int_\Omega \frac{1}{2}(a_{ij})(u_m)_i(u_m)_j + cu_m^2,$$

which implies that

$$\frac{1}{2}\int_\Omega a_{ij}(u_m)_i(u_m)_j + cu_m^2 - \frac{1}{2}\int_\Omega a_{ij}(\phi_m)_i(\phi_m)_j + c\phi_m^2 \le \frac{1}{2}\int_0^T\int_\Omega f_m^2 - \frac{1}{2}\int_0^T\int_\Omega \left(\frac{\partial u_m}{\partial t}\right)^2 .$$

Rearranging the terms, and using strict ellipticity, we have

$$\int_0^T\int_\Omega \left(\frac{\partial u_m}{\partial t}\right)^2 + \lambda_0 \int_\Omega |\nabla u_m|^2 \le \|c\|_{L^\infty(\Omega)} \int_\Omega u_m^2(\mathbf{x},t)d\mathbf{x} + \int_0^T\int_\Omega f_m^2(\mathbf{x},t)d\mathbf{x}dt$$
$$+ C(\|a_{ij}\|_{L^\infty(\Omega)} \|c\|_{L^\infty(\Omega)}) \int_\Omega (|\nabla \phi_m|^2 + |\phi_m|^2)(\mathbf{x},t)d\mathbf{x}dt.$$

Combining this with energy estimate, we obtain

$$\|u_m\|^2_{\overset{o}{W}{}_2^{1,1}(Q_T)} \lesssim \|f_m\|_{L^2(Q_T)} + \|\phi_m\|_{H^1(\Omega)} .$$

Because L is linear, we may replace u_m by $u_m - u_l$, and f_m, ϕ_m by $f_m - f_l$, $\phi_m - \phi_l$ respectively, which shows u_m is Cauchy in $\overset{o}{W}{}_2^{1,1}(Q_T)$ and $C^0([0,T]; H_0^1(\Omega))$. Because these two spaces are Banach, we know u_m converges to some u in these spaces as $m \to \infty$. Using the estimates, one can show that u is a weak solution to the original PDE and B.C.. The initial condition is also satisfied:

$$\|u(\cdot,t) - \phi(\cdot)\|_{H_0^1(\Omega)} \le \|u(\cdot,t) - u_m(\cdot,t)\|_{H_0^1(\Omega)} + \|\phi(\cdot) - u_m(\cdot,t)\|_{H_0^1(\Omega)} + \|\phi - \phi_m\|_{H_0^1(\Omega)} , \quad \circledast$$

Using energy estimate, and that ϕ_m converges to ϕ in $H_0^1(\Omega)$, we know for any $\epsilon > 0$, there corresponds an $M > 0$ such that for any $m > M$, we have

$$\|u(\cdot,t) - u_m(\cdot,t)\|_{H_0^1(\Omega)} + \|\phi - \phi_m\|_{H_0^1(\Omega)} < \epsilon,$$

and hence

$$\limsup_{t\to 0} LHS \text{ of } \circledast \le \epsilon.$$

$\square$

Theorem 5.1.6. Existence Theorem (Full-version) *Consider (IBVP). If $f \in L^2(Q_T)$ and $\phi \in H_0^1(\Omega)$, then (IBVP) has a weak solution*

$$u \in \overset{o}{W}{}_2^{1,1}(Q_T) \cap C^0([0,T]; H_0^1(\Omega)),$$

and

$$u(\cdot,t) \xrightarrow{t\to 0} \phi(\cdot), \text{ in } H_0^1(\Omega).$$

Moreover, we have the estimate

$$\max_{t\in[0,T]} \|u(\cdot,t)\|_{H_0^1(\Omega)} + \|u\|_{W_2^{1,1}(Q_T)} \lesssim \|\phi\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q_T)}.$$

One should also notice that this theorem requires no smoothness on $\partial\Omega$.

$W_2^{2,1}(Q_T)$-regularity

Theorem 5.1.7. *Suppose $a_{ij} \in C^1(\bar{\Omega})$, $c \in L^\infty(\Omega)$,$\phi \in H_0^1(\Omega)$, $\partial\Omega \in C^2$ and L strictly elliptic on Ω. Then the weak solution $u \in W_2^{2,1}(Q_T)$, and*

$$\|u\|_{W_2^{2,1}(Q_T)} \lesssim \|\phi\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q_T)}.$$

证明. Recall $u_m(\mathbf{x},t) = \sum_{k=1}^m c_k(t)e_k(\mathbf{x})$, and $f_m(\mathbf{x},t) = \sum_{k=1}^m d_k(t)e_k(\mathbf{x})$. We may write $c_k(t) = e^{-\lambda_k t}(\phi, e_k)_{L^2(\Omega)} + \int_0^t e^{-\lambda_k(t-s)}d_k(s)ds$. Also recall that for almost every t, u_m is a weak solution of

$$\begin{cases} Lu_m(\mathbf{x},t) = f_m(\mathbf{x},t) - \frac{\partial u_m(\mathbf{x},t)}{\partial t}, & \text{on } \Omega, \\ u_m(\mathbf{x},t) = 0, & \text{on } \partial\Omega, \end{cases}$$

in the sense of $H^{-1}(\Omega)$. By elliptic H^2-estimates, we know that $u_m(\cdot,t) \in H^2(\Omega)$, and

$$\|u_m(\cdot,t)\|_{H^2(\Omega)} \lesssim \|f_m(\cdot,t)\|_{L^2(\Omega)} + \left\|\frac{u_m(\cdot,t)}{\partial t}\right\|_{L^2(\Omega)} + \|u_m(\cdot,t)\|_{L^2(\Omega)},$$

which is equivalent to

$$\|u_m(\cdot,t)\|_{H^2(\Omega)}^2 \lesssim \|f_m(\cdot,t)\|_{L^2(\Omega)}^2 + \left\|\frac{u_m(\cdot,t)}{\partial t}\right\|_{L^2(\Omega)}^2 + \|u_m(\cdot,t)\|_{L^2(\Omega)}^2,$$

and so by energy estimate

$$\begin{aligned} \int_0^T LHS dt &\lesssim \|f_m\|_{L^2(Q_T)}^2 + \left\|\frac{u_m(\cdot,t)}{\partial t}\right\|_{L^2(Q_T)}^2 + \|u_m\|_{L^2(Q_T)}^2 \\ &\lesssim \|f_m\|_{L^2(Q_T)}^2 + \|\phi_m\|_{H_0^1(Q_T)}^2. \end{aligned}$$

Therefore, we have the estimate

$$\|u_m\|_{W_2^{2,1}(Q_t)}^2 \lesssim \|\phi_m\|_{H_0^1(\Omega)}^2 + \|f_m\|_{L^2(Q_T)}^2.$$

The above estimate still holds when u_m is replaced by $u_m - u_l$ and f_m, ϕ_m by $f_m - f_l$, $\phi_m - \phi_l$ respectively. Thus u_m becomes a Cauchy sequence in $W_2^{2,1}(Q_T)$ and converges to some u_∞ in this space. According to Imbedding Theorem, we know that $u_\infty = u$. $\square$

Higher Order Regularity and Compatibility Condition: Assume $a_{ij} \in C^1(\bar{\Omega})$, $c \in L^\infty(\Omega)$, $\partial\Omega \in C^2$, and $f \in W_2^{2,1}(Q_T) \hookrightarrow C^0([0,T]; H^1(\Omega))$. We claim that $d_k \in H^1(0,T)$, and $d_k'(t) = \int_\Omega \frac{\partial f(\mathbf{x},t)}{\partial t} e_k(\mathbf{x}) d\mathbf{x}$ for almost every t. To see this, we recall that $C^\infty(\overline{Q_T})$ is dense in $W_2^{2,1}(Q_T)$ (Because the boundary is smooth), there is a sequence of functions f_m in the prior space that converges to f in $W_2^{2,1}(Q_T)$. Let

$$d_{m,k}(t) = \int_\Omega f_m(\mathbf{x},t) e_k(\mathbf{x}) d\mathbf{x},$$

we know that $d_{m,k} \in C^\infty([0,T])$, and thus $d_{m,k}'(t) = \int_\Omega \frac{\partial f_m(\mathbf{x},t)}{\partial t} e_k(\mathbf{x}) d\mathbf{x}$. We now evaluate

$$\int_0^T (d_{m,k}(t) - d_k(t))^2 dt + \int_0^T \left(d_{m,k}'(t) - \int_\Omega \frac{\partial f(\mathbf{x},t)}{\partial t} e_k(\mathbf{x}) d\mathbf{x} \right)^2 dt \leq \int_{Q_T} (f_m - f)^2 + \int_{Q_T} \left(\frac{\partial f_m}{\partial t} - \frac{\partial f}{\partial t} \right)^2,$$

and because RHS converges to 0 as $m \to \infty$, we have proven the claim.

On the other hand, we recall that

$$c_k(t) = e^{-\lambda_k t}(\phi, e_k)_{L^2(\Omega)} + \int_0^t e^{-\lambda_k(t-s)} d_k(s) ds,$$

we know that $c_k \in C^1([0,T])$, and $c_k'(t) = d_k(t) - \lambda_k c_k(t) \in AC[0,T]$. Thus $c_k'' = d_k' - \lambda_k c_k' \in L^2(0,T)$, which shows that

$$\frac{\partial u_m(\mathbf{x},t)}{\partial t} = \sum_{k=1}^m c_k'(t) e_k(\mathbf{x}) \in W_2^{2,1}(Q_T) \cap \overset{o}{W}{}_2^{1,1}(Q_T). \square$$

The above needs the fact that each $e_k \in H^2(\Omega)$. Observing also that

$$\frac{\partial f_m(\mathbf{x},t)}{\partial t} = \sum_{k=1}^m d_k'(t) e_k(\mathbf{x}) \in L^2(Q_T),$$

we may rewrite the PDE $\frac{\partial u_m}{\partial t} + Lu_m = f_m$ as

$$\sum_{k=1}^m (c_k' + \lambda_k c_k) e_k + Lu_m = f_m = \sum_{k=1}^m d_k(t) e_k(\mathbf{x}).$$

Differentiating both sides, we obtain a new equation

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial t}\right) + L\frac{\partial u_m}{\partial t} = \frac{\partial f_m}{\partial t} \in L^2(Q_T), & a.e. \text{ in } Q_T, \\ \frac{\partial u_m}{\partial t} = 0, & \text{on } S_T, \text{ because of } \square. \end{cases}$$

It is more interesting to talk about its initial condition

$$\begin{aligned} \left.\frac{\partial u_m}{\partial t}\right|_{t=0} &= \sum_{k=1}^m c_k'(0) e_k \\ &= \sum_{k=1}^m (d_k(0) - \lambda_k c_k(0)) e_k \\ &= f_m - L\phi_m \\ &= (f(\cdot,0) - L\phi)_m, \; m \geq 1. \end{aligned}$$

Led by previous results, we need to add a new condition (*Compatibility Condition*):

$$f(\cdot,t) - L\phi \in H_0^1(\Omega).$$

Therefore, we can safely apply $W_2^{2,1}$-regularity theorem, and obtain the estimate

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{W_2^{2,1}(Q_T)} \lesssim \left\|\frac{\partial f_m}{\partial t}\right\|_{L^2(Q_T)} + \|(f(\cdot,0) - L\phi)_m\|_{H_0^1(\Omega)}.$$

We claim that $\frac{\partial f_m}{\partial t}$ converges to $\frac{\partial f}{\partial t}$ in $L^2(Q_T)$ as $m \to \infty$. First observe that

$$\frac{\partial f_m}{\partial t} = \sum_{k=1}^{m} d_k'(t) e_k(\mathbf{x}) = \sum_{k=1}^{m} \int_\Omega \frac{\partial f}{\partial t}(\mathbf{x},t) e_k(\mathbf{x}) d\mathbf{x} e_k(\mathbf{x}).$$

Because $\frac{\partial f}{\partial t} \in L^2(Q_T)$, for almost all t, $\frac{\partial f}{\partial t}(\cdot,t) \in L^2(\Omega)$, we then may write in the sense of $L^2(\Omega)$

$$\frac{\partial f}{\partial t}(\mathbf{x},t) = \sum_{k=1}^{\infty} \left(\frac{\partial f}{\partial t}(\cdot,t), e_k\right)_{L^2(\Omega)} e_k(\mathbf{x}),$$

which clearly shows that for fixed proper t, $\frac{\partial f_m}{\partial t}(\cdot,t)$ converges to $\frac{\partial f}{\partial t}(\cdot,t)$ in $L^2(\Omega)$ as $m \to \infty$. But we also have the estimate

$$\int_\Omega \left|\frac{\partial f_m}{\partial t}(\mathbf{x},t) - \frac{\partial f}{\partial t}(\mathbf{x},t)\right|^2 d\mathbf{x} \leq 4\int_\Omega \left|\frac{\partial f}{\partial t}(\mathbf{x},t)\right|^2 d\mathbf{x} \in L^1(0,T),$$

and so an application of L.D.C.T. shows the convergence.

By , we know that $\frac{\partial u_m}{\partial t}$ is Cauchy in $W_2^{2,1}(Q_T) \hookrightarrow L^2(Q_T)$. It must converge to some function v_∞ in $W_2^{2,1}(Q_T)$, but we already know that u_m converges to u in $W_2^{2,1}(Q_T)$, and so $v_\infty = \frac{\partial u}{\partial t}$. Therefore, we have $\frac{\partial u}{\partial t} \in W_2^{2,1}(Q_T)$, and

$$\left\|\frac{\partial u}{\partial t}\right\|_{W_2^{2,1}(Q_T)} \lesssim \left\|\frac{\partial f}{\partial t}\right\|_{L^2(Q_T)} + \|f(\cdot,0) - L\phi\|_{H_0^1(\Omega)}.$$

To estimate the remaining terms $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u$, with $|\boldsymbol{\alpha}| = 4$, we recall for almost all t we have

$$\begin{cases} Lu_m = f_m - \frac{\partial u_m}{\partial t} \in H^2(\Omega), & a.e. \text{ on } \Omega, \\ u_m = 0, & \text{on } \partial\Omega. \end{cases}$$

Assuming $\partial\Omega \in C^4$, $a_{ij} \in C^3(\bar{\Omega})$, $c \in C^2(\bar{\Omega})$, and by higher order elliptic theory, we have

$$\|u_m(\cdot,t)\|_{H^4(\Omega)}^2 \lesssim \|f_m(\cdot,t)\|_{H^2(\Omega)}^2 + \left\|\frac{\partial u_m(\cdot,t)}{\partial t}\right\|_{H^2(\Omega)}^2 + \|u_m(\cdot,t)\|_{L^2(\Omega)}^2.$$

Integrating this inequality over $(0,T)$, we obtain

$$\int_0^T \|u_m(\cdot,t)\|_{H^4(\Omega)}^2 dt \lesssim \int_{Q_T} |f_m|^2 + \int_{Q_T} \left|\frac{\partial u_m}{\partial t}\right|^2 + \int_{Q_T} |u_m|^2,$$

and so after taking $m \to \infty$, we have

$$\int_0^T \|u(\cdot,t)\|_{H^4(\Omega)}^2 dt \lesssim \int_{Q_T} |f|^2 + \int_{Q_T} \left|\frac{\partial u}{\partial t}\right|^2 + \int_{Q_T} |u|^2.$$

Combining this estimate, and $W_2^{2,1}$-estimate of u, we know that

$$\begin{aligned}\|u\|_{W_2^{4,2}(Q_T)} &\lesssim \|f\|_{W_2^{2,1}(Q_T)} + \|L\phi - f(\cdot,0)\|_{H_0^1} + \|\phi\|_{H_0^1(\Omega)} \\ &\lesssim \|f\|_{W_2^{2,1}(Q_T)} + \|\phi\|_{H^3(\Omega)}\,,\end{aligned}$$

where $\phi \in H^3(\Omega)$ is an implicit requirement, because $L\phi \in H_0^1(\Omega)$ and $T\phi = 0$ imply that $\phi \in H^3(\Omega)$. Higher order regularity is obtained similarly by differentiating the PDE w.r.t. t formally.

Theorem 5.1.8. Higher Order Regularity (Full-version) *Let $m \geq 0$. Suppose $\partial\Omega \in C^{2m+2}$, $a_{ij} \in C^{2m+1}(\bar{\Omega})$, $c \in C^{2m}(\bar{\Omega})$, $f \in W_2^{2m,m}(Q_T)$. We further assume compatibility condition:*

$$\begin{aligned}&f(\cdot,0) - L\phi =: \phi_1 \in H_0^1(\Omega),\\ &\frac{\partial f}{\partial t} - L\phi_1 =: \phi_2 \in H_0^1(\Omega),\\ &\quad\vdots\\ &\frac{\partial^{m-1} f}{\partial^{m-1} t} - L\phi_{m-1} =: \phi_m \in H_0^1(\Omega).\end{aligned}$$

($\implies \phi \in H^{2m+1}(\Omega)$). Then, we have

$$\|u\|_{W_2^{2m+2,m+1}(Q_T)} \lesssim \|f\|_{W_2^{2m,m}(Q_T)} + \|\phi\|_{H^{2m+1}(\Omega)}\,.$$

Theorem 5.1.9. Imbedding Theorem(Ladynskaya, Uracera & Solonikov)

$$W_p^{2l,l}(Q_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T}),\ 0 < \alpha < 2l - \frac{n+2}{p},\ 1 < p < \infty,\ l \geq 1.$$

Special Case: When $0 < \alpha < 1$, then

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})} = \sup_{(\mathbf{x},t)\neq(\mathbf{y},s)} \frac{|u(\mathbf{x},t) - u(\mathbf{y},s)|}{|\mathbf{x}-\mathbf{y}|^\alpha + |t-s|^{\alpha/2}} + \max_{Q_T}|u|.$$

Corollary 5.1.1. *Suppose everything is C^∞-smooth and compatibility holds at all orders, we have the solution $u \in C^\infty(\overline{Q_T})$.*

Remark:

1. Suppose $\partial\Omega, L, f$ satisfies corresponding smoothness conditions, and u is a $W_2^{1,1}(Q_T)$ weak solution of $\frac{\partial u}{\partial t} + Lu = f$ in Q_T. Then for all $Q_T' \subset\subset$ parabolic interior of Q_T, we have $u \in W_2^{2m+2,m+1}(Q_T')$ and

$$\|u\|_{W_2^{2m+2,m+1}(Q_T')} \lesssim \|f\|_{W_2^{2m,m}(Q_T)} + \|u\|_{L^2(Q_T)}\,;$$

2. (Smoothing Effect of Parabolic Equations) Suppose $\partial\Omega, L, f$ satisfies corresponding smoothness conditions, and u is a $\overset{o}{W}{}_2^{1,1}(Q_T)$ weak solution of $\frac{\partial u}{\partial t} + Lu = f$ in Q_T and $u\big|_{S_T} = 0$. Then for any $\delta > 0$, $u \in W_2^{2m+2,m+1}(\Omega \times (\delta, T))$, and

$$\|u\|_{W_2^{2m+2,m+1}(\Omega\times(\delta,T))} \lesssim \|f\|_{W_2^{2m,m}(Q_T)} + \|u\|_{L^2(Q_T)}\,.$$

5.2 Schauder and L^p Theory

Schauder Theory

Consider $Mu = \frac{\partial u}{\partial t} - a_{ij}(\mathbf{x},t)u_{ij} + b_i(\mathbf{x},t)u_i + c(\mathbf{x},t)u$ for $\mathbf{x} \in \Omega \subset\subset \mathbb{R}^n$, $t > 0$, and

$$(DIBVP)\begin{cases} Mu = f(\mathbf{x},t), & (\mathbf{x},t) \in Q_T, \\ u = 0, & \text{on } S_T = \partial\Omega \times (0,T), \\ u(\mathbf{x},0) = \phi(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases}$$

For $P, Q \in \overline{Q_T}$, we define parabolic distance

$$d(P,Q) = \left(|\mathbf{x}-\mathbf{y}|^2 + |t-s|\right)^{1/2}, \, P = (\mathbf{x},t), \, Q = (\mathbf{y},s).$$

Define for $0 < l < 1$

$$C^{l,\frac{l}{2}}(\overline{Q_T}) = \left\{ u \in C^0(\overline{Q_T}); \sup_{P,Q\in\overline{Q_T},\, P\neq Q} \frac{|u(P)-u(Q)|}{d(P,Q)^l} =: [u]_{l,\frac{l}{2};\overline{Q_T}} < \infty \right\}.$$

Then $C^{l,\frac{l}{2}}(\overline{Q_T})$ is Banach, with norm $\|u\|_{C^{l,\frac{l}{2}}(\overline{Q_T})} = \|u\|_{L^\infty(Q_T)} + [u]_{l,\frac{l}{2};\overline{Q_T}}$. Furthermore, we define for $k \geq 0$,

$$C^{2k+l,k+\frac{l}{2}}(\overline{Q_T}) = \left\{ u \in C^{2k,k}(\overline{Q_T}); \ \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta} u \in C^{l,\frac{l}{2}}(\overline{Q_T}), \, |\boldsymbol{\alpha}| + 2\beta = 2k. \right\},$$

which is Banach with norm

$$\|u\|_{C^{2k+l,k+\frac{l}{2}}(\overline{Q_T})} = \|u\|_{C^{2k,k}(\overline{Q_T})} + \sum_{|\boldsymbol{\alpha}|+2\beta=2k} \left[\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta} u\right]_{l,\frac{l}{2};\overline{Q_T}}.$$

Theorem 5.2.1. Embedding Theorem *Suppose $\partial\Omega \in C^1$, if $2k - \frac{n+2}{p} > 0$, then*

$$W_p^{2k,k}(Q_T) \hookrightarrow C^{2k-\frac{n+2}{p},k-\frac{n+2}{2p}}(\overline{Q_T}),$$

provided $2k - \frac{n+2}{p}$ is not an integer. (If it is, then it may be replaced by any $\lambda < 2k - \frac{n+2}{p}$.)

Compatibility Condition: For $k \geq 0$, let

$$u^{(k)}(\mathbf{x}) = \left.\frac{\partial^k u(\mathbf{x},t)}{\partial t^k}\right|_{t=0},$$

then one may write it as the combination of f, ϕ and the PDE:

$$\begin{aligned} u^{(0)}(\mathbf{x}) &= \phi(\mathbf{x}) \\ u^{(1)}(\mathbf{x}) &= \frac{\partial u}{\partial t}(\mathbf{x},0) = f(\mathbf{x},0) - L\phi(\mathbf{x}) \\ u^{(2)}(\mathbf{x}) &= \frac{\partial^2 u}{\partial t^2}(\mathbf{x},0) = f_t(\mathbf{x},0) + \frac{\partial a_{ij}}{\partial t}(\mathbf{x},0)\phi_{ij} - \frac{\partial b_i}{\partial t}(\mathbf{x},0)\phi_i - \frac{\partial c}{\partial t}(\mathbf{x},0)\phi \\ &\vdots \end{aligned}$$

Theorem 5.2.2. *Let $l > 0$, $[l] = 2k$, $k \geq 0$ and l nonintegral. Suppose $\partial\Omega \in C^{2+l}$, $a_{ij}, b_i, c \in C^{l,\frac{l}{2}}(\overline{Q_T})$, $f \in C^{l,\frac{l}{2}}(\overline{Q_T})$ and $\phi \in C^{2+l}(\bar{\Omega})$. Then (DIBVP) has a unique solution $u \in C^{2+l,1+\frac{l}{2}}(\overline{Q_T})$, and*

$$\|u\|_{C^{2+l,1+\frac{l}{2}}(\overline{Q_T})} \lesssim \|f\|_{C^{l,\frac{l}{2}}(\overline{Q_T})} + \|\phi\|_{C^{2+l}(\bar{\Omega})},$$

provided the compatibility condition holds up to order $1 + [l/2]$: $u^{(j)}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$, and $j = 0, 1, \cdots, 1 + [l/2]$.

In the case of Robin/Neumann B.C.: $\beta_i(\mathbf{x}, t)u_i(\mathbf{x}, t) + \beta(\mathbf{x}, t)u(\mathbf{x}, t) = 0$ for $(\mathbf{x}, t) \in S_T$, where $(\beta_1, \cdots, \beta_n)$ is a outward pointing vector field on $\partial\Omega$. The corresponding compatibility condition becomes

$$\frac{\partial^k}{\partial t^k}\left(\beta_i(\mathbf{x}, t)u_i(\mathbf{x}, t) + \beta(\mathbf{x}, t)u(\mathbf{x}, t)\right)\bigg|_{t=0} = (\text{ in terms of } \phi) = 0, \ \forall \mathbf{x} \in \partial\Omega.$$

Theorem 5.2.3. *Assume all conditions in previous theorem, $\beta_i, \beta \in C^{l+1,\frac{l+1}{2}}(\partial\Omega \times [0, T])$. Then (R/NIBVP) has a unique solution $u \in C^{2+l,1+\frac{l}{2}}(\overline{Q_T})$, and*

$$\|u\|_{C^{2+l,1+\frac{l}{2}}(\overline{Q_T})} \lesssim \|f\|_{C^{l,\frac{l}{2}}(\overline{Q_T})} + \|\phi\|_{C^{2+l}(\bar{\Omega})},$$

provided the compatibility condition holds up to order $[(l + 1)/2]$.

Theorem 5.2.4. Schauder Interior Estimate *Let $0 < l < 1$, $q_{ij}, b_i, c, f \in C^{l,\frac{l}{2}}(\overline{Q_T})$. If $u \in C^{2+l,1+\frac{l}{2}}(\overline{Q_T})$ is a solution of $Mu = f(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \overline{Q_T}$. Then for $Q' \subset Q_T$ closed in $\mathbb{R}^n$, and $Q' \cap$ parabolic boundary $\Gamma = \emptyset$. Then,*

$$\|u\|_{C^{2+l,1+\frac{l}{2}}(Q')} \lesssim_{n,l,Q_T,Q',L} \|f\|_{C^{l,\frac{l}{2}}(\overline{Q_T})} + \|u\|_{L^\infty(Q_T)}.$$

Theorem 5.2.5. Boundary Estimate *Assume all conditions in previous result, and more: $\partial\Omega \in C^{2+l}$, $u \in C^{2+l,1+\frac{l}{2}}(\overline{Q_T})$ satisfies*

$$\begin{cases} PDE & \text{in } \overline{Q_T}, \\ u = 0 & \text{on } S_T, \\ \text{No I.C.} \end{cases}$$

Then, for all $\epsilon > 0$, we have

$$\|u\|_{C^{2+l,1+\frac{l}{2}}(\bar{\Omega}\times[\epsilon,T])} \lesssim_{n,l,Q_T,\epsilon,L} \|f\|_{C^{l,\frac{l}{2}}(\overline{Q_T})} + \|u\|_{L^\infty(Q_T)}.$$

The boundary condition can also be replaced by R/N without harming the result.

L^p-theory

Let $1 < p < \infty$, $a_{ij} \in C^0(\overline{Q_T})$, $b_i, c \in L^\infty(Q_T)$, $\partial\Omega \in C^2$. We consider (DIBVP) with $\phi \in W^{1,p}(\Omega)$.

Definition 5.4. *We say u is a strong solution of (DIBVP) if*

- $u \in W_p^{2,1}(Q_T)$ *and satisfies the PDE a.e. in* Q_T*;*
- $u \in W_p^{2,1}(Q_T) \subset W_p^{1,1}(Q_T) = W^{1,p}(Q_T)$*, and* $u\big|_{S_T} = 0$ *in the sense of trace;*
- $(u-\phi)\big|_{\Omega\times\{0\}} = 0$ *in the sense of trace.*

Theorem 5.2.6. Existence & Uniqueness *Suppose* $1 < p < \infty$, $p \neq 3/2$, *then for all* $f \in L^p(Q_T)$, $\phi \in W^{2,p}(\Omega)$ *satisfying for the case of* $p > 3/2$ *the compatibility condition* $\phi\big|_{\partial\Omega} = 0$. *Then, (DIBVP) has a unique solution* $u \in W_p^{2,1}(Q_T)$, *and*

$$\|u\|_{W_p^{2,1}(Q_T)} \lesssim \|f\|_{L^p(Q_T)} + \|\phi\|_{W^{2,p}(\Omega)}.$$

When $p = 3/2$, *see Ladynzenskaya, P342.*

Theorem 5.2.7. Interior Estimate *Let* $1 < p < \infty$, *suppose* $u \in W_p^{2,1}(Q_T)$ *is a strong solution of PDE. Then for all* Q' *closed in* Q_T, $Q_T \cap \Gamma = \emptyset$. *Then*

$$\|u\|_{W_p^{2,1}(Q')} \lesssim_{n,p,L,Q',Q_T} \|f\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)}.$$

Theorem 5.2.8. Boundary Estimate *Let* $\partial\Omega \in C^2$, $u \in W_p^{2,1}(Q_T)$, *satisfying PDE and Dirichlet B.C., then*

$$\|u\|_{W_p^{2,1}(\bar{\Omega}\times[\epsilon,T])} \lesssim_{n,p,L,\epsilon,Q_T} \|f\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)}.$$

For R/N B.C., see Garoni & Solonikov, Communication in PDE, Vol 9. 1323-1372(1984).

Theorem 5.2.9. R/NIBVP *Let* $a_{ij} \in C^0(\overline{Q_T})$, $b_i, c \in L^\infty(Q_T)$, $\beta_i, \beta \in C^1$, $\partial\Omega \in C^2$, $1 < p < \infty$, $p \neq 3$. *Then for all* $f \in L^p(Q_T)$, $\phi \in W^{2,p}(\Omega)$ *satisfying compatibility condition for the case* $p > 3$

$$\beta_i(\mathbf{x},0)\phi_i(\mathbf{x}) + \beta(\mathbf{x},0)\phi(\mathbf{x}) = 0, \quad \text{on } \partial\Omega.$$

Then (R/NIBVP) has a unique solution $u \in W_p^{2,1}(Q_T)$, *and*

$$\|u\|_{W_p^{2,1}(Q_T)} \lesssim_{n,p,L,Q_T,\beta_i,\beta} \|f\|_{L^p(Q_T)} + \|\phi\|_{W^{2,p}(\Omega)}.$$

5.3 Existence and Uniqueness of Nonlinear Heat Equation

Let Ω be bounded, $\partial\Omega \in C^{2+\alpha}$, $0 < \alpha < 1$. We consider the following problem

$$(DIBVP)\begin{cases} u_t - k\Delta u = f(\mathbf{x},t,u), & \mathbf{x} \in \Omega, t > 0,\\ u = 0, & \text{on } S_T,\\ u(\mathbf{x},0) = \phi(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases}$$

Assume f satisfies:

- For each fixed $u \in \mathbb{R}$, $f^u(\mathbf{x},t)$ is measurable;

- For any bounded $B \subset \bar{\Omega} \times [0, \infty) \times \mathbb{R}$, there is a constant $M(B) > 0$ such that

$$|f(\mathbf{x}, t, u)| \leq M(B), \, \forall (\mathbf{x}, t, u) \in B,$$

  and

$$|f(\mathbf{x}, t, u) - f(\mathbf{x}, t, v)| \leq M(B)|u - v|, \, (\mathbf{x}, t, u), \, (\mathbf{x}, t, v) \in B.$$

Theorem 5.3.1. *For any $\phi \in L^\infty(\Omega)$, there is $T_\phi \in (0, \infty]$ such that (DIBVP) has a unique solution u, satisfying*

- *For small $\epsilon > 0$, $u \in W_p^{2,1}(\Omega \times (\epsilon, T_\phi - \epsilon))$, for $1 < p < \infty$, and the PDE holds a.e. on $\Omega \times (0, T_\phi)$;*
- *The map $t \in [0, T_\phi - \epsilon] \mapsto u(\cdot, t) \in L^\infty(\Omega)$ is bounded;*
- *If $T_\phi < \infty$, then*

$$\lim_{t \to T_\phi^-} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty;$$

- *B.C. holds in the classical sense if $t > 0$;*
- *I.C. holds in teh sense*

$$\lim_{t \to 0^+} \|u(\cdot, t) - \phi(\cdot)\|_{L^p(\Omega)} = 0, \, 1 < p < \infty.$$

If Lipschitz condition is strengthened as

$$|f(\mathbf{x}, t, u) - f(\mathbf{y}, s, v)| \leq M(B)(|\mathbf{x} - \mathbf{y}|^\alpha + |t - s|^{\alpha/2} + |u - v|),$$

then

$$u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [\epsilon, T_\phi - \epsilon]).$$

Furthermore, if $\phi \in C^{2+\alpha}(\bar{\Omega})$ and $\phi\big|_{\partial\Omega} = 0$, $k\Delta\phi(\mathbf{x}) + f(\mathbf{x}, 0, \phi(\mathbf{x})) = 0$ for $\mathbf{x} \in \partial\Omega$,then $u \in C^{2+\alpha}(\bar{\Omega} \times [0, T_\phi - \epsilon])$.

Remark:

1. If $f \in C^\infty(\bar{\Omega} \times (0, \infty) \times \mathbb{R})$, $\partial\Omega \in C^\infty$, then $u \in C^\infty(\bar{\Omega} \times (0, T_\phi))$;
2. If $f(\mathbf{x}, t, 0) = 0$, then $\phi \geq 0$ implies $u \geq 0$. See "Global Solutions of Reaction-Diffusion Systems" by Rothe;
3. B.C. can be replaced by

$$k\frac{\partial u}{\partial \gamma} + \beta(\mathbf{x})u = 0$$

   on $\partial\Omega$ for $t > 0$, and $\beta \geq 0$, $\beta \in C^{1+\alpha}(\partial\Omega)$, γ unit outer normal of $\partial\Omega$. Then the corresponding compatibility condition is

$$k\frac{\partial \phi}{\partial \gamma} + \beta(\mathbf{x})\phi = 0$$

   on $\partial\Omega$.

Example:

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u, & \mathbf{x} \in \Omega, t > 0, 1 < p < \infty, \\ u(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, t > 0, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}), & \phi(\mathbf{x}) \geq 0, \phi \not\equiv 0, \mathbf{x} \in \Omega, \end{cases} \tag{5.3.1}$$

where $\partial\Omega \in C^{2+\alpha}$, $\phi \in L^\infty(\Omega)$. By previous theorem, there is a $T_\phi > 0$ such that (5.3.1) has a unique nonnegative solution $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [\epsilon, T_\phi - \epsilon])$. Is $T_\phi < \infty$? The answer is "Yes", when ϕ is very large.

(**Kaplan's Eigenfunction Method**) Let $e_1(\mathbf{x}) > 0$ be the principal eigenfunction, and apply this to the equation, we have

$$\frac{d}{dt}\int_\Omega u(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x} - \int_\Omega \Delta u(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x} = \int_\Omega u^p(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x},$$

and then by I.B.P.

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x} + \lambda_1 \int_\Omega u(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x} &= \int_\Omega u^p(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x} \\ &\geq \left(\int_\Omega u(\mathbf{x}, t)d\mu(\mathbf{x})\right)^p \\ &= \left(\int_\Omega u(\mathbf{x}, t)e_1(\mathbf{x})d\mathbf{x}\right)^p. \end{aligned}$$

Thus, we obtain an ordinary differential inequality

$$\begin{cases} h'(t) + \lambda_1 h(t) \geq h^p(t), & t > 0, \\ h(0, t) = \int_\Omega \phi(\mathbf{x})e_1(\mathbf{x})d\mathbf{x}. \end{cases}$$

Observe that if $\left(\int_\Omega \phi(\mathbf{x})e_1(\mathbf{x})d\mathbf{x}\right)^p - \lambda_1\left(\int_\Omega \phi(\mathbf{x})e_1(\mathbf{x})d\mathbf{x}\right) > 0$, then $h'(t) > 0$ for $t > 0$. (Equivalently it requires that $\int_\Omega \phi e_1 > \lambda^{\frac{1}{p-1}}$). But

$$\frac{h'(t)}{h^p(t) - \lambda_1 h(t)} \geq 1,$$

and hence if $T_\phi = \infty$, we have

$$\int_0^t \frac{h'(t)}{h^p(t) - \lambda_1 h(t)} dt \geq t,$$

and

$$LHS = \int_{h(0^+)}^{h(t)} \frac{dh}{h^p - \lambda_1 h} \geq t.$$

Observe that LHS is bounded, and hence we are done.

Chapter 6

Conservation Law

Let $u(\mathbf{x}, t)$ be the density of substance at $\mathbf{x}$ and t, and $\vec{f}(\mathbf{x}, t, u)$ the flux of substance. The equation we concern about is

$$u_t + \nabla \cdot \vec{f} = 0.$$

Special Case: $x \in \mathbb{R}$, $f = f(u)$ smooth and the equation becomes

$$\begin{cases} u_t + (f(u))_x = 0, & \text{on } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

(**Method of Characteristics**) Let

$$C : \frac{dX}{dt} = f'(u(x,t))$$

be the curve of characteristics, then by chain rule

$$\frac{du(X(t),t)}{dt} = u_t + u_x \frac{dX}{dt} = 0,$$

which implies that u is a constant on C. Then $\frac{dX}{dt} = f'(M)$, and $X = f'(M)t + s$. Recall that $u(x,t) = g(s)$ for $(x,t) \in C$, then u is given implicitly by

$$u = g(X - f'(u)t).$$

Example: When $f(u) = \frac{u^2}{2}$, the equation becomes

$$\begin{cases} u_t + u_x u = 0, \\ u(x,0) = g(x). \end{cases}$$

The curve of characteristics is $X = g(s)t + s$, and $u(x,t) = g(s)$.

Pathological Phenomenon: If there are $s_1 < s_2$ such that $g(s_1) > g(s_2)$, then it is impossible to have classical solutions existing for all $t > 0$. Something must blow up in finite time. Suppose $u = g(x - ut)$, we have

$$u_x(x,t) = g'(x - ut)(1 - u_x(x,t)t),$$

which implies

$$u_x(x,t) = \frac{g'(x-ut)}{1+tg'(x-ut)} = \frac{g'(s)}{1+tg'(s)}.$$

Because $g(s_1) > g(s_2)$, then there should be some $s \in (s_1, s_2)$ such that $g'(s) < 0$. Then, as $t \to^+ -\frac{1}{g'(s)}$, we have

$$u_x(X(t),t) \longrightarrow -\infty,$$

which is also called Steepening Phenomenon, or Shock Wave.

Consider the following problem

$$(IVP)\begin{cases} u_t + (f(u))_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x). \end{cases}$$

Theorem 6.0.1. *Let $f''(u) > 0$ be continuous, and g is increasing in x, g is bounded and C^1-smooth. Then (IVP) has a C^1-smooth solution in $\mathbb{R} \times (0,\infty)$.*

证明. Consider characteristic curve

$$\begin{cases} \frac{dX}{dt} = f'(g(s)), \\ X(0) = s, \end{cases}$$

we have $X = f'(g(s))t + s$. The idea is that for any (x,t), $t > 0$, we find a unique $s \in \mathbb{R}$ such that there is a C.C. C_s passes through it. Define $h(s) = f'(g(s))t_0 + s$ and because g is bounded and f' continuous, $f'(g(s))$ is bounded on $\mathbb{R}$, which gives that $\lim_{s\to\pm\infty} h(s) = \pm\infty$. Obviously h is continuous in s. By I.V.T., there should be some $s_0 \in \mathbb{R}$ such that $h(s_0) = x$. Because $h(s)$ is strictly increasing, such s_0 is unique. Thus $u(x_0,t_0) = g(s_0)$ is well-defined on the upper half plane. Define $u(x_0,0) = g(x_0)$.

To show $u(x,t) \in C^1$, we check

$$f'(g(s))t + s - x =: F(x,t,s) = 0,$$

and observe

- $F \in C^1$;
- $F(x_0,t_0,s_0) = 0$;
- $F_s(x_0,t_0,s_0) = f''(g(s))t_0 g'(s) + 1 \neq 0$.

By Implicit Function Theorem, there should be some neighborhood $N_r(x_0,t_0)$ and function $s = s(x,t)$, for $(x, t \in N_r)$ such that $s \in C^1(N_r)$, $F(x,t,s(x,t)) = 0$ on N_r and hence

$$s_t = -f'(g(s(x,t)))s_x.$$

Moreover, if $F(x,t,s) = 0$ with $(x,t) \in N_r$ and $s \approx s_0$, then $s = s(x,t)$, and so $u(x,t) = g(s(x,t))$ for all (x,t) with $t > 0$. The PDE is automatically satisfied by using the above equalities, and the boundary condition is fulfilled because

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} g(s(x,t)) = g(x).$$

□

Theorem 6.0.2. *If all the other conditions are satisfied, but g is not increasing in x, then (IVP) has no classical solution for all $t > 0$.*

Moral of Story: Have to give up the hope for classical solution.

Weak Solutions

Discussion: Suppose $u \in C^1(\mathbb{R} \times \overline{\mathbb{R}_+})$ is a solution, then for all $\phi \in C_0^\infty(\mathbb{R} \times \overline{\mathbb{R}_+})$, we have

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} \int_0^{\infty} (u_t + (f(u))_x)\, \phi dt dx \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} (u\phi)_t + (f(u)\phi)_x dt dx - \int_{-\infty}^{\infty} \int_0^{\infty} u\phi_t + f(u)\phi_x \\
&= \int_{\partial \mathbb{R}^2_+} (f(u)\phi, u\phi) \cdot \vec{n} ds - \int_{-\infty}^{\infty} \int_0^{\infty} u\phi_t + f(u)\phi_x \\
&= -\int_{-\infty}^{\infty} g(x)\phi(x,0)dx - \int_{-\infty}^{\infty} \int_0^{\infty} u\phi_t + f(u)\phi_x.
\end{aligned}$$

Definition 6.1. *If $u, f(u) \in L^1_{loc}(\mathbb{R}^2_+)$ and the above equality holds for any $\phi \in C_0^1(\mathbb{R} \times \overline{\mathbb{R}_+})$. Then we say u is a weak solution of (IVP).*

An important necessary condition for weak solution of the PDE: *Rankine-Hugoniat* condition: Let $C : X(t)$ be a C^1-smooth curve, and u a classical solution on each side of C. $u_l(X(t), t)$ be the limit of u on left hand side of C on the curve, and $u_r(X(t), t)$ the right hand side. Then, if u is a weak solution of PDE on Ω, we must have by Physics

$$\frac{d}{dt} \int_a^b u(x,t)dx + f(u(b,t)) - f(u(a,t)) = 0,$$

while

$$\begin{aligned}
LHS &= \frac{d}{dt} \left[\int_a^{X(t)} + \int_{X(t)}^b \right] + f(u(b,t)) - f(u(a,t)) \\
&= u_l(X(t),t)X'(t) + \int_a^{X(t)} u_t(x,t)dx - u_r(X(t),t)X'(t) + \int_{X(t)}^b u_t(x,t)dx + f(u(b,t)) - f(u(a,t)) \\
&= X'(t)(u_l - u_r) - \int_a^{X(t)} (f(u))_x dx - \int_{X(t)}^b (f(u))_x dx + f(u(b,t)) - f(u(a,t)) \\
&= X'(t)(u_l - u_r) - f(u_l) + f(u_r) \\
&=: X'(t)[u] - [f(u)]
\end{aligned}$$

which gives that $X'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r}$. "Soul" Proof.

Rigorous proof: Because u is a weak solution of PDE, then for all $\phi \in C_0^1(\Omega)$, we have

$$\int_\Omega (u\phi_t + f(u)\phi_x) dx dt = 0,$$

with Ω a small neighborhood of the curve. Let Ω_l and Ω_r denote the two subdomains that are split by the curve C. Then, using Divergence Theorem, we have

$$\int_{\Omega_l} (\cdots) = \int_C (f(u_l)\phi, u_l\phi) \cdot \frac{(1, -X'(t))}{\sqrt{1 + (X'(t))^2}} ds,$$

and similarly, we have

$$\int_{\Omega_r} (\cdots) = -\int_C (f(u_r)\phi, u_r\phi) \cdot \frac{((1, -X'(t))}{\sqrt{1+(X'(t))^2}} ds,$$

and therefore

$$0 = \int_{t_1}^{t_2} \phi(X(t), t)[f(u)] - X'(t)[u] dt.$$

Remark: If u is C^1 solution of PDE on each side of C and R-H condition is satisfied, then u is a weak solution of the PDE.

Example:

1. Consider the Burger's Equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right) = 0, \\ u(x, 0) = \begin{cases} 1, & x < 0, \\ 0, & x0. \end{cases} \end{cases}.$$

   Then R-H demands that

$$s'(t) = \frac{[f(u)]}{[u]} = \frac{\frac{1}{2} - 0}{1 - 0} = 1/2,$$

   and $s(0)$. Thus $s(t) = t/2$, and $1/2$ is called the *shock wave speed*;

2. Consider the above PDE with

$$g(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

   Then we define

$$u(x, t) = \begin{cases} 0, & x < 0, \\ \frac{x}{t}, & 0 < \frac{x}{t} < 1, \\ 1, & \frac{x}{t} \geq 1. \end{cases}$$

   Then u is a weak solution. There is another weak solution defined using R-H condition at the curve $s'(t) = 1/2$, but it is not Physical. We will talk about that later;

3. Consider the equation with $f(u) = u^2$, and zero initial condition. Set $a < 0 < b$ arbitrary constants. Then we may split the upper half plane into 4 components by three rays: $X = \frac{bt}{2}$, $X = \frac{a+bt}{2}$ and $X = \frac{at}{2}$. Assigning the values $0, b, a, 0$ to each component counterclockwise, we obtain infinitely many weak solutions.

Big Question: Criterion for "Correct Solutions"? The answer is viscosity/entropy solution. We consider

$$\begin{cases} u_t + (f(u))_x = \epsilon u_{xx}, \\ u(x, 0) = g(x), \end{cases}$$

where $\epsilon > 0$ is called the viscosity coefficient. If $\lim_{\epsilon\to 0} u_\epsilon$ exists, then it will be called the viscosity solution of (IVP).

There is an equivalent way of defining entropy solution, which is through entropy condition:

$$u(x+z,t) - u(x,t) \leq \frac{Cz}{t},$$

for all $x, z > 0 \in \mathbb{R}$, $t > 0$ and $C > 0$ some constant. This equivalent to that $u(x,t) - \frac{Cx}{t}$ decreasing in x for any fixed $t > 0$. Therefore, for each fixed $t > 0$, u has at most countably many discontinuities, and at each of which, u jumps down, that is, Example 3. is not an entropy solution for all $a < 0 < b$.

Theorem 6.0.3. *Suppose f is uniformly convex $f'' \geq \theta > 0$. If $g \in L^\infty(\mathbb{R})$, then*

$$(IVP)\begin{cases} u_t + (f(u))_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x), \end{cases}$$

has one and only one bounded weak solution satisfying entropy solution, and

$$\|u\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq \|g\|_{L^\infty(\mathbb{R})}.$$

Existence: Oleinik: AMS translate series 2, 26. 95-173,33; 285-290. If f is not uniformly convex, or if (IVP) is a system, see "Vanishing Viscosity Systems" by Bianchini & Bressan, Annals of Math, 2005.

Riemann Problem: Always assume $f'' \geq \theta > 0$. We consider

$$\begin{cases} u_t + (f(u))_x = 0, \\ u(x,0) = g(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \end{cases}$$

where u_r, u_l are constants. In the case $u_r = u_l$, the entropy solution is exactly $u = u_l = u_r$. When $u_l > u_r$, we then have $f'(u_l) > f'(u_r)$. We consider the curve $s'(t) = \frac{f(u_l)-f(u_r)}{u_l-u_r} =: \sigma$, because f is concave up, then we have

$$f'(u_r) < s'(t) < f'(u_l),$$

and

$$u(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} < \sigma, \\ u_r, & \text{if } \frac{x}{t} \geq \sigma. \end{cases}$$

Because for each fixed $t > 0$, u jumps down, then it's an entropy solution. Here we would like to call $s(t)$ shock wave, and $s'(t)$ the shock speed. In the case $u_l < u_r$, we have $f'(u_l) < f'(u_r)$, and so

$$\frac{dX}{dt} = f'(g(0)) = f'(c), c \in (u_l, u_r).$$

But we also have $x = tf'(c)$, which gives $c = (f')^{-1}\left(\frac{x}{t}\right)$. Thus, we obtain a formula

$$u(x,t) = g(0) = c = (f')^{-1}\left(\frac{x}{t}\right), \ (x,t) \text{ in the conic region.}$$

We claim that u is a classical solution to the PDE in the conic region: Let $G = (f')^{-1}$, then in the region, we have

$$u_t + (f(u))_x = G'\left(\frac{x}{t}\right)\left(\frac{-x}{t^2}\right) + \frac{1}{t}f'\left(G\left(\frac{x}{t}\right)\right)G'\left(\frac{x}{t}\right) = 0.$$

We also claim that u is an entropy solution: Notice that

$$u(x,t) = \begin{cases} u_l, & \frac{x}{t} < f'(u_l), \\ G\left(\frac{x}{t}\right), & f'(u_l) < \frac{x}{t} < f'(u_r), \\ u_r, & f'(u_r) < \frac{x}{t}. \end{cases}$$

Define

$$\tilde{G}(v) = \begin{cases} u_l, & v < f'(u_l), \\ G\left(\frac{x}{t}\right), & f'(u_l) < v < f'(u_r), \\ u_r, & f'(u_r) < v. \end{cases}$$

Then $G(f'(u)) = u$ implies that $G'(f'(u))f''(u) = 1$, and because $f'' \geq \theta > 0$, we have

$$0 < G'(v) = \frac{1}{f''(u)} \leq \frac{1}{\theta},$$

which shows that $\tilde{G}$ is globally Lipschistz on $\mathbb{R}$. Observe $u(x,t) = \tilde{G}\left(\frac{x}{t}\right)$, and then for $z > 0$ and $x \in \mathbb{R}$, $t > 0$

$$\frac{u(x+z) - u(x,t)}{z} = \frac{\tilde{G}\left(\frac{x+z}{t}\right) - \tilde{G}\left(\frac{x}{t}\right)}{z} \leq \frac{L\frac{z}{t}}{z} = \frac{L}{t}.$$

Vanishing Viscosity Method

Consider

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in R, \\ u_0 \in L^\infty(\mathbb{R}). \end{cases}$$

The goal is to find a Physically meaningful solution. The strategy is to replace the equation by $u_t + \left(\frac{u^2}{2}\right) = \mu u_{xx}$, for some small μ. The constant μ is called viscosity coefficient, and we will show that

$$\lim_{\mu \to 0} u_\mu(x,t)$$

exists and is exactly an entropy solution. (The work is done by Hopf.)

Discussion: We rewrite the equation as

$$\begin{aligned} u_t &= \left(\mu u_x - \frac{u}{2}u\right)_x \\ &= \mu\left(u_x - \frac{u}{2\mu}u\right)_x \\ &= \mu\left(e^{\int_0^x \frac{u(y)}{2\mu}dy}\left(e^{-\int_0^x \frac{u(y)}{2\mu}dy}\right)_x\right)_x. \end{aligned}$$

Integrating both sides w.r.t. x, and we have

$$\frac{d}{dt}\int_0^x u(y,t)dy = \mu e^{\int_0^x \frac{u(y)}{2\mu}dy}\left(e^{-\int_0^x \frac{u(y)}{2\mu}dy}\right)_x + c(t),$$

which further gives

$$\frac{d}{dt}\left(e^{-\int_0^x \frac{u(y,t)}{2\mu}dy}\right)\cdot(-2\mu) = \mu\left(e^{-\int_0^x \frac{u(y)}{2\mu}dy}\right)_x + c(t)e^{-\int_0^x \frac{u(y,t)}{2\mu}dy}.$$

Letting (this is called "Hopf transform")

$$\phi(x,t) = e^{-\int_0^x \frac{u(y,t)}{2\mu}dy},$$

we then have

$$-2\mu\phi_t = -2\mu^2\phi_{xx} + c(t)\phi,$$

which means

$$\phi_t = \mu\phi_{xx} + c(t)\phi.$$

Therefore, we have

$$\left(e^{-\int_0^t c(\tau)d\tau}\phi\right)_t = \mu\left(e^{-\int_0^t c(\tau)d\tau}\phi\right)_{xx}.$$

Letting $\psi = e^{-\int_0^t c(\tau)d\tau}\phi$, we know that ψ satisfies

$$\begin{cases} \psi_t = \mu\psi_{xx}, \\ \psi(x,0) = e^{\int_0^x -\frac{u_0(y)}{2\mu}dy} = e^{O(|x|)}. \end{cases}$$

Thus, we have

$$\psi(x,t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\mu t}}\psi(y,0)dy,$$

and

$$\begin{aligned} u(x,t) &= -2\mu\frac{\phi_x}{\phi} \\ &= -2\mu\frac{\psi_x}{\psi} \\ &= \frac{\int_{-\infty}^{\infty}\frac{x-y}{t}\exp\left[\frac{-(x-y)^2}{4\mu t} - \frac{1}{2\mu}\int_0^y u_0(\xi)d\xi\right]dy}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\mu t}}\psi(y,0)dy} \\ &= \frac{\int_{-\infty}^{\infty}\frac{x-y}{t}\exp\left(\frac{-1}{4\mu t}\left[(x-y)^2 + 2t\int_0^y u_0(\xi)d\xi\right]\right)dy}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\mu t}}\psi(y,0)dy}. \end{aligned}$$

We define $F(x,y,t) = (x-y)^2 + 2t\int_0^y u_0(\xi)d\xi$. Fix $(x,t)\in\mathbb{R}\times\mathbb{R}_+$, we know that

$$\frac{F(x,y,t)}{y^2}\longrightarrow 1,$$

as $|y|\to\infty$, then there is a global minimum point $\bar{y}(x,t)$ of F. Therefore, we may write

$$u(x,t) = \frac{\int_{-\infty}^{\infty}\frac{x-y}{t}\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)dy}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2 - F(x,\bar{y},t)}{4\mu t}}\psi(y,0)dy}.$$

If $\bar{y}(x,t)$ is unique, denote

$$G_\mu(y) = \frac{\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2 - F(x,\bar{y},t)}{4\mu t}}\psi(y,0)dy},$$

and we claim that $G_\mu(y)$ converges to $\delta(y-\bar{y}(x,t))$. To see why, we first check that its integral is 1 (this is easy to see), and for every $\delta > 0$, $\int_{|y-\bar{y}|} G_\mu(y)dy$ converges to 0 as $\mu \to 0$. To see the latter one, we first claim $F(x,y,t) - F(x,\bar{y}(x,t),t) \geq C_\delta(x,t)(y-\bar{y})^2$ for all $|y-\bar{y}| \geq \delta$ (Exercise). Then we have

$$\begin{aligned}
\int_{|y-\bar{y}|\geq\delta} G_\mu(y)dy &= \frac{\int_{|y-\bar{y}|\geq\delta}\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)dy}{\int_{-\infty}^{\infty}\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)dy} \\
&\leq \frac{\int_{|y-\bar{y}|\geq\delta} e^{-\frac{C(x,t)}{4\mu t}(y-\bar{y})^2}dy}{\int_{-\infty}^{\infty}\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)dy} \\
&= \frac{\int_{|z|\geq\delta\sqrt{\frac{C(x,t)}{4\mu t}}} e^{-z^2}dz}{\int_{-\infty}^{\infty}\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)dy}\cdot\sqrt{\frac{4t\mu}{C(x,t)}}.
\end{aligned}$$

In the meantime, the bottom is greater than

$$\int_{|y-\bar{y}|<\epsilon}\exp\left(\frac{-1}{4\mu t}\left[F(x,y,t) - F(x,\bar{y},t)\right]\right)dy,$$

and when $\epsilon > 0$ is small, we have $F(x,y,t) - F(x,\bar{y},t) \leq \delta^2 C(x,t)$, which makes the above quantity greater than

$$\exp\left[\frac{-\delta^2 C(x,t)}{4\mu t}\right]\cdot 2\epsilon.$$

Now, the original term is less than

$$\begin{aligned}
\frac{2\sqrt{\frac{4\mu t}{C(x,t)}}\int_{\delta\sqrt{\frac{C(x,t)}{4\mu t}}}^{\infty} e^{-z^2}dz}{\exp\left[\frac{-\delta^2 C(x,t)}{4\mu t}\right]\cdot 2\epsilon} &\leq \frac{2\sqrt{\frac{4\mu t}{C(x,t)}}\int_{\delta\sqrt{\frac{C(x,t)}{4\mu t}}}^{\infty} ze^{-z^2}dz}{\exp\left[\frac{-\delta^2 C(x,t)}{4\mu t}\right]\cdot 2\epsilon} \\
&\leq 2\sqrt{\frac{4\mu t}{C(x,t)}}\cdot\frac{1}{4\epsilon} \\
&\longrightarrow 0,
\end{aligned}$$

as $\mu \to 0$. Thus, we have

$$\lim_{\mu\to 0} u(x,t) = \lim_{\mu\to 0}\int_{-\infty}^{\infty}\frac{x-y}{t}G_\mu(y)dy = \frac{x-\bar{y}(x,t)}{t}.$$

Question: What if $\bar{y}(x,t)$ not unique? Let $y^*(x,t)$ be the largest, and $y_*(x,t)$ the smallest.

We then have

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \frac{x-y}{t} G_\mu(y)dy \\
&= \int_{-\infty}^{\infty} \left(\frac{x-y_*}{t} + \frac{y_*-y}{t} \right) G_\mu(y)dy \\
&= \frac{x-y_*}{t} + \int_{-\infty}^{\infty} \frac{y_*-y}{t} G_\mu(y)dy \\
&\leq \frac{x-y_*}{t} + \int_{-\infty}^{y_*} \frac{y_*-y}{t} G_\mu(y)dy,
\end{aligned}$$

but $G_\mu\big|_{(-\infty,y_*)}(y)$ converges to $\delta(y-y_*)$, and thus, we have

$$\limsup_{\mu\to 0} u(x,t) \leq \frac{x-y_*}{t} + 0.$$

Similarly, we have

$$\liminf_{\mu\to 0} u(x,t) \geq \frac{x-y^*}{t} + 0.$$

Now, we are going to show several facts about the limit.

Proposition 6.0.1. Fact 1. *For $x < x'$, we have*

$$y_*(x,t) \leq y^*(x,t) \leq y_*(x',t) \leq y^*(x',t).$$

证明. Observe that

$$\begin{aligned}
F(x',y,t) - F(x',y^*(x,t),t) &= (x'-y)^2 + 2t\int_0^y u_0(\xi)d\xi - (x'-y^*(x,t))^2 - 2t\int_0^{y^*(x,t)} u_0(\xi)d\xi \\
&= y^2 - (y^*)^2 - 2x'(y-y^*) + 2t\left(\int_0^y u_0(\xi)d\xi - \int_0^{y^*(x,t)} u_0(\xi)d\xi \right) \\
&= F(x,y,t) - F(x,y^*(x,t),t) - 2(x'-x)(y-y^*(x,t)) \\
&\geq 0.
\end{aligned}$$

□

Fact 1. shows that both y^* and y_* are increasing in x and hence have at most countably jump discontinuities for each $t > 0$. Moreover if y^* (y_*) is continuous at x_0, then so is y_* (y^*), and $y_*(x_0,t) = y^*(x_0,t)$. Thus for at most countably many points, $y^* = y_*$ for each fixed $t > 0$.

Thus, we may define vanishing viscosity solution as

$$v(x,t) = \begin{cases} \frac{x-y_*}{t}, & \text{if } y_* \text{ is continuous at } x, \\ \frac{x-y_*(x-0,t)}{t}, & \text{otherwise.} \end{cases}$$

Then, we see that v satisfies entropy condition:

$$\frac{v(x+z,t)-v(x,t)}{z} = \frac{z - y_*(x+z-0,t) + y_*(x-0,t)}{tz} \leq \frac{1}{t}.$$

Proposition 6.0.2. Fact 2. *We have the bound*

$$\frac{|x-y_*|}{t}, \frac{|x-y^*|}{t} \le \|u_0\|_{L^\infty(\mathbb{R})}.$$

In particular, both y_* and y^* converges to x uniformly as $t \to 0$.

证明. Let $\bar{y}(x,t) = y_*$ or y^*. Then we have

$$(x-\bar{y}(x,t))^2 + 2t\int_0^{\bar{y}(x,t)} u_0(\xi)d\xi \le (x-y)^2 + 2t\int_0^y u_0(\xi)d\xi,\ \forall y \in \mathbb{R}.$$

For $\alpha \in (0,1)$, we take y such that $x - y = \alpha(x-\bar{y})$, then

$$(x-\bar{y})^2 \le \alpha^2(x-\bar{y})^2 + 2t\int_{\bar{y}}^y u_0(\xi)d\xi,$$

and so

$$(1-\alpha)(1+\alpha)(x-\bar{y})^2 \le 2t\|u_0\|_{L^\infty(\mathbb{R})}|y-\bar{y}|,$$

where by definition, we have $y - \bar{y} = (1-\alpha)(x-\bar{y})$, and hence

$$\frac{|x-\bar{y}|}{t} \le \frac{2}{1+\alpha}\|u_0\|_{L^\infty(\mathbb{R})}.$$

Sending α to 1, we are done. □

Now, we turn to the boundary condition.

Proposition 6.0.3. Fact 3. *For $a < b$ real numbers, we have*

$$\int_a^b v(x,t)dx \xrightarrow{t\to 0} \int_a^b u_0(\xi)d\xi.$$

This implies that for all $\phi \in C_0^0(\mathbb{R})$

$$\int_{-\infty}^\infty v(x,t)\phi(x)dx \xrightarrow{t\to 0} \int_{-\infty}^\infty u_0(\xi)\phi(\xi)d\xi.$$

证明. The latter statement is obtain by taking a sequence of step functions that approaches the chosen continuous function. To prove the prior one, we first study the case that $u_0 \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We then see

$$F(x,y,t) = (x-y)^2 + 2t\int_0^y u_0(\xi)d\xi$$

has critical point $y_*(x,t)$, which implies that

$$2(y_*-x) + 2tu_0(y_*) = 0,$$

and so $u_0(y_*(x,t)) = \frac{x-y_*}{t} = v(x,t),\ x-a.e.$ on $\mathbb{R}$. Now, we have

$$\int_a^b v(x,t)dx = \int_a^b u_0(y_*(x,t))dx \longrightarrow \int_a^b u_0(x)dx,$$

as $t \to 0$.

For general $u_0 \in L^\infty(\mathbb{R})$, because y_* is monotone in x, we have

$$\int_a^b v(x,t)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{x_i + \Delta x - y_*(x_i + \Delta x, t)}{t} \Delta x.$$

Observe

$$\begin{aligned}
2(x - y_*(x,t))\Delta x &= (x + \Delta x - y_*(x,t))^2 - (x - y_*(x,t))^2 - (\Delta x)^2 \\
&= F(x + \Delta x, y_*(x,t), t) - 2t \int_0^{y_*(x,t)} u_0(\xi) d\xi \\
&\quad - \left(F(x, y_*(x,t), t) - 2t \int_0^{y_*(x,t)} u_0(\xi) d\xi \right) - (\Delta x)^2 \\
&\geq F(x + \Delta x, y_*(x + \Delta x, t), t) - F(x, y_*(x,t), t) - (\Delta x)^2,
\end{aligned}$$

and then replace x by x_i, we have

$$2 \sum_{i=0}^{n-1} (x_i - y_*(x_i, t)) \Delta x \geq -\Delta x (b - a) + F(x_n, y_*(x_n, t), t) - F(x_0, y_*(x_0, t), t).$$

But LHS converges to $2t \int_a^b v(x,t)dx$, while LHS converges to $F(b, y_*(b,t), t) - F(a, y_*(a,t), t)$. On the other hand, we have

$$\begin{aligned}
2t \int_0^b u_0(\xi) d\xi = F(b,b,t) &\geq F(b, y_*(b,t), t) \\
&= (b - y_*)^2 + 2t \int_0^{y_*(b,t)} u_0(\xi) d\xi \\
&\geq 2t \int_0^{y_*(b,t)} u_0(\xi) d\xi.
\end{aligned}$$

Similarly, we have

$$-2t \int_0^a u_0(\xi) d\xi \leq -F(a, y_*(a,t), t) \leq -2t \int_0^{y_*(a,t)} u_0(\xi) d\xi.$$

Now, we obtain

$$\int_a^{y_*(b,t)} u_0(\xi) d\xi \leq \int_a^b v(x,t)dx \leq \int_{y_*(a,t)}^b u_0(\xi) d\xi.$$

By Fact 2., we are done. □

Proposition 6.0.4. Fact 4. *For all $t > 0$, we have*

$$\|u_\mu(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}.$$

证明. Mollify u_0 by $u_\epsilon^o = j_\epsilon * u_0$, then $u_\epsilon^o \in C^\infty(\mathbb{R})$, and

$$\|u_\epsilon^o\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

and u_ϵ^o converges to u pointwise *a.e.* on $\mathbb{R}$. Consider Cauchy problem

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = \mu u_{xx}, \\ u(x,0) = u_\epsilon^0(x). \end{cases}$$

We know that this problem admits a unique solution $u_{\epsilon,\mu} \in C^\infty(\mathbb{R} \times [0,\infty))$. By Maximum Principle we know that

$$\|u_{\epsilon,\mu}\|_{L^\infty(\mathbb{R}\times[0,\infty))} \leq \|u_\epsilon^o\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}.$$

Because $u_{\epsilon,\mu}$ converges to u_μ pointwise, as $\epsilon \to 0$, we know the bound also hold for u_μ. $\square$

Finally we have the tools to show that v is a weak solution. Recall that $u_\mu \in C^\infty(\mathbb{R} \times (0,\infty))$ is a solution to the equation with viscosity μ. Fix $\delta > 0$, and $w \in C_0^1(\mathbb{R} \times [0,\infty))$, we have

$$\int_\delta^\infty \int_{-\infty}^\infty \left(u_t + \left(\frac{u^2}{2}\right)_x\right) w dx dt = \int_\delta^\infty \int_{-\infty}^\infty \mu u_{xx} w dx dt.$$

Observe that

$$LHS = -\int_\delta^\infty \int_{-\infty}^\infty u w_t + \frac{u^2}{2} w_x dx dt - \int_{-\infty}^\infty u(x,\delta) w(x,\delta) d\mathbf{x}.$$

Using the facts: $u(x,t)$ converges to $v(x,t)$ $x - a.e.$ for all $t > 0$ and Fact 4., and applying L.D.C.T., we know LHS converges to

$$-\int_\delta^\infty \int_{-\infty}^\infty v w_t + \frac{v^2}{2} w_x dx dt - \int_{-\infty}^\infty v(x,\delta) w(x,\delta) dx.$$

Also observe $\int v(x,\delta) w(x,0) dx$ converges to $\int u_0(x) w(x,0) dx$ by Fact 3., and $\int v(x,\delta)(w(x,\delta) - w(x,0)) dx$ converges to 0 as $\delta \to 0$ because w has compact support, and so we obtain

$$\int_0^\infty \int_{-\infty}^\infty v w_t + \frac{v^2}{2} w_x dx dt + \int_{-\infty}^\infty v(x,0) w(x,0) dx = 0.$$

The Function $F(x,y,t)$: We the solution to the PDE with $f(u) = u^2/2$ is of the form

$$v(x,t) = \frac{x - y_*(x,t)}{t},$$

with y_* the smallest minimum point of

$$F(x,y,t) = (x-y)^2 + 2t \int_0^y u_0(\xi) d\xi.$$

In the Riemann Problem that $u_0(x) = \chi_{(0,\infty)}$, we see the corresponding function

$$F(x,y,t) = (x-y)^2 + 2ty\chi_{(0,\infty)}.$$

For fixed (x,t), the critical points in y are

- $x \leq 0$, $y_*(x,t) = x$;

- $x > t$, $y_*(x,t) = x - t$;
- $0 < x \leq t$, $y_*(x,t) = 0$.

Large-time Behaviour of Entropy Solution: Suppose $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Because y_* is the smallest minimum point of F, we know that

$$(x - y_*)^2 + 2t \int_0^{y_*} u_0(\xi) d\xi \leq F(x,x,t) = 2t \int_0^x u_0(\xi) d\xi,$$

which gives

$$(x - y_*)^2 \leq 2t \int_{y_*}^x u_0(\xi) d\xi,$$

and thus

$$v^2(x,t) \leq \frac{2}{t} \int_{\mathbb{R}} |u_0(\xi)| d\xi.$$

Uniqueness of Entropy Solution:

Theorem 6.0.4. *Suppose $f \in C^2(\mathbb{R})$, $f'' \geq 0$. Let u, v be $L^\infty(\mathbb{R} \times (0,\infty))$ two entropy solutions of*

$$\begin{cases} w_t + (f(w))_x = 0, \\ w(x,0) = u_0(x). \end{cases}$$

Then $u = v$, a.e. on $\mathbb{R} \times (0, \infty)$.

证明. **Idea**: By definition of weak solution, for any $\phi \in C_0^1(\mathbb{R} \times [0,\infty))$, we have

$$\iint_{t>0} (u\phi_t + f(u)\phi_x)\, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0) dx = 0,$$

and a similar one for v. Let $z = u - v$, we then have

$$\iint_{t>0} (z\phi_t + (f(u) - f(v))\phi_x)\, dxdt = 0,$$

but we also have

$$f(u) - f(v) = \int_0^1 \frac{d}{dr} f(ru + (1-r)v) dr = \int_0^1 f'(ru + (1-r)v) dr z =: b(x,t) z,$$

and hence

$$\iint_{t>0} (z\,(\phi_t + b(x,t)\phi_x))\, dxdt = 0.$$

If one can show for all $\psi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, there is a $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ such that

$$\phi_t + b(x,t)\phi_x = \psi, \text{ on } \mathbb{R} \times [0,\infty),$$

then we have $z = 0$ *a.e.*. However, the agony is that $b(x,t)$ is bad. To resolve this, we will mollify u, v. Let u, v vanish on the lower half plane, and

$$u_\epsilon(x,t) = j_\epsilon *_{x,t} u(x,t), v_\epsilon(x,t) = j_\epsilon *_{x,t} v(x,t).$$

Because $|u|, |v|$ are bounded by M, so are u_ϵ, v_ϵ, and

$$u_\epsilon \longrightarrow u;\ v\epsilon \longrightarrow v,$$

pointwise as $\epsilon \to 0$. Thus, we have

$$b_\epsilon(x,t) := \int_0^1 f'(ru_\epsilon + (1-r)v_\epsilon)dr \in C^1(\mathbb{R}^2).$$

We now have to solve $\phi_t + b_\epsilon(x,t)\phi_x = \psi$. For $x \in \mathbb{R}$, $t > 0$, the C.C. passing through (x,t) is given by

$$\begin{cases} \frac{dX(s)}{ds} = b_\epsilon(X(s),s), \\ X(s)\big|_{s=t} = x. \end{cases}$$

Observe that there is no finite time blow up because b_ϵ is bounded for $\epsilon > 0$, and the solution $X(s;x,t)$ exists and is unique. By ODE, we know that $X \in C^1(\mathbb{R}^3)$. Take large T such that $\psi \equiv 0$ if $t \geq T$. Solve

$$\begin{cases} \phi_t + b_\epsilon(x,t)\phi_x = \psi, & (x,t) \in \mathbb{R}^2, \\ \phi(x,T) = 0, & x \in \mathbb{R}. \end{cases}$$

Integrating the PDE along the path $X(s) = X(s;x,t)$, we know

$$\phi(X(T;x,t),T) - \phi(X(t;x,t),t) = \int_t^T \psi(X(s;x,t),s)ds,$$

and so

$$\phi(x,t) = -\int_t^T \psi(X(s;x,t),s)ds.$$

Let A be a large number such that $Supp\{\psi\} \subset\subset (-A,A) \times (0,T)$ and $K \not\sim \epsilon, (x,t)$ an upper bound for b_ϵ, we then observe

$$X(s;A+KT,0) \geq A + KT - sK \geq A,\ 0 \leq s \leq T,$$

and

$$X(s;-A-KT,0) \leq -A - KT + sK \leq -A,\ 0 \leq s \leq T.$$

Thus $\phi \in C_0^1(\mathbb{R} \times [0,\infty)$. We discuss the properties of ϕ in two steps.

Step 1. For any small $\delta > 0$ such that $\psi \equiv 0$, $0 \leq t < \delta$, we have $|\phi_x|$ is bounded by a number independent of ϵ on $\mathbb{R} \times [\delta, T]$: Recall that

$$\begin{aligned} \phi_x(x,t) &= -\int_t^T \left(\psi(X(s;x,t),s)\right)_x ds \\ &= -\int_T^t \psi_x(X(s;x,t),s)\frac{\partial X}{\partial x}ds. \end{aligned}$$

Observing that

$$\begin{cases} \frac{d}{ds}\frac{\partial X}{\partial x}(s;x,t) = \frac{\partial b_\epsilon}{\partial x}\frac{\partial X}{\partial x}, \\ \frac{\partial X}{\partial x}(t;x,t) = 1, \end{cases}$$

we know

$$\frac{\partial X}{\partial x} = e^{\int_t^s \frac{\partial b_\epsilon}{\partial x}(X(\tau;x,t),\tau)d\tau}.$$

But

$$\begin{aligned}\frac{\partial b_\epsilon}{\partial x}(X(\tau;x,t),\tau) &= \frac{\partial}{\partial x}\left(\int_0^1 f'(ru_\epsilon + (1-r)v_\epsilon)dr\right)\Bigg|_{x=X,t=\tau} \\ &= \int_0^1 f''(\cdots)\left(r\frac{\partial u_\epsilon}{\partial x} + (1-r)\frac{\partial v_\epsilon}{\partial x}\right)dr\Bigg|_{x=X,t=\tau}.\end{aligned}$$

By entropy condition, we have for all $z \geq 0$, $\tau \geq t \geq \delta$,

$$\frac{u(x+z,\tau) - u(x,\tau)}{z} \leq \frac{C}{\tau} \leq \frac{C}{\delta},$$

which implies that

$$\frac{\partial u_\epsilon}{\partial x}(x,\tau) \leq \frac{C}{\delta},$$

and

$$\frac{\partial v_\epsilon}{\partial x}(x,\tau) \leq \frac{C}{\delta}.$$

Thus b_ϵ is bounded above for $x \in \mathbb{R}$, and so are $\frac{\partial X}{\partial x}$ and $|\phi_x|$ in $\mathbb{R} \times [\delta, T]$;

Step 2. For all small $\delta > 0$ such that $\psi(\cdot,t) \equiv 0$ for $t \leq \delta$, we have

$$\int_{-\infty}^{\infty} |\phi_x|(x,t)dx \leq C \not\sim \epsilon,$$

for all $0 < t \leq \delta$. Let $a = A + KT$, we have

$$\begin{aligned}\int_{\mathbb{R}} |\phi_x(x,t)|dx &= \int_{-a}^{a} |\phi_x(x,t)|dx \\ &= V_{-a}^{a}\phi(\cdot,t) \\ &= \sup\left\{\sum_{i=0}^{m} |\phi(x_{i+1},t) - \phi(x_i,t)|;\ \{x_i\} \text{ is a partition of } [-a,a]\right\},\ 🔧.\end{aligned}$$

Notice that

$$\phi(x,t) = -\int_t^T \psi(X(s;x,t),s)ds,$$

we know when $t \leq \delta$, it is a constant along each C.C. (may depend on ϵ). Therefore 🔧 equals to

$$\begin{aligned}\sup\left\{\sum_{i=0}^{m} |\phi(x_{i+1},\delta) - \phi(x_i,\delta)|;\ \{x_i\} \text{ is a partition of } [-a,a]\right\} &= \int_{-a}^{a} |\phi_x|(x,\delta)dx \\ &\leq 2a\,\|\phi_x(\cdot,\delta)\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Coming back to

$$\iint_{t>0} z\left(\phi_t + b_\epsilon\phi_x\right)dxdt = \iint_{t>0} (b_\epsilon - b)\phi_x z dxdt.$$

We already know

$$LHS = \iint\limits_{t>0} z\psi dxdt,$$

and hence it suffices to show RHS converges to 0 as $\epsilon \to 0$, which is easy to obtain using Step 1. and 2.. $\square$

Appendices

附录 A

Some Regularity Theories

A.1 Interior Regularity of Distributional Solutions

The problem comes from geometric analysis, and if you are not interested in the geometry, you may start with Equation (A.1.1): Let (M, g) be a real oriented compact Riemannian manifold of dimension $n \geq 1$. On a chart $\mathbf{x} : U' \subset M \hookrightarrow U \subset \mathbb{R}^n$, we know that $g = (g_{ij}) : U \to \mathbb{R}^{n \times n}$. Given any point $p \in U'$, the tangent space T_pM is defined to be the set $\{(\mathbf{x}, v); v \in T_{\mathbf{x}(p)}U\}$ modulo the equivalence relation: $(\mathbf{x}, v) \sim (\mathbf{y}, w)$ if and only if $w = d(\mathbf{y} \circ \mathbf{x}^{-1})v$. Now g naturally induces an inner product on T_pM by $([(\mathbf{y}, v)], [(\mathbf{x}, u)]) = (d(\mathbf{y} \circ \mathbf{x}^{-1})v)^T \cdot gu$, which is well-defined because g is assumed compatible with transition map. On the dual space T_pM^* of T_pM, there is also an inner product induced by $g^{-1} = (g^{ij})$, which is also compatible with transition maps (it is also natural in the viewpoint of linear algebra). The dual basis with respect to $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ will be denoted by $\{dx_i\}_{i=1}^n$.

A k-form is a smooth global section to the bundle $\bigwedge^k TM^*$, and we denote the collection of it by $\mathcal{A}^k(M)$. We are now able to define inner product on $\mathcal{A}^k(M)$:

$$(\alpha, \beta) \coloneqq \int_M \alpha \wedge *\beta,$$

where $*$ is the Hodge star operator. With the differential $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ we define $d^* = (-1)^{n(k+1)+1} * d* : \mathcal{A}^k \to \mathcal{A}^{k-1}$. The form Laplacian is then

$$\Delta_{form} = dd^* + d^*d.$$

When restricted to $\mathcal{A}^0$, the form Laplacian is exactly negative sign Laplace-Beltrami operator

Δ_{L-B}: for $u, v \in \mathcal{A}^0$, and $\text{supp}\{v\} \subset\subset U'$ for $(U', \mathbf{x})$ a chart, we have

$$\begin{aligned}
(\Delta_{form} u, v) &= (d^* du, v) \\
&= (du, dv) \\
&= \int_U g^{ij} (du)_i (dv)_j \sqrt{\det(g)} dx_1 \cdots dx_n \\
&= \int_U g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \sqrt{\det(g)} dx_1 \cdots dx_n \\
&= -\int_U \frac{\partial}{\partial x_j} \left(\sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x_i} u \right) v dx_1 \cdots dx_n \\
&= -(\Delta_{L-B} u, v),
\end{aligned}$$

where we have used Einstein's notations. From now on, we simply consider $\Delta = \Delta_{L-B}$.

Definition A.1. *A k-form α is called **harmonic** if $\Delta\alpha = 0$, and we denote by H^k the collection of harmonic k-form.*

Theorem A.1.1. (**Hodge Decomposition Theorem**) *We have the following decomposition of $\mathcal{A}^k$:*

$$\begin{aligned}
\mathcal{A}^k &= \Delta(\mathcal{A}^k) \bigoplus H^k \\
&= dd^*(\mathcal{A}^k) \bigoplus d^* d(\mathcal{A}^k) \bigoplus H^k \\
&= d(\mathcal{A}^{k-1}) \bigoplus d^*(\mathcal{A}^{k+1}) \bigoplus H^k.
\end{aligned}$$

The orthogonality is clear to see and the essential PDE problem is: given a k-form α, is there another k-form ω such that

$$\Delta\omega = \alpha ? ☺$$

It's not hard to notice that a k-form is harmonic if and only if all its coefficients are harmonic, and so to solve the PDE problem, one may simply consider scalar functions. The first problem we meet is how to define a solution. In a classical manner it is not clear to see the existence of solution, and so we need the theory of distributions. Notice that a classical solution ω to ☺ satisfies for every $\nu \in \mathcal{A}^k$,

$$(\alpha, \nu) = (\Delta\omega, \nu) = (\omega, \Delta\nu).$$

The right hand side thus defines a bounded linear functional on the subspace $\Delta(\mathcal{A}^k)$ of $\mathcal{A}^k$. This functional has a natural extension $l(\cdot) = (\omega, \cdot)$ on the whole $\mathcal{A}^k$.

Definition A.2. *A **distributional solution** to ☺ is defined to be a bounded linear functional $l \in (\mathcal{A}^k)^*$ such that*

$$l(\Delta\phi) = (\alpha, \phi),$$

for all $\phi \in \mathcal{A}^k$.

Before everything starts, we state two important theorems, details of which will be discussed later.

Theorem A.1.2. (**A**) *A distributional solution always have a smooth incarnation. That is to say, if l is a distributional solution then there is an $\omega \in \mathcal{A}^k$ such that $l(\phi) = (\omega, \phi)$ for all $\phi \in \mathcal{A}^k$. This also forces ω to be a classical solution.*

Theorem A.1.3. (**B**) *Any sequence of k-forms α_m such that $\|\alpha_m\| + \|\Delta\alpha_m\| \leq C$ for some $C > 0$ independent of m has a Cauchy subsequence.*

proof of the big guy. It is clear by theorem B, the dimension of H^k should be finite, otherwise there will be an infinite sequence of orthonormal basis, which contains no Cauchy subsequence. Now, H^k naturally becomes a closed subspace of $\mathcal{A}^k$ and so we may write

$$\mathcal{A}^k = H^k \bigoplus (H^k)^\perp.$$

Because Δ is self-adjoint, we find that $\Delta(\mathcal{A}^k) \subset (H^k)^\perp$. To show that $(H^k)^\perp \subset \Delta(\mathcal{A}^k)$, we need to prove the following inequality

$$\|\beta\| \leq C \|\Delta\beta\|, \forall \beta \in (H^k)^\perp.$$

Suppose the contrary, we obtain a sequence $\|\beta_j\| = 1$ and $\|\Delta\beta_j\| \to 0$ as $j \to \infty$. Define $l(\phi) = \lim_{j\to\infty}(\beta_j, \phi)$ for each $\phi \in \mathcal{A}^k$, we have that l is bounded and linear on $\mathcal{A}^k$ and has norm 1. On the other hand, we have

$$l(\Delta\phi) = \lim(\beta_j, \Delta\phi) = \lim(\Delta\beta_j, \phi) = 0.$$

Now l is a distributional solution to ☺ with $\alpha = 0$. By theorem A, there is a k-form β such that $\Delta\beta = 0$. Each β_j is orthogonal to H^k, and so is β. Therefore $\beta = 0$, but the convergence in L^2-norm of β_j to β forces $\|\beta\| = 1$.

Let $H : \mathcal{A}^k \to H^k$ be the natural projection, we have

$$l(\Delta\phi) = (\alpha, \phi - H(\phi))$$

is well defined, and

$$|l(\Delta\phi)| \leq \|\alpha\| \|\phi - H(\phi)\| \leq C \|\alpha\| \|\Delta\phi\|.$$

By Hahn-Banach Theorem, there should be an extension of l to the whole space $\mathcal{A}^k$. Now, a simple application of theorem A establishes the solution. □

proof of B. Let $U'' \subset\subset U' \subset M$ and $(U', \mathbf{x})$ be a chart ($U := \mathbf{x}(U')$). Suppose $\phi \in C_0^\infty(U')$ such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on U'', we then have (by considering the inner product component-wise)

$$\begin{aligned}
|(\Delta\alpha_m, \alpha_m\phi^2)| &= \left| \int_U \partial_i \left(g^{ij}\sqrt{\det g}\,\partial_j\alpha_m \right) \alpha_m\phi^2 dx_1 \cdots dx_n \right| \\
&= \left| \int_U \left(g^{ij}\sqrt{\det g}\,\partial_j\alpha_m \right) \partial_i\left(\alpha_m\phi^2\right) dx_1 \cdots dx_n \right| \\
&\geq \left| \int_U \phi^2 g^{ij}\sqrt{\det g}\,\partial_i\alpha_m\partial_j\alpha_m \right| - 2\left| \int_U \phi\alpha_m g^{ij}\sqrt{\det g}\,\partial_j\alpha_m\partial_i\phi \right|.
\end{aligned}$$

Since (g^{ij}) is positive definite and $2|ab| \le |a|/\epsilon + \epsilon|b|$ for all $\epsilon > 0$, we have, by taking $\epsilon = 1/2$

$$
\begin{aligned}
RHS &\ge \frac{1}{2}\left|\int_U \phi^2 g^{ij}\sqrt{\det g}\partial_i\alpha_m\partial_j\alpha_m\right| - 2\left|\int_U \alpha_m^2 g^{ij}\sqrt{\det g}\partial_i\phi\partial_j\phi\right| \\
&\ge \frac{C}{2}\int_{\mathbf{x}(U'')} |\nabla\alpha_m|^2 - 2C\int_U \alpha_m^2\sqrt{\det g}.
\end{aligned}
$$

On the other hand,

$$
|(\Delta\alpha_m, \alpha_m\phi^2)| \le \|\alpha_m\|\,\|\Delta\alpha_m\|\,,
$$

and then know that α_m is a bounded sequence in $W^{1,2}(\mathbf{x}(U''))$. By Relich-Kondrakov compact embedding, we know that the inclusion $W_0^{1,2}(V) \hookrightarrow L^2(V) \cong L^2(V, \sqrt{\det g})$ is compact for every bounded $V \subset\subset U$, which means that α_m restricted to some $U''' \subset\subset U''$ should have a Cauchy subsequence. By compactness of M, we know that there must be a Cauchy sequence in the original sequence α_m.

□

proof of A. By taking a chart $(U', \mathbf{x})$ $(U := \mathbf{x}(U'))$, we localizes the distributional solution l so that now it is a bounded linear functional on $L^2(U, \sqrt{\det g})$. By Riesz representation theorem, there is an element w in $L^2(U, \sqrt{\det g})$ such that $l(\phi) = \int_U w\phi\sqrt{\det g}d\vec{x}$ for all $\phi \in C_0^\infty(U)$ (clearly $\phi \circ \mathbf{x} \in \mathcal{A}^0$). Replacing ϕ by $\Delta\phi$, we obtain

$$
(\alpha, \phi) = l(\Delta\phi) = \int_U w\partial_i\left(g^{ij}\sqrt{\det g}\partial_j\phi\right)d\vec{x}.
$$

On the other hand,

$$
LHS = \int_U \alpha\phi\sqrt{\det g}d\vec{x}.
$$

Denoting $a_{ij} = g^{ij}\sqrt{\det g}$ and $c = \alpha\sqrt{\det g}$, we know $w \in L^2(U, \sqrt{\det g}) \cong L^2(U)$ satisfies integral equation

$$
\int_U w\partial_i(a_{ij}\partial_j\phi) = \int_U c\phi,\ \forall\phi \in C_0^\infty(U). \tag{A.1.1}
$$

To show the smoothness of w it suffices to show the existence of its L^2 first order derivatives (see Chapter 4). Since smoothness is a local property we restrict our attention to one point $\vec{y} \in U$ and some small ball $B_r := B(\vec{y}, r) \subset\subset U$.

The idea is to solve an auxiliary Dirichlet problem:

$$
☹ \begin{cases} \partial_i(a_{ij}\partial_j\phi)(\vec{x}) = \psi(\vec{x}), & \vec{x} \in B_r, \\ \phi(\vec{x}) = 0, & \vec{x} \in \partial B_r. \end{cases}
$$

This problem has a unique smooth solution $\phi \in C^\infty(B_r) \cap C^0\left(\overline{B_r}\right)$ for each $\psi \in C_0^\infty(B_r)$. We define for $h \neq 0$ and $k = 1, \cdots, n$,

$$
T_k^h f(x_1, \cdots, x_k, \cdots, x_n) := f(x_1, \cdots, x_k + h, \cdots, x_n),
$$

and difference quotient

$$
D_k^h f := \frac{T_k^h f - f}{h}.
$$

Now, we have for $\psi \in C_0^\infty(B_{r/2})$ and $0 < |h| << 1$, the problem ☹ has a unique solution $\phi^{k,h}$ for each $T_k^h\psi$. By linearity of the PDE and if we denote $P^{k,h} = \frac{\phi^{j,h}-\phi}{h}$, we obtain

$$\begin{cases} \partial_i(a_{ij}\partial_j P^{k,h})(\vec{x}) = D_k^h\psi(\vec{x}), & \vec{x} \in B_r, \\ P^{k,h}(\vec{x}) = 0, & \vec{x} \in \partial B_r. \end{cases} \tag{A.1.2}$$

Applying $P^{k,h}$ to both sides we have

$$\begin{aligned} \int_{B_r} a_{ij}\partial_i P^{k,h}\partial_j P^{k,h} &= -\int_{B_r} D_k^h\psi P^{k,h} \\ &= \int_{B_r} \psi D_k^{-h} P^{k,h} \\ &\leq \|\psi\| \left\|D_k^{-h}P^{k,h}\right\| \\ &\lesssim \|\psi\| \left\|\nabla P^{k,h}\right\|. \end{aligned}$$

By strict ellipticity of a_{ij} we obtain

$$\left\|\nabla P^{k,h}\right\| \lesssim \|\psi\|.$$

Moreover, by Poincaré inequality,

$$\left\|\nabla P^{k,h}\right\| + \left\|P^{k,h}\right\| \lesssim \|\psi\|. \tag{A.1.3}$$

Roughly speaking: equation (A.1.1) can be reformulated as

$$\mathcal{L}_k^h(\psi) := \int_{B_r} wD_k^h\psi = \int_{B_r} cP^{k,h}.$$

According to (A.1.3), the sequence of functionals $\mathcal{L}_k^h$ is uniformly bounded in $(L^2(B_{r/2}))^*$. By Banach-Alaoglu theorem, there is an $\mathcal{L}_k \in (L^2(B_{r/2}))^*$ such that $\mathcal{L}_k^h \rightharpoonup \mathcal{L}_k$ as $h \to 0$. This $\mathcal{L}_k$ has an $L^2(B_{r/2})$ incarnation $-w_k$. In particular, we have

$$\int_{B_r} wD_k^h\psi \xrightarrow{h\to 0} -\int_{B_r} w_k\psi.$$

On the other hand, by LDCT,

$$LHS \xrightarrow{h\to 0} \int_{B_r} w\partial_k\psi.$$

One Last Problem: This rough idea seems feasible but there is still one remaining problem that each $\phi^{k,h}$ is not smooth when extended naturally to U. This requires one to cut off each $\phi^{k,h}$. Let $\eta \in C_0^\infty(B_{4r/5})$, $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $B_{3r/5}$. Define $Q^{k,h} = P^{k,h}\eta \in C_0^\infty(B_r)$ and we have

$$\begin{aligned} \partial_i(a_{ij}\partial_j Q^{k,h}) &= \partial_i(a_{ij}\partial_j \eta P^{k,h}) \\ &= \partial_i(a_{ij}\partial_j P^{k,h})\eta + 2a_{ij}\partial_j P^{k,h}\partial_i\eta + \partial_i(a_{ij}\partial_j\eta)P^{k,h} \\ &= D_k^h\psi\eta + 2a_{ij}\partial_j P^{k,h}\partial_i\eta + \partial_i(a_{ij}\partial_j\eta)P^{k,h} \\ &= D_k^h\psi + trash. \end{aligned}$$

Again, by equation (A.1.1) we still have a proper estimate

$$
\begin{aligned}
\left|\int wD_k^h\psi\right| &= \left|\int cQ^{k,h} - 2\int wa_{ij}\partial_j P^{k,h}\partial_i\eta - \int wP^{k,h}\partial_i(a_{ij}\partial_j\eta)\right| \\
&\lesssim \left\|P^{k,h}\right\| + \left\|\nabla P^{k,h}\right\| \\
&\lesssim \|\psi\|.
\end{aligned}
$$

□

A.2 Marcinkiewicz Interpolation and L^p Estimates

The following two sections are copies of the corresponding contents in the book "Elliptic Partial Differential Equations of Second Order". The differential operator L we mainly consider in this section is of the form

$$Lu = a^{ij}(\mathbf{x})D_{ij}u + b^i(\mathbf{x})D_iu + c(\mathbf{x})u.$$

A.2.1 Cube Decomposition

Let K_0 be a cube in $\mathbb{R}^n$, f nonnegative integrable function on K_0 and $t > 0$ satisfying

$$\int_{K_0} f \leq t|K_0|.$$

By bisecting edges of K_0, we obtain 2^n congruent subcubes with disjoint interiors. For those cubes K satisfying

$$\int_K f \leq t|K|, \tag{A.2.1}$$

we subdivide them in a similar manner to K_0. We then collect other cubes that are not subdivided and denote the class by $\mathcal{I}$. For $K \in \mathcal{I}$, we call $\tilde{K}$ the cube whose subdivision gives K. By definition $\tilde{K} \notin \mathcal{I}$, and therefore

$$t < \frac{1}{|K|}\int_K f \leq 2^n t. \tag{A.2.2}$$

Furthermore, setting $F = \cup_{K\in\mathcal{I}}K$ and $G = K_0 \backslash F$, we have

$$f \leq t \text{ a.e. in } G, \tag{A.2.3}$$

which is obtained by Lebesgue's Differentiation Theorem, because almost every point in G is contained in a decreasing sequence of parallel cubes with diameters shrinking to 0. Letting $\tilde{F} = \cup_{K\in\mathcal{I}}\tilde{K}$, we have by (A.2.1),

$$\int_F f \leq t|\tilde{F}|. \tag{A.2.4}$$

In particular, when $f = \chi_\Gamma$ for some measurable subset $\Gamma \subset K_0$, we have

$$|\Gamma| = |\Gamma \cap \tilde{F}| \leq t|\tilde{F}|. \tag{A.2.5}$$

A.2.2 Marcinkiewicz Interpolation

For f a nonnegative measurable function on a domain Ω (bounded or unbounded) in $\mathbb{R}^n$. The *distribution function* $\mu = \mu_f$ is defined by

$$\mu(t) = |\{\mathbf{x} \in \Omega \mid f(\mathbf{x}) > t\}|.$$

This distribution function measures the relative size of f.

Lemma A.2.1. *For $p > 0$ and $f \in L^p(\Omega)$, we have*

$$\mu(t) \le t^{-p} \int_\Omega |f|^p, \tag{A.2.6}$$

and

$$\int_\Omega |f|^p = p \int_0^\infty t^{p-1} \mu(t) dt. \tag{A.2.7}$$

We now prove the following restricted Marcinkiewicz Interpolation.

Theorem A.2.1. (**Marcinkiewicz Interpolation Theorem**) *Let T be a linear mapping from $L^q(\Omega) \cap L^r(\Omega)$ into itself, $1 \le q \le r < \infty$ and suppose there are constants T_1 and T_2 such that*

$$\mu_{Tf} \le \left(\frac{T_1 \|f\|_q}{t}\right)^q, \quad \mu_{Tf} \le \left(\frac{T_2 \|f\|_r}{t}\right)^r \tag{A.2.8}$$

for all $f \in L^q(\Omega) \cap L^r(\Omega)$ and $t > 0$. Then T extends as a bounded linear mapping from $L^p(\Omega)$ into itself for any p such that $q < p < r$, and

$$\|Tf\|_p \le C T_1^\alpha T_2^{1-\alpha} \|f\|_p \tag{A.2.9}$$

for all $f \in L^q(\Omega) \cap L^r(\Omega)$, where

$$\frac{1}{p} = \frac{\alpha}{q} + \frac{1-\alpha}{r}$$

and C depends only on p, q and r.

证明. For $f \in L^q(\Omega) \cap L^r(\Omega)$ and $s > 0$, we write

$$f = f_1 + f_2,$$

where

$$f_1(\mathbf{x}) = f(\mathbf{x}) \chi_{|f|>s}.$$

Then $|Tf| \le |Tf_1| + |Tf_2|$, and hence

$$\begin{aligned} \mu_{Tf}(t) &\le \mu_{Tf_1}(t/2) + \mu_{Tf_2}(t/2) \\ &\le \left(\frac{2T_1}{t}\right)^q \int_\Omega |f_1|^q + \left(\frac{2T_2}{t}\right)^r \int_\Omega |f_2|^r. \end{aligned}$$

Therefore, by Lemma A.2.1, we have

$$\begin{aligned}\int_\Omega |Tf|^p &= p\int_0^\infty t^{p-1}\mu(t)dt \\ &\le p(2T_1)^q\int_0^\infty t^{p-1-q}\left(\int_{|f|>s}|f|^q\right)dt \\ &\quad + p(2T_2)^r\int_0^\infty t^{p-1-r}\left(\int_{|f|\le s}|f|^r\right)dt.\end{aligned}$$

Now, we choose $t = As$ for A some positive number to be fixed later. Thus, we obtain

$$\begin{aligned}\int_\Omega |Tf|^p \le p(2T_1)^q A^{p-q}\int_0^\infty s^{p-1-q}\left(\int_{|f|>s}|f|^q\right)ds \\ + p(2T_2)^r A^{p-r}\int_0^\infty s^{p-1-r}\left(\int_{|f|\le s}|f|^r\right)ds.\end{aligned}$$

But

$$\begin{aligned}\int_0^\infty s^{p-1-q}\left(\int_{|f|>s}|f|^q\right)ds &= \int_\Omega |f|^q\int_0^{|f|} s^{p-1-q}ds \\ &= \frac{1}{p-q}\int_\Omega |f|^p,\end{aligned}$$

and similarly,

$$\begin{aligned}\int_0^\infty s^{p-1-r}\left(\int_{|f|\le s}|f|^r\right)ds &= \int_\Omega |f|^r\left(\int_{|f|}^\infty s^{p-1-r}ds\right) \\ &= \frac{1}{r-p}\int_\Omega |f|^p.\end{aligned}$$

Consequently, we have

$$\int_\Omega |Tf|^p \le \left[\frac{p}{p-q}(2T_1)^q A^{p-q} + \frac{p}{r-p}(2T_2)^r A^{p-r}\right]\int_\Omega |f|^p$$

for any positive A. By taking the value of A for which the expression embraces a minimum, namely

$$A = 2T_1^{q/(r-q)}T_2^{r/(r-q)},$$

we thus obtain

$$\|Tf\|_p \le 2\left(\frac{p}{p-q}+\frac{p}{r-p}\right)^{1/p} T_1^\alpha T_2^{1-\alpha}\|f\|_p.$$

□

A.2.3 The Calderon-Zygmund Inequality

Let Ω be a bounded domain in $\mathbb{R}^n$ and f a function in $L^p(\Omega)$ for some $p \ge 1$. Recall the Newtonian potential of f is the function

$$w(\mathbf{x}) = \int_\Omega \Gamma(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y},$$

where Γ is the fundamental solution of Laplace's equation. The following result is a special case of Calderon-Zygmund inequality.

Theorem A.2.2. *Let $f \in L^p(\Omega)$, $1 < p < \infty$, and let w be the Newtonian potential of f. Then $w \in W^{2,p}(\Omega)$, $\Delta w = f$ a.e. and*

$$\left\|D^2 w\right\|_p \leq C \left\|f\right\|_p, \tag{A.2.10}$$

where C depends only on n and p. Furthermore, when $p = 2$ we have

$$\int_{\mathbb{R}^2} |D^2 w|^2 = \int_\Omega f^2. \tag{A.2.11}$$

证明. i. Let us first deal with the case $p = 2$. If $f \in C_0^\infty(\mathbb{R}^n)$, we have $w \in C^\infty(\mathbb{R}^n)$ and $\Delta w = f$. Consequently, for any ball B_R containing the support of f,

$$\int_{B_R} (\Delta w)^2 = \int_{B_R} f^2.$$

Applying Green's first identity twice, we have

$$\begin{aligned} \int_{B_R} |D^2 w|^2 &= \int_{B_R} \sum (w_{ij})^2 \\ &= \int_{B_R} f^2 + \int_{\partial B_R} Dw \cdot \frac{\partial Dw}{\partial \nu}. \end{aligned}$$

It's clear to see that

$$Dw = O(R^{1-n}),\ D^2 w = O(R^{-n}),$$

uniformly on ∂B_R as $R \to \infty$, whence the identity follows. Recalling that $N : f \mapsto w$ is a bounded linear mapping from $L^p(\Omega)$ to itself for $1 \leq p < \infty$, the proof is then finished by applying Interpolation Inequality 4.1.2 and a density argument;

ii. For fixed i, j, we now define the linear operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$Tf = D_{ij} w.$$

By the above equality, we have

$$\mu_{Tf}(t) \leq \left(\frac{\|f\|_2}{t}\right)^2, \tag{A.2.12}$$

for all $t > 0$ and $f \in L^2(\Omega)$. We now show that, in addition,

$$\mu(t) \leq C \frac{\|f\|_1}{t}, \tag{A.2.13}$$

for all $t > 0$ and $f \in L^2(\Omega)$, thereby making possible the application of Marcinkiewicz interpolation theorem. To accomplish this we extend f to vanish outside Ω and fix a cube $\Omega \subset K_0$, so that for fixed $t > 0$ we have

$$\int_{K_0} |f| \leq t|K_0|.$$

The cube is now decomposed according to the procedure described in the first Subsection A.2.1 giving a sequence of parallel subcubes $\{K_l\}_{l=1}^\infty$ such that

$$t < \frac{1}{|K_l|} \int_{K_l} |f| < 2^n t, \tag{A.2.14}$$

and

$$|f| \leq t \text{ a.e. on } G = K_0 \backslash \cup K_l.$$

The function is now split into a "good part" g defined by

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{for } \mathbf{x} \in G, \\ \frac{1}{|K_l|} \int_{K_l} |f|, & \text{for } \mathbf{x} \in K_l,\ l = 1, 2, \ldots, \end{cases}$$

and a "bad part" $b = f - g$. Clearly,

$$|g| \leq 2^n t, \text{ a.e.},$$

$$b(\mathbf{x}) = 0, \text{ for } \mathbf{x} \in G,$$

$$\int_{K_l} b = 0, \text{ for } l = 1, 2, \ldots.$$

Since T is linear, $Tf = Tg + Tb$; hence,

$$\mu_{Tf}(t) \leq \mu_{Tg}(t/2) + \mu_{Tb}(t/2);$$

iii. Estimation of Tg: By (A.2.12), we have

$$\begin{aligned} \mu_{Tg}(t/2) &\leq \frac{4}{t^2} \int g^2 \\ &\leq \frac{2^{n+2}}{t} \int |g| \\ &\leq \frac{2^{n+2}}{t} \int |f|; \end{aligned}$$

iv. Estimation of Tb: Writing

$$b_l = b\chi_{K_l} = \begin{cases} b, & \text{on } K_l, \\ 0, & \text{elsewhere,} \end{cases}$$

we have

$$Tb = \sum_{l=1}^{\infty} Tb_l.$$

Let us now fix some l and a sequence $\{b_{lm}\} \subset C_0^\infty(K_l)$ converging to b_l in $L^2(\Omega)$ and satisfying

$$\int_{K_l} b_{lm} = \int_{K_l} b_l = 0.$$

Then for $\mathbf{x} \notin K_l$, we have the formula

$$\begin{aligned} Tb_{lm}(\mathbf{x}) &= \int_{K_l} D_{ij}\Gamma(\mathbf{x} - \mathbf{y}) b_{lm}(\mathbf{y}) d\mathbf{y} \\ &= \int_{K_l} \{D_{ij}\Gamma(\mathbf{x} - \mathbf{y}) - D_{ij}\Gamma(\mathbf{x} - \bar{\mathbf{y}})\} b_{lm}(\mathbf{y}) d\mathbf{y}, \end{aligned}$$

where $\bar{\mathbf{y}} = \bar{\mathbf{y}}_l$ denotes the center of K_l. Letting $\delta = \delta_l$ denote the diameter of K_l, we then obtain

$$|Tb_{lm}(\mathbf{x})| \leq C(n)\frac{\delta}{\operatorname{dist}(\mathbf{x}, K_l)^{n+1}}\int_{K_l}|b_{lm}(\mathbf{y})|d\mathbf{y}.$$

Letting $B_l = B_\delta(\bar{\mathbf{y}})$ denote the concentric ball of radius δ, we obtain by integration

$$\begin{aligned}\int_{K_0\backslash B_l}|Tb_{lm}| &\leq C(n)\delta\int_{|\mathbf{x}|\geq\delta/2}\frac{d\mathbf{x}}{|\mathbf{x}|^{n+1}}\int_{K_l}|b_{lm}|\\ &\leq C(n)\int_{K_l}|b_{lm}|.\end{aligned}$$

Consequently, letting $m \to \infty$, writing $F^* = \cup B_l$ ($F = \cup K_l$), $G^* = K_0\backslash F^*$ and summing over l, we get

$$\int_{G^*}|Tb| \leq C(n)\int|b| \leq C(n)\int|f|,$$

so that

$$|\{\mathbf{x}\in G^* \mid |Tb| > t/2\}| \leq C\frac{\|f\|_1}{t}.$$

However, by (A.2.14),

$$|F^*| \leq \omega_n n^{n/2}|F| \leq C\frac{\|f\|_1}{t};$$

v. To conclude the proof we apply Marcinkiewicz Interpolation Theorem for $q = 1$, $r = 2$. Consequently,

$$\|Tf\|_p \leq C(n,p)\,\|f\|_p\,, \tag{A.2.15}$$

for all $1 < p \leq 2$ and $f \in L^2(\Omega)$. This inequality is extended to $p > 2$ by duality.

□

Remark: The operator T can be defined as a bounded operator on $L^p(\Omega)$ even when Ω is unbounded when $n \geq 3$.

Corollary A.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $u \in W_0^{2,p}(\Omega)$, $1 < p < \infty$. Then*

$$\left\|D^2u\right\|_p \leq C\,\|\Delta u\|_p\,, \tag{A.2.16}$$

where $C = C(n,p)$. If $p = 2$,

$$\left\|D^2u\right\|_2 = \|\Delta u\|_2\,. \tag{A.2.17}$$

A.2.4 L^p Estimates

This subsection focuses on the derivation of local and global L^p estimates.

Theorem A.2.3. (**Local L^p Estimates**) *Let Ω be an open set in $\mathbb{R}^n$ and $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, a strong solution of the equation $Lu = f$ in Ω where the coefficients of L satisfy, for positive constants λ, Λ,*

$$\begin{aligned} &a^{ij} \in C^0(\Omega),\ b^i, c \in L^\infty(\Omega), f \in L^p(\Omega); \\ &a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n; \\ &|a^{ij}|, |b^i|, |c| \leq \Lambda, \end{aligned} \tag{A.2.18}$$

where $i, j = 1, 2, \ldots, n$. Then for any domain $\Omega' \subset\subset \Omega$,

$$\|u\|_{2,p;\Omega'} \leq C\left(\|u\|_{p,\Omega} + \|f\|_{p,\Omega}\right), \tag{A.2.19}$$

where C depends on $n, p, \lambda, \Lambda, \Omega', \Omega$ and the moduli of continuity of the coefficients a^{ij} on Ω'.

证明. For a fixed point $\mathbf{x}_0 \in \Omega'$, we let L_0 denote the constant coefficient operator given by

$$L_0 u = a^{ij}(\mathbf{x}_0) D_{ij} u.$$

By means of linear transformation, we have by Corollary A.2.1

$$\left\|D^2 v\right\|_{p;\Omega} \leq \frac{C}{\lambda}\left\|L_0 v\right\|_{p,\Omega}, \tag{A.2.20}$$

for any $v \in W^{2,p}_0(\Omega)$, where $C = C(n,p)$. Consequently, if v has support in a ball $B_R = B_R(\mathbf{x}_0) \subset\subset \Omega$, we have

$$L_0 v = (a^{ij}(\mathbf{x}_0) - a^{ij}) D_{ij} v + a^{ij} D_{ij} v,$$

and hence

$$\left\|D^2 v\right\|_p \leq \frac{C}{\lambda}\left(\sup_{B_R}|a - a(\mathbf{x}_0)| \left\|D^2 v\right\|_p + \left\|a^{ij} D_{ij} v\right\|_p\right),$$

where $a = (a^{ij})$. Since a is uniformly continuous on Ω', there is a positive number δ such that

$$|a - a(\mathbf{x}_0)| \leq \lambda/2C$$

if $|\mathbf{x} - \mathbf{x}_0| < \delta$, and hence

$$\left\|D^2 v\right\|_p \leq C\left\|a^{ij} D_{ij} v\right\|_p,$$

provided $R \leq \delta$, where $C = C(n, p, \lambda)$.

For $\sigma \in (0,1)$, we now introduce a cut-off function $\eta \in C^2_0(B_R)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\sigma R}$, $\eta = 0$ for $|\mathbf{x}| \geq \sigma' R$, $\sigma' = (1+\sigma)/2$, $|D\eta| \leq 4/(1-\sigma)R$, $|D^2\eta| \leq 16/(1-\sigma)^2 R^2$. Then, if $u \in W^{2,p}_{loc}(\Omega)$ satisfies $Lu = f$ in Ω and $v = \eta u$, we obtain

$$\begin{aligned} \left\|D^2 u\right\|_{p;B_{\sigma R}} &\leq C\left\|\eta a^{ij} D_{ij} u + 2a^{ij} D_i\eta D_j u + u a^{ij} D_{ij}\eta\right\|_{p;B_R} \\ &\leq C\left(\|f\|_{p;B_R} + \frac{1}{(1-\sigma)R}\|Du\|_{p;B_{\sigma' R}} + \frac{1}{(1-\sigma)^2 R^2}\|u\|_{p;B_R}\right), \end{aligned}$$

provided $R \leq \delta \leq 1$, where $C = C(n, p, \lambda, \Lambda)$. Introducing the weighted seminorms

$$\Phi_k = \sup_{1<\sigma<1}(1-\sigma)^k R^k \left\|D^k u\right\|_{p;B_{\sigma R}}, \quad k = 0, 1, 2,$$

we, therefore have

$$\Phi_2 \leq C\left(R^2 \|f\|_{p;B_R} + \Phi_1 + \Phi_0\right). \tag{A.2.21}$$

We claim now that Φ_k satisfy an interpolation inequality

$$\Phi_1 \leq \epsilon\Phi_2 + \frac{C}{\epsilon}\Phi_0,$$

for any $\epsilon > 0$, where $C = C(n)$. By its invariance under coordinate stretching it suffices to prove for the case $R = 1$.

For $\gamma > 0$, we fix $\sigma = \sigma_\gamma$ so that

$$\begin{aligned}\Phi_1 &\leq (1-\sigma_\gamma)\|Du\|_{p;B_\sigma} + \gamma \\ &\leq \epsilon(1-\sigma)^2\left\|D^2u\right\|_{p;B_\sigma} + \frac{C}{\epsilon}_{p;B_\sigma} + \gamma,\end{aligned} \tag{A.2.22}$$

by interpolation inequality. Sending $\gamma \to 0$, we obtain

$$\Phi_2 \leq C\left(R^2 \|f\|_{p;B_R} + \Phi_0\right),$$

that is

$$\left\|D^2u\right\|_{p;B_{\sigma R}} \leq \frac{C}{(1-\sigma)^2R^2}\left(R^2\|f\|_{p;B_R} + \|u\|_{p;B_R}\right), \tag{A.2.23}$$

where $C = C(n, p, \lambda, \Lambda)$ and $0 < \sigma < 1$.

The desired estimate follows by taking $\sigma = 1/2$ and covering Ω' with a finte number of balls of radius $R/2$ for $R \leq \min\{\delta, \operatorname{dist}(\Omega', \partial\Omega)\}$.

□

Theorem A.2.4. (**Global L^p Estimates**) *Let Ω be a domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial\Omega$. Let $u \in W^{2,p}(\Omega)$, $1 < p < \infty$ be a strong solution of $Lu = f$ in Ω with $u = 0$ on T (in the sense of trace), where L satisfies conditions in local estimates with $a^{ij} \in C^0(\Omega \cup T)$. Then, for any domain $\Omega' \subset\subset \Omega \cup T$,*

$$\|u\|_{2,p;\Omega'} \leq C\left(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}\right), \tag{A.2.24}$$

where C depends on $n, p, \lambda, \Lambda, T, \Omega', \Omega$ and the moduli of continuity of the coefficients a^{ij} on Ω'.

Theorem A.2.5. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and suppose the operator L satifies the conditions in local estimates with $a^{ij} \in C^0(\bar{\Omega})$, $i, j = 1, 2, \ldots, n$. Then if $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $1 < p < \infty$, we have*

$$\|u\|_{2,p;\Omega} \leq C\|Lu - \sigma u\|_{p;\Omega}, \tag{A.2.25}$$

for all $\sigma \geq \sigma_0$, where C and σ_0 are positive constants depending only on $n, p, \lambda, \Lambda, \Omega$ and the moduli of continuity of the coeffients a^{ij}.

证明. We define a domain Ω_0 in $\mathbb{R}^{n+1}(\mathbf{x}, t)$ by

$$\Omega_0 = \Omega \times (-1, 1),$$

together with the operator L_0, given by

$$L_0 v = Lv + D_{tt} v,$$

for $v \in W^{2,p}(\Omega_0)$. Then, if $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, the function v, given by

$$v(\mathbf{x}, t) = u(\mathbf{x}) \cos \sigma^{1/2} t$$

belongs to $W^{2,p}(\Omega_0)$ and vanishes on $\partial\Omega \times (-1, 1)$ in the sense of trace. Furthermore,

$$L_0 v = \cos \sigma^{1/2} t \, (Lu - \sigma u) .$$

By global estimate, we have for $\Omega' = \Omega \times (-\epsilon, \epsilon)$, $0 < \epsilon \leq 1/2$, we get

$$\|D_{tt} u\|_{p;\Omega'} \leq C \left(\|Lu - \sigma u\|_{p;\Omega} + \|u\|_{p;\Omega} \right),$$

where C depends on quantities that are described before. But now, taking $\epsilon = \pi/3\sigma^{1/2}$, we have

$$\begin{aligned} \|D_{tt} v\|_{p;\Omega'} &= \sigma \|v\|_{p;\Omega'} \\ &\geq \sigma \cos(\sigma^{1/2} \epsilon)(2\epsilon)^{1/p} \|u\|_{p;\Omega} \\ &\geq \frac{1}{2} \left(\frac{2\pi}{3} \right)^{1/p} \sigma^{1-1/2p} \|u\|_{p;\Omega} , \end{aligned}$$

so that if σ is sufficiently large

$$\|u\|_{p;\Omega} \leq C \|Lu - \sigma u\|_{p;\Omega} . \tag{A.2.26}$$

The desired estimate follows from global estimates. □

A.3 Schauder Theory

A.3.1 Symbols

Let $\mathbf{x}_0$ be a point in $\mathbb{R}^n$, and f a function defined on a bounded set D containing $\mathbf{x}_0$. If $0 < \alpha < 1$, we say that f is *Hölder continuous with exponent α at $\mathbf{x}_0$* if the quantity

$$[f]_{\alpha;\mathbf{x}_0} = \sup_D \frac{|f(\mathbf{x}) - f(\mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|^\alpha} < \infty.$$

The notion of Hölder continuity is immediately extended to the whole of D:

$$f \text{ is } \textit{uniformly Hölder continuous} \text{ if } [f]_{\alpha;D} = \sup_{\substack{\mathbf{x},\mathbf{y} \in D \\ \mathbf{x} \neq \mathbf{y}}} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha} < \infty.$$

On the other hand, f is called *locally uniformly Hölder continuous* if it is uniformly Hölder continuous on compact subsets of D. For $\Omega \subset \mathbb{R}^n$ an open set, we define the Hölder spaces $C^{k,\alpha}(\bar{\Omega})$ ($C^{k,\alpha}(\Omega)$) as the subspaces of $C^k(\bar{\Omega})$ ($C^k(\Omega)$) consisting of functions whose k-th derivatives are (locally) uniformly Hölder continuous with exponent α in Ω. For simplicity, we write

$$
\begin{aligned}
&C^{0,\alpha}(\Omega) = C^{\alpha}(\Omega),\ C^{0,\alpha}(\bar{\Omega}) = C^{\alpha}(\bar{\Omega});\\
&C^{k,0}(\Omega) = C^{k}(\Omega),\ C^{k,0}(\bar{\Omega}) = C^{k}(\bar{\Omega}).
\end{aligned}
$$

Moreover, $C_0^{k,\alpha}(\Omega)$ is defined to be the subspace of $C^{k,\alpha}(\Omega)$ composed of functions with compact supports in Ω.

Now we set

$$
[u]_{k,\alpha;\Omega} = [D^k u]_{\alpha;\Omega} = \sup_{|\beta|=k} [D^{\beta} u]_{\alpha;\Omega}, \tag{A.3.1}
$$

and the related norms

$$
\|u\|_{C^{k,\alpha}(\bar{\Omega})} = |u|_{k,\alpha;\Omega} = \|u\|_{C^k(\bar{\Omega})} + [u]_{k,\alpha;\Omega}. \tag{A.3.2}
$$

The spaces $C^{k,\alpha}(\bar{\Omega})$ equipped with these norms are Banach.

To work with interior estimates we introduce certain interior norms which will be useful later. For $\mathbf{x}, \mathbf{y} \in \Omega$, which is a proper open subset of $\mathbb{R}^n$, let us write $d_{\mathbf{x}} = \operatorname{dist}(\mathbf{x}, \partial\Omega)$, $d_{\mathbf{x},\mathbf{y}} = \min(d_{\mathbf{x}}, d_{\mathbf{y}})$. We define for $u \in C^k(\Omega)$, $C^{k,\alpha}(\Omega)$ the following quantities

$$
\begin{aligned}
&[u]^*_{k,0;\Omega} = [u]^*_{k;\Omega} = \sup_{\substack{\mathbf{x}\in\Omega\\ |\beta|=k}} d^k_{\mathbf{x}} |D^{\beta} u(\mathbf{x})|,\ k = 1, 2, \ldots;\\
&|u|^*_{k,0;\Omega} = |u|^*_{k;\Omega} = \sum_{j=0}^{k} [u]^*_{j;\Omega};\\
&[u]^*_{k,\alpha;\Omega} = \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega\\ \mathbf{x}\neq\mathbf{y}}} d^{k+\alpha}_{\mathbf{x},\mathbf{y}} \frac{|D^{\beta} u(\mathbf{x}) - D^{\beta} u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}},\ 0 < \alpha \le 1;\\
&|u|^*_{k,\alpha;\Omega} = |u|^*_{k;\Omega} + [u]^*_{k,\alpha;\Omega}.
\end{aligned} \tag{A.3.3}
$$

In this notation,

$$
[u]^*_{0;\Omega} = |u|^*_{0;\Omega} = |u|_{0;\Omega}.
$$

We note that $|u|^*_{k;\Omega}$ and $|u|^*_{k,\alpha;\Omega}$ are norms on the subspaces of $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ respectively for which they are finite. If $\Omega' \subset\subset \Omega$ and $\sigma = \operatorname{dist}(\Omega', \partial\Omega)$, then

$$
\min(1, \sigma^{k+\alpha}) |u|_{k,\alpha;\Omega'} \le |u|^*_{k,\alpha;\Omega}. \tag{A.3.4}
$$

To develop a global theory we need to introduce some quantities that involve boundary values, and here we simply start with Ω some proper open subset of $\mathbb{R}^n_+$ with open boundary portion T on $x_n = 0$. For $\mathbf{x}, \mathbf{y} \in \Omega$ let us write

$$
\bar{d}_{\mathbf{x}} = \operatorname{dist}(\mathbf{x}, \partial\Omega \backslash T),\ \bar{d}_{\mathbf{x},\mathbf{y}} = \min(\bar{d}_{\mathbf{x}}, \bar{d}_{\mathbf{y}}).
$$

The quantities are:

$$\begin{aligned}
&[u]^*_{k,0;\Omega\cup T}=[u]^*_{k;\Omega\cup T}=\sup_{\substack{\mathbf{x}\in\Omega\\|\beta|=k}}\bar{d}^k_{\mathbf{x}}|D^\beta u(\mathbf{x})|,\ k=0,1,2,\ldots;\\
&|u|^*_{k,0;\Omega\cup T}=|u|^*_{k;\Omega\cup T}=\sum_{j=0}^{k}[u]^*_{j;\Omega\cup T};\\
&[u]^*_{k,\alpha;\Omega\cup T}=\sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega\\|\beta|=k}}\bar{d}^{k+\alpha}_{\mathbf{x},\mathbf{y}}\frac{|D^\beta u(\mathbf{x})-D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha},\ 0<\alpha\le 1;\\
&|u|^*_{k,\alpha;\Omega\cup T}=|u|^*_{k;\Omega\cup T}+[u]^*_{k,\alpha;\Omega\cup T};\\
&|u|^{(k)}_{0,\alpha;\Omega\cup T}=\sup_{\mathbf{x}\in\Omega}\bar{d}^k_{\mathbf{x}}|u(\mathbf{x})|+\sup_{\mathbf{x},\mathbf{y}\in\Omega}\bar{d}^{k+\alpha}_{\mathbf{x},\mathbf{y}}\frac{|u(\mathbf{x})-u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}.
\end{aligned}\tag{A.3.5}$$

A.3.2 Hölder Estimates—the Preliminaries

Hölder Interior Estimates

In this subsection, we consider a special case–Poisson's equation: $\Delta u=f$.

Lemma A.3.1. *Let f be bounded and integrable in Ω, and let w be the Newtonian potential of f. Then $w\in C^1(\mathbb{R}^n)$ and for any $\mathbf{x}\in\Omega$,*

$$D_iw(\mathbf{x})=\int_\Omega D_i\Gamma(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y},\ i=1,\ldots,n.\tag{A.3.6}$$

证明. By the formulation of $D_i\Gamma$, the function

$$v(\mathbf{x})=\int_\Omega D_i\Gamma(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y}$$

is well-defined. To show that $v=D_iw$, we fix a function $\eta\in C^1(\mathbb{R})$ satisfying $0\le\eta\le 1$, $0\le\eta'\le 2$, $\eta(t)=0$ for $t\le 1$, $\eta(t)=1$ for $t\ge 2$ and define for $\epsilon>0$,

$$w_\epsilon(\mathbf{x})=\int_\Omega \Gamma\eta_\epsilon f(\mathbf{y})d\mathbf{y},\ \Gamma=\Gamma(\mathbf{x}-\mathbf{y}),\eta_\epsilon=\eta(|\mathbf{x}-\mathbf{y}|/\epsilon).$$

Clearly, $w_\epsilon\in C^1(\mathbb{R}^n)$ and

$$\begin{aligned}
|v(\mathbf{x})-D_iw_\epsilon(\mathbf{x})|&=\left|\int_{|\mathbf{x}-\mathbf{y}|\le 2\epsilon}D_i\left[(1-\eta_\epsilon)\Gamma\right]f(\mathbf{y})d\mathbf{y}\right|\\
&\le\|f\|_\infty\times\int_{|\mathbf{x}-\mathbf{y}|\le 2\epsilon}\left(|D_i\Gamma|+\frac{2}{\epsilon}|\Gamma|\right)d\mathbf{y}\\
&\le\|f\|_\infty\times\begin{cases}\frac{2n\epsilon}{n-2} & \text{for } n>2\\ 4\epsilon(1+|\log 2\epsilon|) & \text{for } n=2.\end{cases}
\end{aligned}$$

The results then follow from the above estimates. □

Lemma A.3.2. *Let f be bounded and locally Hölder continuous (with exponent $\alpha\le 1$) in Ω, and let w be the Newtonian potential of f. Then $w\in C^2(\Omega)$, $\Delta w=f$ in Ω, and for any*

$\mathbf{x} \in \Omega$,

$$D_{ij}w(\mathbf{x}) = \int_{\Omega_0} D_{ij}\Gamma(\mathbf{x}-\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}))d\mathbf{y} \\ - f(\mathbf{x}) \int_{\partial\Omega_0} D_i\Gamma(\mathbf{x}-\mathbf{y})\nu_j(\mathbf{y})ds_{\mathbf{y}},\ i,j = 1,\dots,n. \tag{A.3.7}$$

Here Ω_0 is any domain containing Ω for which the divergence theorem holds and f is extended to vanish outside Ω.

证明. The proof is similar to that of the preceding lemma. □

Corollary A.3.1. *Let Ω be a bounded domain and suppose that each point of $\partial\Omega$ is regular (with respect to the Laplacian). Then if f is bounded and locally Hölder continuous on Ω, the classical Dirichlet problem: $\Delta u = f$ in Ω, $u = \phi$ on $\partial\Omega$, is uniquely solvable for any continuous boundary values ϕ.*

The following estimates are the starting points of the future theory.

Lemma A.3.3. *Let $B_1 = B_R(\mathbf{x}_0)$, $B_2 = B_{2R}(\mathbf{x}_0)$ be concentric balls in $\mathbb{R}^n$. Suppose $f \in C^\alpha(\overline{B_2})$, $0 < \alpha < 1$, and let w be the Newtonian potential of f in B_2. Then $w \in C^{2,\alpha}(\overline{B_1})$ and*

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \le C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}), \tag{A.3.8}$$

where $C = C(n,\alpha)$.

证明. For any $\mathbf{x} \in B_1$, we have by formula (A.3.7),

$$D_{ij}w(\mathbf{x}) = \int_{B_2} D_{ij}\Gamma(\mathbf{x}-\mathbf{y})(f(\mathbf{y}) - f(\mathbf{x}))d\mathbf{y} - f(\mathbf{x}) \int_{\partial B_2} D_i\Gamma(\mathbf{x}-\mathbf{y})\nu_j(\mathbf{y})ds_{\mathbf{y}},$$

and so by direct computation

$$\begin{aligned} |D_{ij}w(\mathbf{x})| &\le \frac{|f(\mathbf{x})|}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_{\mathbf{y}} + \frac{[f]_{\alpha;\mathbf{x}}}{\omega_n} \int_{B_2} |\mathbf{x}-\mathbf{y}|^{\alpha-n} d\mathbf{y} \\ &\le 2^{n-1}|f(\mathbf{x})| + \frac{n}{\alpha}(3R)^\alpha [f]_{\alpha;\mathbf{x}} \\ &\le C_1 \left(|f(\mathbf{x})| + R^\alpha [f]_{\alpha;\mathbf{x}}\right), \end{aligned} \tag{A.3.9}$$

where $C_1 = C_1(n,\alpha)$.

Next, for any other point $\bar{\mathbf{x}} \in B_1$ we have again by (A.3.7),

$$D_{ij}w(\bar{\mathbf{x}}) = \int_{B_2} D_{ij}\Gamma(\bar{\mathbf{x}}-\mathbf{y})(f(\mathbf{y}) - f(\bar{\mathbf{x}}))d\mathbf{y} - f(\bar{\mathbf{x}}) \int_{\partial B_2} D_i\Gamma(\bar{\mathbf{x}}-\mathbf{y})\nu_j(\mathbf{y})ds_{\mathbf{y}}.$$

Writing $\delta = |\mathbf{x} - \bar{\mathbf{x}}|$, $\xi = (\mathbf{x} + \bar{\mathbf{x}})/2$, we consequently obtain by subtraction

$$D_{ij}w(\bar{\mathbf{x}}) - D_{ij}w(\mathbf{x}) = f(\mathbf{x})I_1 + (f(\mathbf{x}) - f(\bar{\mathbf{x}}))I_2 + I_3 + I_4 \\ + (f(\mathbf{x}) - f(\bar{\mathbf{x}}))I_5 + I_6,$$

where the integrals I_1, I_2, I_3, I_4, I_5 and I_6 are given by

$$\begin{aligned}
I_1 &= \int_{\partial B_2} \left(D_i\Gamma(\mathbf{x}-\mathbf{y}) - D_i\Gamma(\bar{\mathbf{x}}-\mathbf{y})\right) \nu_j(\mathbf{y}) ds_{\mathbf{y}} \\
I_2 &= \int_{\partial B_2} D_i\Gamma(\bar{\mathbf{x}}-\mathbf{y})\nu_j(\mathbf{y}) ds_{\mathbf{y}} \\
I_3 &= \int_{B_\delta(\xi)} D_{ij}\Gamma(\mathbf{x}-\mathbf{y})(f(\mathbf{x})-f(\mathbf{y}))d\mathbf{y} \\
I_4 &= \int_{B_\delta(\xi)} D_{ij}\Gamma(\bar{\mathbf{x}}-\mathbf{y})(f(\mathbf{y})-f(\bar{\mathbf{x}}))d\mathbf{y} \\
I_5 &= \int_{B_2\setminus B_\delta(\xi)} D_{ij}\Gamma(\mathbf{x}-\mathbf{y})d\mathbf{y} \\
I_6 &= \int_{B_2\setminus B_\delta(\xi)} \left(D_{ij}\Gamma(\mathbf{x}-\mathbf{y}) - D_{ij}\Gamma(\bar{\mathbf{x}}-\mathbf{y})\right)\left(f(\bar{\mathbf{x}})-f(\mathbf{y})\right) d\mathbf{y}.
\end{aligned}$$

The estimation of these integrals can be achieved as follows:

- $$\begin{aligned}
|I_1| &\le |\mathbf{x}-\bar{\mathbf{x}}| \int_{\partial B_2} |DD_i\Gamma(\hat{\mathbf{x}}-\mathbf{y})| ds_{\mathbf{y}} \quad \text{for some } \mathbf{x} \text{ between } \mathbf{x} \text{ and } \bar{\mathbf{x}}, \\
&\le \frac{n^2 2^{n-1}|\mathbf{x}-\bar{\mathbf{x}}|}{R}, \text{ since } |\hat{\mathbf{x}}-\mathbf{y}| \ge R, \forall \mathbf{y} \in \partial B_2, \\
&\le n^2 2^{n-\alpha}\left(\frac{\delta}{R}\right)^{\alpha}, \text{ since } \delta = |\mathbf{x}-\bar{\mathbf{x}}| < 2R.
\end{aligned}$$

- $$|I_2| \le \frac{1}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_{\mathbf{y}} = 2^{n-1}.$$

- $$\begin{aligned}
|I_3| &\le \int_{B_\delta(\xi)} |D_{ij}\Gamma(\mathbf{x}-\mathbf{y})||f(\mathbf{x})-f(\mathbf{y})|d\mathbf{y} \\
&\le \frac{1}{\omega_n}[f]_{\alpha;\mathbf{x}} \int_{B_{3\delta/2}} |\mathbf{x}-\mathbf{y}|^{\alpha-n} d\mathbf{y} \\
&\le \frac{n}{\alpha}\left(\frac{3\delta}{2}\right)^{\alpha} [f]_{\alpha;\mathbf{x}}.
\end{aligned}$$

- $$|I_4| \le \frac{n}{\alpha}\left(\frac{3\delta}{2}\right)^{\alpha} [f]_{\alpha;\bar{\mathbf{x}}}, \text{ as in the estimation of } I_3.$$

- Integration by parts gives

$$\begin{aligned}
|I_5| &= \left|\int_{\partial(B_2\setminus B_\delta(\xi))} D_i\Gamma(\mathbf{x}-\mathbf{y})\nu_j(\mathbf{y}) ds_{\mathbf{y}}\right| \\
&\le \left|\int_{\partial B_2} D_i\Gamma(\mathbf{x}-\mathbf{y})\nu_j(\mathbf{y}) ds_{\mathbf{y}}\right| + \left|\int_{\partial B_\delta(\xi)} D_i\Gamma(\mathbf{x}-\mathbf{y})\nu_j(\mathbf{y}) ds_{\mathbf{y}}\right| \\
&\le 2^{n-1} + \frac{1}{n\omega_n}\left(\frac{\delta}{2}\right)^{1-n} \int_{\partial B_\delta(\xi)} ds_{\mathbf{y}} = 2^n.
\end{aligned}$$

•

$$\begin{aligned}|I_6| &\leq |\mathbf{x}-\bar{\mathbf{x}}| \int_{B_2\setminus B_\delta(\xi)} |DD_{ij}\Gamma(\hat{\mathbf{x}}-\mathbf{y})||f(\bar{\mathbf{x}})-f(\mathbf{y})|d\mathbf{y}\\ &\leq c\delta \int_{|\mathbf{y}-\xi|\geq\delta} \frac{|f(\bar{\mathbf{x}})-f(\mathbf{y})|}{|\hat{\mathbf{x}}-\mathbf{y}|^{n+1}} d\mathbf{y},\ c=n(n+5)/\omega_n\\ &\leq c\delta[f]_{\alpha;\bar{\mathbf{x}}} \int_{|\mathbf{y}-\xi|\geq\delta} \frac{|\bar{\mathbf{x}}-\mathbf{y}|^\alpha}{|\bar{\mathbf{x}}-\mathbf{y}|^{n+1}} d\mathbf{y},\\ &\leq c\left(\frac{3}{2}\right)^\alpha 2^{n+1}\delta[f]_{\alpha;\bar{\mathbf{x}}} \int_{|y-\xi|\geq\delta} |\xi-\mathbf{y}|^{\alpha-n-1} d\mathbf{y}, \quad \circ\\ &\leq \frac{n^2(n+5)}{1-\alpha} 2^{n+1}\left(\frac{3}{2}\right)^\alpha \delta^\alpha [f]_{\alpha;\bar{\mathbf{x}}},\end{aligned}$$

where "$\circ$" holds because $|\bar{\mathbf{x}}-\mathbf{y}| \leq \frac{3}{2}|\xi-\mathbf{y}| \leq 3|\hat{\mathbf{x}}-\mathbf{y}|$.

Now, collecting terms we have

$$|D_{ij}w(\bar{\mathbf{x}}) - D_{ij}w(\mathbf{x})| \leq C_2\left(R^{-\alpha}|f(\mathbf{x})| + [f]_{\alpha;\mathbf{x}} + [f]_{\alpha;\bar{\mathbf{x}}}\right)|\mathbf{x}-\bar{\mathbf{x}}|^\alpha, \tag{A.3.10}$$

where constant C_2 depends only on n and α. □

Now, for bounded domains Ω with $d = \mathrm{diam}\Omega$ we define non-dimensional norms on $C^k(\bar{\Omega})$ and $C^{k,\alpha}(\bar{\Omega})$

$$\|u\|'_{C^k(\bar{\Omega})} = |u|'_{k;\Omega} = \sum_{j=0}^{k} d^j [u]_{j,0;\Omega};$$

$$\|u\|'_{C^{k,\alpha}(\bar{\Omega})} = |u|'_{k;\Omega} + d^{k+\alpha}[D^k u]_{\alpha;\Omega}.$$

Theorem A.3.1. *Let Ω be a domain in $\mathbb{R}^n$ and let $u \in C^2(\Omega)$, $f \in C^\alpha(\Omega)$, satisfy Poisson's equation $\Delta u = f$ in Ω. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 = B_R(\mathbf{x}_0)$, $B_2 = B_{2R}(\mathbf{x}_0) \subset\subset \Omega$ we have*

$$|u|'_{2,\alpha;B_1} \leq C(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2}), \tag{A.3.11}$$

where $C = C(n,\alpha)$.

证明. We can write for $\mathbf{x} \in B_2$, $u(\mathbf{x}) = v(\mathbf{x}) + w(\mathbf{x})$, where v is harmonic in B_2 and w is the Newtonian potential of f in B_2. By previous estimates and the representation of Dw in terms of f we have

$$R|Dw|_{0,B_1} + R^2|D^2w|'_{0,\alpha;B_1} \leq CR^2|f|'_{0,\alpha;B_2}.$$

Since v is harmonic, we have

$$R|Dv|_{0,B_1} + R^2|D^2v|'_{0,\alpha;B_1} \leq C|v|_{0;B_2} \leq C(|u|_{0;B_2} + R^2|f|_{0,B_2}).$$

The last inequality reuses the formula $v = u - w$. □

Theorem A.3.2. (**Hölder Interior Estimates**) *Let $u \in C^2(\Omega)$, $f \in C^\alpha(\Omega)$ satisfy $\Delta u = f$ in an open set Ω of $\mathbb{R}^n$. Then*

$$|u|^*_{2,\alpha;\Omega} \le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}), \tag{A.3.12}$$

where $C = C(n, \alpha)$.

证明. If either of $|u|_{0;\Omega}$ or $|f|^{(2)}_{0,\alpha;\Omega}$ is infinite, we are done. Otherwise for $\mathbf{x} \in \Omega$, $R = d_{\mathbf{x}}/3$, $B_1 = B_R(\mathbf{x})$, $B_2 = B_{2R}(\mathbf{x})$, we have for any first derivative Du and second derivative D^2u

$$\begin{aligned} d_{\mathbf{x}}|Du(\mathbf{x})| + d^2_{\mathbf{x}}|D^2u(\mathbf{x})| &\le (3R)|Du|_{0;B_1} + (3R)^2|D^2u|_{0;B_1} \\ &\le C(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2}) \\ &\le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}). \end{aligned}$$

Hence we obtain

$$|u|^*_{2;\Omega} \le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}).$$

To estimate $[u]^*_{2,\alpha;\Omega}$ we let $\mathbf{x}, \mathbf{y} \in \Omega$ with $d_{\mathbf{x}} \le d_{\mathbf{y}}$. Then

$$\begin{aligned} d^{2+\alpha}_{\mathbf{x},\mathbf{y}} \frac{|D^2u(\mathbf{x}) - D^2(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} &\le (3R)^{2+\alpha}[D^2u]_{\alpha;B_1} + 3^\alpha(3R)^2(|D^2u(\mathbf{x})| + |D^2u(\mathbf{y})|) \\ &\le C(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2}) + 6[u]'_{2;\Omega} \\ &\le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}). \end{aligned}$$

□

Hölder Boundary Estimates near Hyperplane boundary portion

In what follows, $\mathbb{R}^n_+$ will denote the half-space, $x_n > 0$, and T the hyperplane, $x_n = 0$; $B_2 = B_{2R}(\mathbf{x}_0)$, $B_1 = B_R(\mathbf{x}_0)$ with center $\mathbf{x}_0 \in \overline{\mathbb{R}^n_+}$, and we let $B_2^+ = B_2 \cap \mathbb{R}^n_+$, $B_1^+ = B_1 \cap \mathbb{R}^n_+$.

Lemma A.3.4. *Let $f \in C^\alpha(\overline{B_2^+})$, and let w be the Newtonian potential of f in B_2^+. Then $w \in C^{2,\alpha}(\overline{B_1^+})$ and*

$$|D^2w|'_{0,\alpha;B_1^+} \le C|f|'_{0,\alpha;B_2^+} \tag{A.3.13}$$

where $C = C(n, \alpha)$.

证明. We assume that B_2 intersects T since otherwise the result is already discussed before. The integral representation (A.3.7) of $D_{ij}w$ still holds for $\Omega = B_2^+$. If either i or $j \ne n$, then the portion of the boundary integral

$$\int_{\partial B_2^+} D_i\Gamma(\mathbf{x}-\mathbf{y})\nu_j(\mathbf{y})ds_{\mathbf{y}}$$

on T vanishes since ν_i or $\nu_j = 0$ there. The previous methods in estimating $D_{ij}w$ (i or $j \ne n$) still work. Finally $D_{nn}w$ can be estimated from the equation $\Delta w = f$ and the estimates on $D_{kk}w$ for $k = 1, \ldots, n-1$. □

Theorem A.3.3. *Let $u \in C^2(B_2^+) \cap C^0(\overline{B_2^+})$, $f \in C^\alpha(\overline{B_2^+})$, satisfy $\Delta u = f$ in B_2^+, $u = 0$ on T. Then $u \in C^{2,\alpha}(\overline{B_1^+})$ and we have*

$$|u|'_{2,\alpha;B_1^+} \leq C(|u|_{0,\alpha;B_2^+} + R^2|f|'_{0,\alpha;B_2^+}) \tag{A.3.14}$$

where $C = C(n,\alpha)$.

证明. Let $\mathbf{x}' = (x_1, \ldots, x_{n-1})$, $\mathbf{x}^* = (\mathbf{x}', -x_n)$ and define

$$f^*(\mathbf{x}) = f^*(\mathbf{x}', x_n) = \begin{cases} f(\mathbf{x}', x_n) & \text{if } x_n \geq 0 \\ f(\mathbf{x}', -x_n) & \text{if } x_n \leq 0. \end{cases}$$

We assume that B_2 intersects T or we are done. We set $B_2^- = \{\mathbf{x} \in \mathbb{R}^n \ ; \ \mathbf{x}^* \in B_2^+\}$ and $D = B_2^+ \cup B_2^- \cup (B_2 \cap T)$. Then $f^* \in C^\alpha(\bar{D})$ and $|f^*|'_{0,\alpha;D} \leq 2|f|'_{0,\alpha;B_2^+}$. Now, defining

$$\begin{aligned} w(\mathbf{x}) &= \int_{B_2^+} [\Gamma(\mathbf{x}-\mathbf{y}) - \Gamma(\mathbf{x}^*-\mathbf{y})]\, f(\mathbf{y})d\mathbf{y} \\ &= \int_{B_2^+} [\Gamma(\mathbf{x}-\mathbf{y}) - \Gamma(\mathbf{x}-\mathbf{y}^*)]\, f(\mathbf{y})d\mathbf{y}, \end{aligned} \tag{A.3.15}$$

we have $w(\mathbf{x}', 0) = 0$ and $\Delta w = f$ in B_2^+. Noting that

$$\int_{B_2^+} \Gamma(\mathbf{x}-\mathbf{y}^*) f(\mathbf{y}) d\mathbf{y} = \int_{B_2^-} \Gamma(\mathbf{x}-\mathbf{y}) f^*(\mathbf{y}) d\mathbf{y},$$

we then obtain

$$w(\mathbf{x}) = 2\int_{B_2^+} \Gamma(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} - \int_D \Gamma(\mathbf{x}-\mathbf{y}) f^*(\mathbf{y}) d\mathbf{y}.$$

Letting $w^*(\mathbf{x}) = \int_D \Gamma(\mathbf{x}-\mathbf{y}) f^*(\mathbf{y}) d\mathbf{y}$, we have

$$|D^2 w^*|'_{0,\alpha;B_1^+} \leq C|f^*|'_{0,\alpha;D} \leq 2C|f|'_{0,\alpha;B_2^+}.$$

Combining this with previous lemma, we obtain

$$|D^2 w^*|'_{0,\alpha;B_1^+} \leq C|f|'_{0,\alpha;B_2^+}. \tag{A.3.16}$$

Now let $v = u - w$. Then $\Delta v = 0$ in B_2^+ and $v = 0$ on T. By reflection v may be extended to a harmonic function on B_2 and hence the desired estimate follows from the interior derivative estimate for harmonic functions.

□

We can now state:

Theorem A.3.4. *Let Ω be an open set in $\mathbb{R}^n_+$ with a boundary portion T on $x_n = 0$, and let $u \in C^2(\Omega) \cap C^0(\Omega \cup T)$, $f \in C^\alpha(\Omega \cup T)$ satisfy $\Delta u = f$ in Ω, $u = 0$ on T. Then*

$$|u|^*_{2,\alpha;\Omega\cup T} \leq C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega\cup T}), \tag{A.3.17}$$

where $C = C(n,\alpha)$.

Theorem A.3.5. *Let B be a ball in $\mathbb{R}^n$ and $u \in C^2(B) \cap C^0(\bar{B})$, $f \in C^\alpha(\bar{B})$, $\Delta u = f$ in B, $u = 0$ on ∂B. Then $u \in C^{2,\alpha}(\bar{B})$.*

证明. By a translation we may assume that ∂B passes through the origin. The inversion mapping $\mathbf{x} \mapsto \mathbf{x}^* = \mathbf{x}/|\mathbf{x}|^2$ is a homeomorphic, smooth mapping of $\mathbb{R}^n \backslash \{\mathbf{0}\}$ onto itself which maps B onto a half-space, B^*. Furthermore, the *Kelvin transform* of u defined by

$$v(\mathbf{x}) = |\mathbf{x}|^{2-n} u\left(\frac{\mathbf{x}}{|\mathbf{x}|^2}\right) \tag{A.3.18}$$

belongs to $C^2(B^*) \cap C^0(\overline{B^*})$ and satisfies

$$\begin{aligned} \Delta_{\mathbf{x}^*} v(\mathbf{x}^*) &= |\mathbf{x}^*|^{-n-2} \Delta_{\mathbf{x}} u(\mathbf{x}),\ \mathbf{x}^* \in B^*,\ \mathbf{x} \in B \\ &= |\mathbf{x}^*|^{-n-2} f\left(\frac{\mathbf{x}^*}{|\mathbf{x}^*|^2}\right). \end{aligned} \tag{A.3.19}$$

Hence we may now apply the preceding estimates. Since by translation any point of ∂B can be taken for the origin we obtain $u \in C^{2,\alpha}(\bar{B})$.

□

A.3.3 Schauder Interior Estimates

We denote by $Lu = f$ the equation

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f,\ a^{ij} = a^{ji},$$

where the coefficients and f are defined in an open set $\Omega \subset \mathbb{R}^n$ and the operator L is *strictly elliptic* if otherwise stated:

$$a^{ij}(\mathbf{x}) \xi_i \xi_j \geq \lambda |\xi|^2,\ \forall \mathbf{x} \in \Omega,\ \xi \in \mathbb{R}^n,$$

for some positive constant λ.

To obtain estimates of the interior norm $|u|^*_{2,\alpha;\Omega}$ of solutions of $Lu = f$ in Ω, it suffices to bound only $|u|_{0;\Omega}$ and the seminorm $[u]^*_{2,\alpha;\Omega}$. That this is so is a consequence of the following *interpolation inequalities*: Let $u \in C^{2,\alpha}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. Then for any $\epsilon > 0$ there is a constant $C = C(\epsilon)$ such that

$$[u]^*_{j,\beta;\Omega} \leq C|u|_{0;\Omega} + \epsilon [u]^*_{2,\alpha;\Omega}, \tag{A.3.20}$$

$$|u|^*_{j,\beta;\Omega} \leq C|u|_{0;\Omega} + \epsilon [u]^*_{2,\alpha;\Omega}, \tag{A.3.21}$$

where $j = 0, 1, 2$, $0 \leq \alpha, \beta \leq 1$ and $j + \beta < 2 + \alpha$ (see Appendix 1 in Chapter 6 of Gilbarg-Trudinger).

For future convenience we define the following quantities on the spaces $C^k(\Omega), C^{k,\alpha}(\Omega)$.

For σ a real number and k a nonnegative integer we define

$$\begin{aligned}
[f]^{(\sigma)}_{k,0;\Omega} &= [f]^{(\sigma)}_{k;\Omega} = \sup_{\substack{\mathbf{x}\in\Omega \\ |\beta|=k}} d_{\mathbf{x}}^{k+\sigma}|D^\beta f(\mathbf{x})|; \\
[f]^{(\sigma)}_{k,\alpha;\Omega} &= \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega \\ |\beta|=k}} d_{\mathbf{x},\mathbf{y}}^{k+\alpha+\sigma}\frac{|D^\beta f(\mathbf{x}) - D^\beta f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha},\ 0<\alpha\le 1; \\
|f|^{(\sigma)}_{k;\Omega} &= \sum_{j=0}^{k}[f]^{(\sigma)}_{j;\Omega}; \\
|f|^{(\sigma)}_{k,\alpha;\Omega} &= |f|^{(\sigma)}_{k;\Omega} + [f]^{(\sigma)}_{k,\alpha;\Omega}.
\end{aligned} \tag{A.3.22}$$

It is easy to verify that

$$|fg|^{(\sigma+\tau)}_{0,\alpha;\Omega} \le |f|^{(\sigma)}_{0,\alpha;\Omega}|g|^{(\tau)}_{0,\alpha;\Omega},\ \text{for}\ \sigma+\tau\ge 0. \tag{A.3.23}$$

We now establish the basic Schauder interior estimates.

Theorem A.3.6. (**Schauder Interior Estimates**) *Let Ω be an open subset of $\mathbb{R}^n$, and let $u\in C^{k,\alpha}(\Omega)$ be a bounded solution in Ω of the equation $Lu=f$, where $f\in C^\alpha(\Omega)$ and there are positive constants λ,Λ such that*

$$a^{ij}(\mathbf{x})\xi_i\xi_j \ge \lambda|\xi|^2,\ \forall \mathbf{x}\in\Omega,\ \xi\in\mathbb{R}^n,$$

and

$$|a^{ij}|^{(0)}_{0,\alpha;\Omega},\ |b^i|^{(1)}_{0,\alpha;\Omega},\ |c|^{(2)}_{0,\alpha;\Omega}\le\Lambda.$$

Then

$$|u|^*_{2,\alpha;\Omega}\le C(|u|_{0;\Omega}+|f|^{(2)}_{0,\alpha;\Omega}), \tag{A.3.24}$$

where $C=C(n,\alpha,\lambda,\Lambda)$.

证明. By interpolation inequalities, it suffices to proof the inequality for $[u]^*_{2,\alpha;\Omega}$ and a further observation shows that we only have to prove the latter for compact subsets of Ω. Namely, let $\{\Omega_i\}$ be a sequence of open subsets of Ω such that $\Omega_i\subset\Omega_{i+1}\subset\subset\Omega$ and $\cup\Omega_i=\Omega$. We have that $[u]^*_{2,\alpha;\Omega_i}$ is finite for each i. Now, if the desired inequality is true for Ω_i we have for $\mathbf{x},\mathbf{y}\in\Omega$ and sufficiently large i and any second derivative D^2u

$$\begin{aligned}
(d^{(i)}_{\mathbf{x},\mathbf{y}})^{2+\alpha}\frac{|D^2u(\mathbf{x})-D^2u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} &\le [u]^*_{2,\alpha;\Omega_i} \\
&\le C(|u|_{0;\Omega_i}+|f|^{(2)}_{0,\alpha;\Omega_i}) \\
&\le C(|u|_{0;\Omega}+|f|^{(2)}_{0,\alpha;\Omega}),
\end{aligned}$$

where $d^{(i)}_{\mathbf{x},\mathbf{y}}=\min(\mathrm{dist}(\mathbf{x},\partial\Omega_i),\mathrm{dist}(\mathbf{y},\partial\Omega_i))$. Sending $i\to\infty$, we obtain the inequality

$$d^{2+\alpha}_{\mathbf{x},\mathbf{y}}\frac{|D^2u(\mathbf{x})-D^2u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}\le C(|u|_{0;\Omega}+|f|^{(2)}_{0,\alpha;\Omega}).$$

Now, we may without losing generality assume that $[u]^*_{2,\alpha;\Omega}$ is finite. For notational convenience we use C to be a universal constant that only depend on $n, \alpha, \lambda, \Lambda$.

Let $\mathbf{x}_0, \mathbf{y}_0$ be any two distinct points in Ω and suppose $d_{\mathbf{x}_0} = d_{\mathbf{x}_0,\mathbf{y}_0}$. Let $\mu \leq 1/2$ be a positive constant to be fixed later, and set $d = \mu d_{\mathbf{x}_0}$, $B = B_d(\mathbf{x}_0)$. We rewrite $Lu = f$ in the form

$$a^{ij}(\mathbf{x}_0)D_{ij}u = (a^{ij}(\mathbf{x}_0) - a^{ij}(\mathbf{x}))D_{ij}u - b^i D_i u - cu + f \equiv F(\mathbf{x}), \tag{A.3.25}$$

and we consider this as an equation in B with constant coefficients $a^{ij}(\mathbf{x}_0)$. Applying Hölder estimates with a proper linear transformation to this equation we know that if $\mathbf{y}_0 \in B_{d/2}(\mathbf{x}_0)$, then for any second order derivative D^2u

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2u(\mathbf{x}_0) - D^2u(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \leq C\left(|u|_{0;B} + |F|^{(2)}_{0,\alpha;B}\right);$$

and thus

$$d^{2+\alpha}_{\mathbf{x}_0} \frac{|D^2u(\mathbf{x}_0) - D^2u(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \leq \frac{C}{\mu^{2+\alpha}}\left(|u|_{0;B} + |F|^{(2)}_{0,\alpha;B}\right).$$

On the other hand, if $|\mathbf{x}_0 - \mathbf{y}_0| \geq d/2$,

$$\begin{aligned}
d^{2+\alpha}_{\mathbf{x}_0} \frac{|D^2u(\mathbf{x}_0) - D^2u(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} &\leq \left(\frac{2}{\mu}\right)^\alpha \left[d^2_{\mathbf{x}_0}|D^2u(\mathbf{x}_0)| + d^2_{\mathbf{y}_0}|D^2u(\mathbf{y}_0)|\right] \\
&\leq \frac{4}{\mu^\alpha}[u]^*_{2;\Omega}.
\end{aligned}$$

Therefore, combining these inequalities we have

$$d^{2+\alpha}_{\mathbf{x}_0} \frac{|D^2u(\mathbf{x}_0) - D^2u(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \leq \frac{C}{\mu^{2+\alpha}}\left(|u|_{0;B} + |F|^{(2)}_{0,\alpha;B}\right) + \frac{4}{\mu^\alpha}[u]^*_{2;\Omega}. \tag{A.3.26}$$

We proceed to estimate $|F|^{(2)}_{0,\alpha;B}$ in terms of $|u|_{0;\Omega}$ and $[u]^*_{2,\alpha;\Omega}$. We have

$$\begin{aligned}
|F|^{(2)}_{0,\alpha;B} \leq \sum_{i,j} |(a^{ij}(\mathbf{x}_0) - a^{ij}(\mathbf{x}))D_{ij}u|^{(2)}_{0,\alpha;B} + |\sum_i b^i D_i u|^{(2)}_{0,\alpha;B} \\
+ |cu|^{(2)}_{0,\alpha;B} + |f|^{(2)}_{0,\alpha;B}.
\end{aligned} \tag{A.3.27}$$

It will be useful in estimating these terms to have the following inequality. Recalling that for all $\mathbf{x} \in B$, $d_\mathbf{x} > (1-\mu)d_{\mathbf{x}_0} \geq d_\mathbf{x}/2$, we have for $g \in C^\alpha(\Omega)$

$$\begin{aligned}
|g|^{(2)}_{0,\alpha;B} &\leq d^2|g|_{0;B} + d^{2+\alpha}[g]_{\alpha;B} \\
&\leq \frac{\mu^2}{(1-\mu)^2}[g]^{(2)}_{0;\Omega} + \frac{\mu^{2+\alpha}}{(1-\mu)^{2+\alpha}}[g]^{(2)}_{0,\alpha;\Omega} \\
&\leq 4\mu^2[g]^{(2)}_{0;\Omega} + 8\mu^{2+\alpha}[g]^{(2)}_{0,\alpha;\Omega} \\
&\leq 8\mu^2|g|^{(2)}_{0,\alpha;\Omega}.
\end{aligned} \tag{A.3.28}$$

By (A.3.23) and the above inequality, we obtain

$$\begin{aligned}
|(a(\mathbf{x}_0) - a(\mathbf{x}))D^2u|^{(2)}_{0,\alpha;B} &\leq |a(\mathbf{x}_0) - a(\mathbf{x})|^{(0)}_{0,\alpha;B}|D^2u|^{(2)}_{0,\alpha;B} \\
&\leq |a(\mathbf{x}_0) - a(\mathbf{x})|^{(0)}_{0,\alpha;B}(4\mu^2[u]^{(2)}_{2;\Omega} + 8\mu^{2+\alpha}[u]^{(2)}_{2,\alpha;\Omega}),
\end{aligned}$$

where we write $a(\cdot)D^2u = a^{ij}(\cdot)D_{ij}u$. Since

$$|a(\mathbf{x}_0) - a(\mathbf{x})|^{(0)}_{0,\alpha;B} \le \sup_{\mathbf{x}\in B} |a(\mathbf{x}_0) - a(\mathbf{x})| + d^\alpha [a]_{\alpha;B} \le 4\Lambda\mu^\alpha,$$

we arrive at the following estimate for the principal term in (A.3.27),

$$\begin{aligned} |(a^{ij}(\mathbf{x}_0) - a^{ij}(\mathbf{x}))D_{ij}u|^{(2)}_{0,\alpha;B} &\le 32n^2\Lambda\mu^{2+\alpha}\left([u]^*_{2;\Omega} + \mu^\alpha [u]^*_{2,\alpha;\Omega}\right) \\ &\le 32n^2\Lambda\mu^{2+\alpha}\left(C(\mu)|u|_{0;\Omega} + 2\mu^\alpha [u]^*_{2,\alpha;\Omega}\right). \end{aligned} \tag{A.3.29}$$

The last inequality is obtained by setting $\epsilon = \mu^\alpha$ in the interpolation inequality.

Writing $bDu = b^i D_i u$ for each i, we obtain

$$\begin{aligned} |bDu|^{(2)}_{0,\alpha;B} &\le 8\mu^2 |bDu|^{(2)}_{0,\alpha;\Omega} \\ &\le 8\mu^2 |b|^{(1)}_{0,\alpha;\Omega} |Du|^{(1)}_{0,\alpha;\Omega} \\ &\le 8\mu^2 \Lambda |u|^*_{1,\alpha;\Omega} \\ &\le 8\mu^2\Lambda\left(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]^*_{2,\alpha;\Omega}\right). \end{aligned} \tag{A.3.30}$$

The last inequality is obtained by setting $\epsilon = \mu^{2\alpha}$ in the interpolation inequality.

Similarly, we have

$$\begin{aligned} |cu|^{(2)}_{0,\alpha;B} &\le 8\mu^2 |c|^{(2)}_{0,\alpha;\Omega} |u|^{(0)}_{0,\alpha;\Omega} \\ &\le 8\Lambda\mu^2\left(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]^*_{2,\alpha;\Omega}\right). \end{aligned} \tag{A.3.31}$$

Finally,

$$|f|^{(2)}_{0,\alpha;B} \le 8\mu^2 |f|^{(2)}_{0,\alpha;\Omega}. \tag{A.3.32}$$

Letting C denote constant that depends only on $n, \alpha, \lambda, \Lambda$ and $C(\mu)$ constants depending also on μ, we find

$$|F|^{(2)}_{0,\alpha;B} \le C\mu^{2+2\alpha}[u]^*_{2,\alpha;\Omega} + C(\mu)(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}).$$

Inserting this into the right member of (A.3.26), and using interpolation with $\epsilon = \mu^{2\alpha}$ to estimate $[u]^*_{2;\Omega}$, we obtain

$$d^{2+\alpha}_{\mathbf{x}_0,\mathbf{y}_0} \frac{|D^2u(\mathbf{x}_0) - D^2u(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \le C\mu^\alpha [u]^*_{2,\alpha;\Omega} + C(\mu)\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right).$$

Taking the supremum over all $\mathbf{x}_0, \mathbf{y}_0 \in \Omega$, we obtain

$$[u]^*_{2,\alpha;\Omega} \le C\mu^\alpha [u]^*_{2,\alpha;\Omega} + C(\mu)\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right).$$

Then we arrive at the desired inequality if we set μ to be small.

□

It is usually adequate to know equicontinuity of solutions and their derivatives up to second order on compact subsets.

Corollary A.3.2. *Let $u \in C^{2,\alpha}(\Omega)$, $f \in C^{\alpha}(\bar{\Omega})$ satisfy $Lu = f$ in a bounded domain Ω where L is strictly elliptic and its coefficients are $C^{\alpha}(\bar{\Omega})$. Then if $\Omega' \subset\subset \Omega$ with dist$(\Omega', \partial\Omega) \geq d$, there is a constant C such that*

$$d|Du|_{0;\Omega'} + d^2|D^2u|_{0;\Omega'} + d^{2+\alpha}[D^2u]_{\alpha;\Omega'} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right), \tag{A.3.33}$$

where C depends only on $n, \alpha, \lambda, \Lambda$, diam Ω.

Remark: An immediate consequence is that uniformly bounded solutions to the equation $Lu = f$ are equicontinuous with their first and second derivatives on compact subsets. This is also true for any family of equations with close constants listed in the above corollary.

A.3.4 Boundary and Global Estimates

Definition A.3. *A bounded domain Ω in $\mathbb{R}^n$ and its boundary are of class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, if at each point $\mathbf{x}_0 \in \partial\Omega$ there is a ball $B = B(\mathbf{x}_0)$ and a one-to one mapping ψ of B onto $D \subset \mathbb{R}^n$ such that*

(i) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$;

(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;

(iii) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

A domain Ω will be said to have a boundary portion $T \subset \partial\Omega$ of class $C^{k,\alpha}$ if at each point $\mathbf{x}_0 \in T$ there is a ball centered at it, in which the above conditions are satisfied. We shall say that the diffeomorphism ψ straightens the boundary *near $\mathbf{x}_0$.*

We note in particular that Ω is a $C^{k,\alpha}$ domain if each point of $\partial\Omega$ has a neighborhood in which $\partial\Omega$ is the graph of a $C^{k,\alpha}$ function of $n-1$ of the coordinates $x_1, \dots, x_n$. The converse is also true for $k \geq 1$.

A function ϕ defined on a $C^{k,\alpha}$ boundary portion T of a domain Ω will be said to be in class $C^{k,\alpha}(T)$ if $\phi \circ \psi^{-1}_{\mathbf{x}_0} \in C^{k,\alpha}(D \cap \partial\mathbb{R}^n_+)$ for each $\mathbf{x}_0 \in T$. It is important to note that if $\partial\Omega$ is of $C^{k,\alpha}$ $(k \geq 1)$, then a function $\phi \in C^{k,\alpha}(\partial\Omega)$ can be extended to a function in $C^{k,\alpha}(\bar{\Omega})$ (see Appendix 2 in Chapter 6 of Gilbarg-Trudinger). Conversely, any function in $C^{k,\alpha}(\bar{\Omega})$ has boundary values in $C^{k,\alpha}(\partial\Omega)$.

It is also possible to define a boundary norm on $C^{k,\alpha}(\partial\Omega)$, in various ways. For example, if $\phi \in C^{k,\alpha}(\partial\Omega)$, let Φ denote an extension of ϕ to $\bar{\Omega}$ and define

$$\|\phi\|_{C^{k,\alpha}(\partial\Omega)} = \inf_{\Phi} \|\Phi\|_{C^{k,\alpha}(\bar{\Omega})}.$$

Equipped with this norm, the space $C^{k,\alpha}(\partial\Omega)$ becomes Banach.

In obtaining boundary estimates for $Lu = f$ in domains with a $C^{2,\alpha}$ $(\alpha > 0)$ boundary portion we first establish such an estimate in domains with a hyperplane boundary portion. Let us first introduce the corresponding interpolation inequality: Let Ω be and open subset of $\mathbb{R}^n_+$ with a boundary portion T on $x_n = 0$ and assume $u \in C^{2,\alpha}(\Omega \cup T)$. Then for any $\epsilon > 0$ and some constant $C(\epsilon)$ we have

$$[u]^*_{j,\beta;\Omega\cup T} \le C|u|_{0;\Omega} + \epsilon[u]^*_{2,\alpha;\Omega\cup T}, \tag{A.3.34}$$

$$|u|^*_{j,\beta;\Omega\cup T} \le C|u|_{0;\Omega} + \epsilon[u]^*_{2,\alpha;\Omega\cup T}, \tag{A.3.35}$$

where $j = 0, 1, 2,\ 0 \le \alpha, \beta \le 1$ and $j + \beta \le 2 + \alpha$. These inequalities are proved in Appendix 1 in Chapter 6 of Gilbarg-Trudinger.

Lemma A.3.5. *Let Ω be an open subset of $\mathbb{R}^n_+$, with a boundary portion T on $\mathbf{x}_n = 0$. Suppose that $u \in C^{2,\alpha}(]co \cup T)$ is a bounded solution in Ω of $Lu = f$ satisfying the boundary condition $u = 0$ on T. In addition we assume*

$$|a^{ij}|^{(0)}_{0,\alpha;\Omega\cup T}, |b^i|^{(1)}_{0,\alpha;\Omega\cup T}, |c|^{(2)}_{0,\alpha;\Omega\cup T} \le \Lambda; \quad |f|^{(2)}_{0,\alpha;\Omega\cup T} < \infty. \tag{A.3.36}$$

Then

$$|u|^*_{2,\alpha;\Omega\cup T} \le C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega\cup T}\right), \tag{A.3.37}$$

where $C = C(n, \alpha, \lambda, \Lambda)$.

证明. The proof is identical with that of the interior estimates if we replace $d_{\mathbf{x}}$ and the interpolation inequalities by $\bar{d}_{\mathbf{x}}$ and the one exhibited above respectively. □

In order to extend the preceding lemma to domains with a curved boundary, we introduce the relevant seminorms and norms, in obvious generalizations of (A.3.5). Let Ω be an open set in $\mathbb{R}^n$ with $C^{k,\alpha}$ boundary portion T. For $\mathbf{x}, \mathbf{y} \in \Omega$ let us write

$$\bar{d}_{\mathbf{x}} = \mathrm{dist}(\mathbf{x}, \partial\Omega\backslash T),\ \bar{d}_{\mathbf{x},\mathbf{y}} = \min(\bar{d}_{\mathbf{x}}, \bar{d}_{\mathbf{y}}).$$

The quantities are:

$$\begin{aligned}
&[u]^*_{k,0;\Omega\cup T} = [u]^*_{k;\Omega\cup T} = \sup_{\substack{\mathbf{x}\in\Omega\\|\beta|=k}} \bar{d}^k_{\mathbf{x}}|D^\beta u(\mathbf{x})|,\ k = 0, 1, 2, \ldots;\\
&|u|^*_{k,0;\Omega\cup T} = |u|^*_{k;\Omega\cup T} = \sum_{j=0}^{k}[u]^*_{j;\Omega\cup T};\\
&[u]^*_{k,\alpha;\Omega\cup T} = \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega\\|\beta|=k}} \bar{d}^{k+\alpha}_{\mathbf{x},\mathbf{y}} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha},\ 0 < \alpha \le 1;\\
&|u|^*_{k,\alpha;\Omega\cup T} = |u|^*_{k;\Omega\cup T} + [u]^*_{k,\alpha;\Omega\cup T};\\
&|u|^{(k)}_{0,\alpha;\Omega\cup T} = \sup_{\mathbf{x}\in\Omega} \bar{d}^k_{\mathbf{x}}|u(\mathbf{x})| + \sup_{\mathbf{x},\mathbf{y}\in\Omega} \bar{d}^{k+\alpha}_{\mathbf{x},\mathbf{y}} \frac{|u(\mathbf{x}) - u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}.
\end{aligned} \tag{A.3.38}$$

Let Ω be a bounded domain with $C^{k,\alpha}$ boundary portion T, $k \ge 1$, $0 \le \alpha \le 1$. Suppose that $\Omega \subset\subset D$, where D is a domain that is mapped by a $C^{k,\alpha}$ diffeomorphism ψ onto D'. Letting $\Omega' = \psi(\Omega)$ and $\psi(T) = T'$, we can define all the quantities described before with respect to Ω' and T'. It is not hard to show that the transformation $\mathbf{x}' = \psi(\mathbf{x})$ induces a mapping of functions $u(\mathbf{x}) \to u(\psi^{-1}\psi(\mathbf{x})) =: \tilde{u}(\mathbf{x}')$, and the corresponding quantities of u and $\tilde{u}$ are equivalent in the sense that $K^{-1}|\tilde{u}|_{☺} \le |u|_{☺} \le K|\tilde{u}|_{☺}$ with K depending on ψ and Ω.

Lemma A.3.6. *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$, and let $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$, where $f \in C^{\alpha}(\bar{\Omega})$. It is assumed that the coefficients of L is strictly elliptic and*

$$|a^{ij}|_{0,\alpha;\Omega}, |b^i|_{0,\alpha;\Omega}, |c|_{0,\alpha;\Omega} \leq \Lambda.$$

Then for some δ there is a ball $B = B_\delta(\mathbf{x}_0)$ at each point $\mathbf{x}_0 \in \partial\Omega$ such that

$$|u|_{2,\alpha;B\cap\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right), \tag{A.3.39}$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.

证明. By the definition of a $C^{2,\alpha}$ domain, at each point $\mathbf{x}_0 \in \partial\Omega$ there is a neighborhood N of $\mathbf{x}_0$ and a $C^{2,\alpha}$ diffeomorphism that straightens the boundary in N. Let $B_\rho(\mathbf{x}_0) \subset\subset N$ and set $B' = B_\rho(\mathbf{x}_0) \cap \Omega$, $D' = \psi(B')$, $T = B_\rho(\mathbf{x}_0) \cap \partial\Omega \subset \partial B'$ and $T' = \psi(T) \subset \partial D'$ (T' is a hyperplane portion of $\partial D'$). Under the mapping $\mathbf{y} = \psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \ldots, \psi_n(\mathbf{x}))$, let $\tilde{u}(\mathbf{y}) = u(\mathbf{x})$ and $\tilde{L}\tilde{u}(\mathbf{y}) = Lu(\mathbf{x})$, where

$$\tilde{L}\tilde{u} \equiv \tilde{a}^{ij} D_{ij}\tilde{u} + \tilde{b}^i D_i \tilde{u} + \tilde{c}\tilde{u} = \tilde{f}(\mathbf{y}),$$

and

$$\begin{aligned}
&\tilde{a}^{ij}(\mathbf{y}) = \frac{\partial \psi_i}{\partial x_r}\frac{\partial \psi_j}{\partial x_s} a^{rs}(\mathbf{x}), \quad \tilde{b}^i(\mathbf{y}) = \frac{\partial^2 \psi_i}{\partial x_r \partial x_s} a^{rs}(\mathbf{x}) + \frac{\partial \psi_i}{\partial x_r} b^r(\mathbf{x}), \\
&\tilde{c}(\mathbf{y}) = c(\mathbf{x}), \quad \tilde{f}(\mathbf{y}) = f(\mathbf{x}).
\end{aligned}$$

We observe that in D'

$$\bar{\lambda}|\xi|^2 \leq \tilde{a}^{ij}\xi_i\xi_j, \ \forall \xi \in \mathbb{R}^n,$$

where $\bar{\lambda} = \lambda/K$ for a suitable positive constant K depending only on the mapping ψ on B'. It is not hard to observe that

$$|\tilde{a}^{ij}|_{0,\alpha;D'}, |\tilde{b}^i|_{0,\alpha;D'}, |\tilde{c}|_{0,\alpha;D'} \leq \bar{\Lambda} = K\Lambda; \ |\tilde{f}|_{0,\alpha;D'} < \infty.$$

Thus the conditions of Lemma A.3.5 are satisfied for the equation $\tilde{L}\tilde{u} = \tilde{f}$ in D' with the hyperplane portion T'. We can therefore assert

$$|\tilde{u}|^*_{2,\alpha;D'\cup T'} \leq C\left(|\tilde{u}|_{0;D'} + |\tilde{f}|^{(2)}_{0,\alpha;D'\cup T'}\right),$$

where the constant $C = C(n, \alpha, \bar{\lambda}, \bar{\Lambda})$. It follows from the (semi)norm equivalence under the mapping ψ that

$$\begin{aligned}
|u|^*_{2,\alpha;B'\cup T} &\leq C\left(|u|_{0;B'} + |f|^{(2)}_{0,\alpha;B'\cup T}\right) \\
&\leq C\left(|u|_{0;B'} + |f|_{0,\alpha;B'}\right) \\
&\leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right),
\end{aligned}$$

where C now depends on $n, \alpha, \lambda, \Lambda$ and B'. Letting $B'' = B_{\rho/2}(\mathbf{x}_0) \cap \Omega$ and observing that

$$\min(1, (\rho/2)^{2+\alpha})|u|_{2,\alpha;B''} \leq |u|^*_{2,\alpha;B'\cup T},$$

we obtain

$$|u|_{2,\alpha;B''} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$

The radius ρ appearing in this estimates depends essentially on $\mathbf{x}_0 \in \partial\Omega$. Consider now the collection of balls $B_{\rho/4}(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$, we know by compactness of $\partial\Omega$ there is a finite subcollection $B_{\rho_i/4}(\mathbf{x}_i)$, $1 \leq i \leq N$ that covers $\partial\Omega$. Letting $\min(\rho_i/4)$ it's not hard to see that for this δ the conclusion of the lemma is true. □

We remark here that the dependence of the constant C in the above lemma on the domain Ω is through the constants K, which are essentially related to the $C^{2,\alpha}$ bounds on the family of mappings $\psi_{\mathbf{x}}$, the local representations of $\partial\Omega$ near $\mathbf{x} \in \partial\Omega$. If the bounds on the mappings ψ can be stated uniformly on the boundary, then the uniform bound K can replace Ω in the statement of the estimate and the domain may also be unbounded.

Theorem A.3.7. (**Global Schauder Estimates**) *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution of $Lu = f$ in Ω, where $f \in C^{\alpha}(\bar{\Omega})$ and the coefficients of L satisfy, for positive constants λ, Λ*

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \ \forall \mathbf{x} \in \Omega,\ \xi \in \mathbb{R}^n,$$

and

$$|a^{ij}|_{0,\alpha;\Omega}, |b^i|_{0,\alpha;\Omega}, |c|_{0,\alpha;\Omega} \leq \Lambda.$$

Let $\phi \in C^{2,\alpha}(\bar{\Omega})$, and suppose $u = \phi$ on $\partial\Omega$. Then

$$|u|_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |\phi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}\right), \tag{A.3.40}$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.

证明. We start with $u = 0$ on $\partial\Omega$ and $\phi = 0$, and if this is done we set $v = u - \phi$ and observe that $Lv = f - L\phi$, $v = 0$ on $\partial\Omega$ and $|L\phi|_{0,\alpha;\Omega} \leq C|\phi|_{2,\alpha;\Omega}$, then we obtain

$$|u|_{2,\alpha;\Omega} \leq |v|_{2,\alpha;\Omega} + |\phi|_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |\phi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}\right).$$

Let $\mathbf{x} \in \Omega$. We consider the two possibilities: (i) $\mathbf{x} \in B_0 = B_{2\sigma}(\mathbf{x}_0) \cap \Omega$ for some $\mathbf{x}_0 \in \partial\Omega$, where $\delta = 2\sigma$ is the radius in the preceding lemma; (ii) $\mathbf{x} \in \Omega_\sigma = \{\mathbf{x} \in \Omega\ ;\ \mathrm{dist}\mathbf{x}, \partial\Omega > \sigma\}$. Using boundary estimate for (i) and interior estimate for (ii), we have

$$|Du(\mathbf{x})| + |D^2u(\mathbf{x})| \leq \max(C^{(i)}, C^{(ii)})(|u|_0 + |f|_{0,\alpha}),$$

where $C^{(i)}$ is the bound coefficients from the boundary estimate and $C^{(ii)}$ the interior one.

Now, let $\mathbf{x}, \mathbf{y}$ be distinct points in Ω and consider the following three possibilities: (i) $\mathbf{x}, , \mathbf{y} \in B_0$ for some $\mathbf{x}_0$; (ii) $\mathbf{x}, \mathbf{y} \in \Omega_\sigma$; (iii) $\mathbf{x}$ or $\mathbf{y}$ is in $\Omega \backslash \Omega_\sigma$ but not both $\mathbf{x}$ and $\mathbf{y}$ are in the same ball B_0 for any $\mathbf{x}_0 \in \partial\Omega$. These exhaust all the possibilities. Case (i) and (ii) are addressed by the boundary and interior estimates with bound coefficients C_1, C_2 respectively.

In case (iii), $\text{dist}(\mathbf{x}, \mathbf{y}) > \sigma$, so that

$$\begin{aligned}\frac{|D^2u(\mathbf{x}) - D^2u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} &\le \frac{1}{\sigma^\alpha}\left(|D^2u(\mathbf{x})| + |D^2u(\mathbf{y})|\right) \\ &\le C_3\left(|u|_0 + |f|_{0,\alpha}\right).\end{aligned}$$

Letting $C = \max(C_1, C_2, C_3)$, and taking the supremum over all $\mathbf{x}, \mathbf{y} \in \Omega$, we obtain

$$[D^2u]_\alpha \le C\left(|u|_0 + |f|_{0,\alpha}\right).$$

Combining this estimate with the bound for $|u|_2$, we are done.

$\square$

Remark: The typical application is that for any bounded set of solutions to a family of equations is also bounded in the space $C^{2,\alpha}(\bar{\Omega})$ and hence precompact in $C^2(\bar{\Omega})$.

A.4 de Giorgi-Nash-Moser Estimate

In this section we review the proof of the famous *de Giorgi-Nash-Moser* estimate. Of concern is the regularity problem of a weak solution $u \in H^1(\Omega)$ to the equation

$$\partial_i\left(a^{ij}(\mathbf{x})\partial_j u(\mathbf{x})\right) = 0,\ \mathbf{x} \in \Omega,$$

i.e.

$$\int_\Omega a^{ij} u_i \phi_j = 0 \tag{A.4.1}$$

for all $\phi \in H_0^1(\Omega)$. Here for some $\lambda > 0$

$$\lambda^{-1}|\xi|^2 \le a^{ij}(\mathbf{x})\xi_i\xi_j \le \lambda|\xi|^2,\ \forall \mathbf{x} \in \Omega,\ \xi \in \mathbb{R}^n.$$

Let $\Omega' \subset\subset \Omega$ be a subdomain and $\delta = \text{dist}(\Omega', \partial\Omega)$. We have the following result.

Theorem A.4.1. (de Giorgi-Nash-Moser) *If $\int_\Omega |\nabla u|^2 \le 1$, then there are two positive constants α, β depending only on n, λ and δ such that*

$$|u(\mathbf{x}) - u(\mathbf{y})| \le \beta|\mathbf{x}-\mathbf{y}|^\alpha,\ \ \textit{for } \mathbf{x}, \mathbf{y} \in \Omega'. \tag{A.4.2}$$

Remark: Recall that this result was partially covered by the interior H^2 estimate in Chapter 4. In that chapter, we assumed that $a^{ij} \in C^1(\Omega) \cap L^\infty(\Omega)$, and obtained the following estimates

$$\|u\|_{H^2(\Omega')} \le C\,\|u\|_{L^2(\Omega)}.$$

By Sobolev Imbedding theorem, we have when $n \le 3$, $k - n/p = 2 - n/2 > 0$, and so for some $1 \ge \alpha > 0$, $H^2(\Omega') \hookrightarrow C^\alpha(\overline{\Omega'})$. It is evident to notice that de Giorgi-Nash-Moser estimate extends the result in every sense. The proof is mainly from (J. Moser 1960).

A.4.1 Subsolutions and Moser Iteration

It is well-known that the weak solution $v = u$ of (A.4.1) satisfy an inequality for $\sigma, \rho > 0$,

$$\int_{B_\rho(\mathbf{x})} |\nabla v|^2 \leq \frac{4\lambda^4}{\sigma^2} \int_{B_{\rho+\sigma}(\mathbf{x})} v^2, \tag{A.4.3}$$

whenever the ball $B_{\rho+\sigma}(\mathbf{x})$ is entirely in Ω (this can be done by using a proper auxiliary function $\phi = \eta^2 u$). A crucial observation made by Moser was that this result can be extended to nonnegative *subsolutions*. A nonnegative *subsolution* v to (A.4.1) satisfies $(a^{ij} v_i)_j \geq 0$ in the weak sense. That is to say, for any $0 \leq \psi \in H_0^1(\Omega)$, we have

$$\int_\Omega a^{ij} v_i \psi_j \leq 0. \tag{A.4.4}$$

Lemma A.4.1. *The estimate (A.4.3) also holds for any nonnegative subsolutions of (A.4.1), and in particular for*

$$v = f(u),$$

where u is a solution to (A.4.1) and f is a nonnegative convex function so that integrals in (A.4.3) are both finite. If v is a subsolution so is $f(v)$, provided f is nonnegative, convex and monotone increasing.

Remark:

1. A convex function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz by the following estimate

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq \max \left(\left| \frac{f(x) - f(b)}{x - b} \right| + \left| \frac{f(a) - f(y)}{a - y} \right| \right),$$

for $-\infty < a < x < y < b < +\infty$. Moreover, if f is of sublinear growth (i.e. for some $\alpha, \beta > 0$, $|f(x)| \leq \alpha |x| + \beta$), then f is globally Lipschitz on $\mathbb{R}$;

2. The square integrability of $v = f(u)$ naturally requires that f to be of sublinear growth ($u \in H^1(\Omega)$ is in general not bounded on Ω), and hence it is natural to assume that f is globally Lipschitz;

3. It is important to make precise the pointwise definition of $f'(u)$ because for some $E \subset \mathbb{R}$ of zero measure, $u^{-1}(E)$ may possess positive measure in Ω. To do so we first observe that f is Lipschitz, and hence it has an almost everywhere defined derivative

$$g(u),\ u \in D$$

where D is dense in $\mathbb{R}$. By convexity, $g(u)$ must be a non-decreasing function, and hence we may define for every $u \in D$, $g^+(u) = g^-(u) = g(u)$, and elsewhere $g^+(u) = \lim_{\substack{\epsilon \to 0^+ \\ u+\epsilon \in D}} g(u+\epsilon)$ and $g^-(u) = \lim_{\substack{\epsilon \to 0^+ \\ u-\epsilon \in D}} g(u-\epsilon)$. Now, we define $f'(u) = (g^+(u) + g^-(u))/2$ for $u \in \mathbb{R}$. It is worthwhile to mention that g^+ and g^- are monotone functions on $\mathbb{R}$, and hence both of them have at most countably many discontinuities, which means that D

can be chosen to satisfy that $\mathbb{R}\backslash D$ is a countable subset. Once the pointwise definition of $f'(u)$ is given, we can define $\nabla v = \nabla\left(f(u)\right) = f'(u)\nabla u$, which is therefore well-defined almost everywhere. The rigorous arguments will be presented in the proof.

証明. Without loss of generality we consider the balls centered at $\mathbf{0}$ assuming that $|\mathbf{x}| < \rho+\sigma$ is contained in Ω. By $\eta(\mathbf{x})$ we denote a function of compact support in $|\mathbf{x}| < \rho+\sigma$ with a piecewise continuous derivative. Then in (A.4.4) let

$$\psi(\mathbf{x}) = v\eta^2$$

which is nonnegative and of compact support. Therefore, we have

$$\int a^{ij}\eta^2 v_i v_j + 2\int a^{ij}\eta v \eta_i v_j \leq 0. \tag{A.4.5}$$

Using Schwarz inequality one finds

$$\int \eta^2|\nabla v|^2 \leq 4\lambda^4 \int v^2|\nabla\eta|^2. \tag{A.4.6}$$

Choosing for $\eta(\mathbf{x})$ a function which is piecewise linear in $|\mathbf{x}|$ and is equal to 0 for $|\mathbf{x}| > \rho+\sigma$ and equal to 1 for $|\mathbf{x}| < \rho$, we obtain $|\nabla\eta| \leq \sigma^{-1}$ and

$$\int_{|\mathbf{x}|<\rho} |\nabla v|^2 \leq \frac{4\lambda^4}{\sigma^2}\int_{|\mathbf{x}|<\rho+\sigma} v^2, \tag{A.4.7}$$

which proves the estimate for subsolutions.

We show now that every nonnegative convex function $v = f(u)$ yields such a subsolution. We first start with functions having a continuous second derivative $f''(u)$ which vanishes for $|u| > M$; the convexity implies that $f''(u) \geq 0$. Let $\psi(\mathbf{x}) \geq 0$ be of compact support and

$$\phi(\mathbf{x}) = f'(u)\psi(\mathbf{x}).$$

Then

$$a^{ij}\phi_i u_j = a^{ij}\psi_i v_j + f''\psi a^{ij}u_i u_j \geq a^{ij}\psi_i v_j. \tag{A.4.8}$$

Integrating over $\mathbf{x} \in \Omega$ gives

$$0 \geq \int a^{ij}\psi_i v_j.$$

This proves (A.4.4) for any $\psi \geq 0$ of compact support which is indefinitely differentiable. In fact the assumption $f'' = 0$ for $|u| > M$ insures that

$$\phi_i = f'\psi_i + f'' u_i \psi$$

is square integrable (this is because f' is bounded on $\mathbb{R}$). If here u is only a subsolution and $f' \geq 0$, then $\phi = f'(u)\psi(\mathbf{x}) \geq 0$, and the above arguments also work in this case, which shows that $v = f(u)$ is still a subsolution.

To extend the above results to general nonnegative convex function, we first give the following claim.

Claim 1: Given any nonnegative convex function f on $\mathbb{R}$, there is a sequence of nonnegative convex functions $\{f_m\} \subset C^2(\mathbb{R})$ and $f_m''(u) = 0$ for $|u| > M^{(m)}$ so that $f_m \to f$ in $C^0_{loc}(\mathbb{R})$ and $f_m'(u) \to f'(u)$ for each fix $u \in \mathbb{R}$ as $m \to \infty$, where $f'(u)$ is the one defined in the remark.

proof of Claim 1. Define the following mollifier

$$J(x) = \begin{cases} e^{-1/(1-x^2)}/I & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases},$$

where $I = \int_{-1}^{1} e^{-1/(1-x^2)} dx$ is a normalization constant. We further define $J_\epsilon(x) = J(x/\epsilon)/\epsilon$. It is not hard to see that $\{J_\epsilon\}$ is a family of nonnegative indefinitely differentiable functions that have global integration 1 and support within $[-\epsilon, \epsilon]$ respectively.

Given $\epsilon > 0$, we observe that f' is a bounded function on $[-1/\epsilon, 1/\epsilon]$, and so there is a positive constant A such that $A > f'(x) > -A$ for $x \in [-1/\epsilon, 1/\epsilon]$. Now we define $f^\epsilon(x) = f(x)$ for $x \in [-1/\epsilon, 1/\epsilon]$ satisfying $(f^\epsilon)'(x) = A$ if $x > 1/\epsilon$ and $-A$ if $x < -1/\epsilon$. This definition insures that all f^ϵ are convex functions satisfying $(f^\epsilon)''(x) = 0$ for all $|x| > 1/\epsilon$.

Now we check that

$$f_\epsilon(u) = \int_{\mathbb{R}} f^\epsilon(u-x) J_\epsilon(x) dx$$

satisfies the requirements in the claim. At first, it is not hard to see that f_ϵ's are all nonnegative convex and C^∞ smooth all over $\mathbb{R}$.

Let $M > 0$ and then for $u \in [-M, M]$ we have for small $\epsilon > 0$ (say $1/\epsilon > \max(5M, 2)$),

$$\begin{aligned}
|f_\epsilon(u) - f(u)| &= \left| \int_{\mathbb{R}} (f^\epsilon(u-z) - f(u)) J_\epsilon(z) dz \right| \\
&\leq \int_{|z-u| \leq 1/\epsilon} |f(u-z) - f(u)| J_\epsilon(z) dz \\
&\quad + \int_{u-z \geq 1/\epsilon} |A(u-z-1/\epsilon) + f(1/\epsilon) - f(u)| J_\epsilon(z) dz \\
&\quad + \int_{u-z \leq -1/\epsilon} |-A(u-z-1/\epsilon) + f(-1/\epsilon) - f(u)| J_\epsilon(z) dz \\
&\leq \int_{|z| \leq \epsilon} |f(u-z) - f(z)| J_\epsilon(z) dz \\
&\leq \sup_{x,y \in [-M-1, M+1]} \frac{|f(x) - f(y)|}{|x-y|} \epsilon.
\end{aligned}$$

This proves that f_ϵ converges to f locally uniformly as $\epsilon \to 0^+$.

To see the pointwise convergence of $(f_\epsilon)'$ to f', we first observe that for every $u \in \mathbb{R}$,

$$(f')_\epsilon(u) = (f_\epsilon)'(u). \tag{A.4.9}$$

Consider for $h \neq 0$,

$$\begin{aligned}
\frac{f_\epsilon(u+h) - f_\epsilon(u)}{h} &= \int_{\mathbb{R}} \frac{f(u+h-z) - f(u-z)}{h} J_\epsilon(z) dz \\
&= \int_{-\epsilon}^{\epsilon} \frac{f(u+h-z) - f(u-z)}{h} J_\epsilon(z) dz.
\end{aligned}$$

For small $|h|$, we know that

$$\left| \frac{f(u+h-z) - f(u-z)}{h} J_\epsilon(z) \right| \leq \sup_{x,y \in [|u|-100, |u|+100]} \frac{|f(x) - f(y)|}{|x-y|} J_\epsilon(z),$$

the left hand side of which is integrable. On the other hand, because f is Lipschitz, the integrands converges pointwise to $f'(u-z)J_\epsilon(z)$ for almost every $z \in \mathbb{R}$. Therefore, by LDCT, (A.4.9) holds.

Now, if f' is continuous at some $u \in \mathbb{R}$, we have

$$\begin{aligned} |(f')_\epsilon(u) - f'(u)| &\leq \int_{\mathbb{R}} |f'(u-z) - f'(u)| J_\epsilon(z) dz \\ &\leq \int_{-\epsilon}^{\epsilon} |f'(u-z) - f'(u)| J_\epsilon(z) dz \\ &\leq \sup_{-\epsilon<z<\epsilon} |f'(u-z) - f'(u)|. \end{aligned}$$

Taking $\limsup_{\epsilon\to 0^+}$ on both sides, we obtain the fact that $(f')_\epsilon(u) \to f'(u)$ as $\epsilon \to 0$. If for some $u \in \mathbb{R}$, f' is not continuous, then by monotonicity, it has left and right limits at u. In fact, by using the notations in Remark 3., we know that $f'(x) = g^+(x) = g^-(x)$ except for countably many points in $\mathbb{R}$, and at these exceptional points u, $f'(u) = (g^+(u) + g^-(u))/2 = (f'(u+) + f'(u-))/2$. Furthermore, we have

$$\begin{aligned} |(f')_\epsilon(u) - f'(u)| &= \left| (f')_\epsilon(u) - \frac{f'(u+) + f'(u-)}{2} \right| \\ &\leq \int_0^{\epsilon} |f'(u-z) - f'(u-)| J_\epsilon(z) dz + \int_{-\epsilon}^{0} |f'(u-z) - f'(u-)| J_\epsilon(z) dz \\ &\leq \frac{1}{2} \sup_{0<z<\epsilon} |f'(u-z) - f'(u-)| + \frac{1}{2} \sup_{-\epsilon<z<0} |f'(u-z) - f'(u+)|. \end{aligned}$$

This completes the proof of Claim 1. □

Claim 2: Given any nonnegative convex function f on $\mathbb{R}$, if for general $u \in H^1(\Omega)$ (Ω bounded), $v = f(u) \in H^1(\Omega)$, then f should be globally Lipschitz with coefficient $L > 0$ and $\|f(u)\|_{H^1(\Omega)} \leq L \|u\|_{H^1(\Omega)}$. Furthermore, we have

$$\nabla v(\mathbf{x}) = f'(u(\mathbf{x})) \nabla u(\mathbf{x}),$$

for almost all $\mathbf{x} \in \Omega$.

proof of Claim 2. If f is not globally Lipschitz, then it is not of sublinear growth and so by simply taking $u(\mathbf{x}) \equiv C \to \infty$, we have

$$\frac{\int_\Omega (f(u))^2}{\int_\Omega u^2} = \frac{f(C)^2}{C^2} \to \infty,$$

which shows that $f : H^1 \to H^1$ is not a bounded operator.

Now, we may without losing generality assume that f is Lipschitz. By Claim 1, we have a sequence of nonnegative convex functions $\{f_m\} \subset C^2(\mathbb{R})$ and $f_m''(u) = 0$ for $|u| > M^{(m)}$ so that $f_m \to f$ in $C^0_{loc}(\mathbb{R})$ and $f_m'(u) \to f'(u)$ for each fix $u \in \mathbb{R}$ as $m \to \infty$. By the proof of the claim, the Lipschitz coefficients of f_m's are all bounded by L, and so according to Theorem 2.2.2, all $f_m(u) \in H^1(\Omega)$ and $\|f_m(u)\|_{H^1} \leq L \|u\|_{H^1}$. By local uniform convergence of f_m to f

and $|f_m(u)| \leq L|u| + C$, which is square integrable on Ω, we have by LDCT, f_m converges to f in $L^2(\Omega)$. Moreover, by u-pointwise convergence of $f_m'(u)$ to $f'(u)$, we have $\mathbf{x}$-a.e. pointwise convergence of $f_m'(u(\mathbf{x}))$ to $f'(u(\mathbf{x}))$, and hence by Fatou's lemma

$$\int_\Omega |f'(u)\nabla u|^2 \leq \liminf_{m\to\infty} \int_\Omega |\nabla\left(f_m(u)\right)|^2 \leq L\int_\Omega |\nabla u|^2.$$

Moreover, $|\nabla\left(f_m(u(\mathbf{x}))\right)| = |f_m'(u(\mathbf{x}))\nabla u(\mathbf{x})| \leq L|\nabla u(\mathbf{x})|$ for almost every $\mathbf{x} \in \Omega$, and hence also by LDCT, $\nabla\left(f_m(u)\right)$ converges in $L^2(\Omega)$ to $f'(u)\nabla u$, which shows that

$$\nabla v(\mathbf{x}) = f'(u(\mathbf{x}))\nabla u(\mathbf{x}),\ \mathbf{x}\text{-a.e.}$$

and hence $v \in H^1(\Omega)$.

□

Returning to the proof of the lemma, we have by setting $v_m = f_m(u)$

$$\int_{|\mathbf{x}|<\rho} |\nabla v_m|^2 \leq \frac{4\lambda^4}{\sigma^2}\int_{|\mathbf{x}|<\rho+\sigma} v_m^2.$$

Using Claim 1 and 2, we may send $m \to \infty$ and obtain the desired estimate.

□

Now, let us consider a Sobolev type inequality without giving proof.

Lemma A.4.2. *Let $w \in H^1(\Omega)$. Then there is a constant c_n which depends on n and the choice of c_0 such that*

$$\left(\rho^{-n}\int_{|\mathbf{x}|<\rho} w^{2\kappa}\right)^{1/\kappa} \leq c_n\left(\rho^{-n+2}\int_{|\mathbf{x}|<\rho}|\nabla w|^2 + \rho^{-n}\int_N w^2 dx\right) \tag{A.4.10}$$

for every $1 \leq \kappa \leq n/(n-1)$. Here N is any measurable set in $|\mathbf{x}| < \rho$ of measure $m(N) \geq c_0^{-1}\rho^n$. In the following c_0^{-1} will be chosen to be half the volume of the unit ball and c_n, $n \geq 2$, depends on n only.

Remark:

1. The exponent κ in (A.4.10) does not have to be $\leq n/(n-1)$, however the limitation $\kappa \leq n/(n-2)$ is essential;
2. The existence of the integral in (A.4.10) follows from the finiteness of the integrals on the right.

Using Lemmas A.4.1 and A.4.2 it is possible to estimate the square integral of w^κ in terms of the square integral of w for any nonnegative subsolution w. For this purpose let N be the sphere $|\mathbf{x}| < \rho$ in Lemma A.4.2 and apply (A.4.3) to w:

$$\begin{aligned}\left(\rho^{-n}\int_{|\mathbf{x}|<\rho} w^{2\kappa}\right)^{1/\kappa} &\leq c_n\left(\rho^{-n+2}\int_{|\mathbf{x}|<\rho}|\nabla w|^2 + \rho^{-n}\int_{|\mathbf{x}|<\rho} w^2 dx\right)\\ &\leq c\left(1+\frac{\rho^2}{\sigma^2}\right)\rho^{-n}\int_{|\mathbf{x}|<\rho+\sigma} w^2\\ &\leq c\left(1+\frac{\rho^2}{\sigma^2}\right)\left(1+\frac{\sigma}{\rho}\right)^n\left[(\rho+\sigma)^{-n}\int_{|\mathbf{x}|<\rho+\sigma} w^2\right].\end{aligned}$$

Assuming $\sigma \le \rho$ we have with a new c

$$\left(\rho^{-n}\int_{|\mathbf{x}|<\rho} w^{2\kappa}\right)^{1/\kappa} \le c\left(1+\frac{\rho^2}{\sigma^2}\right)\left[(\rho+\sigma)^{-n}\int_{|\mathbf{x}|<\rho+\sigma} w^2\right] \tag{A.4.11}$$

which is valid for any nonnegative subsolution w.

Finally, we have the following simple observation called *Moser Iteration*: If $\phi_0 > 0$ and

$$0 < \phi_\nu \le c^\nu \phi_{\nu-1}^\kappa,\ \nu = 1,2,\ldots;\ \kappa > 1, \tag{A.4.12}$$

then

$$\limsup_{\nu\to\infty} \phi^{\kappa^{-\nu}} \le c_1\phi_0, \tag{A.4.13}$$

where $c_1 = c^\kappa/(\kappa-1)^2$. Defining the sequence ψ_ν by

$$\psi_\nu = c_1^{\nu+1-\kappa^{-1}\nu}\phi_\nu,$$

(A.4.12) goes over into the inequality

$$0 < \psi_\nu \le \psi_{\nu-1}^\kappa$$

which implies

$$\psi_\nu \le \psi_0^{\kappa^\nu}.$$

This makes (A.4.13) evident.

A.4.2 The Core Theorems

Theorem A.4.2. *Let $v(\mathbf{x}) \ge 0$ be a subsolution in the weak sense which is defined in $|\mathbf{x}| < 2r$. Then*

$$v^2(\mathbf{x}) \le cr^{-n}\int_{|\mathbf{x}|<2r} v^2 d\mathbf{x} \tag{A.4.14}$$

for almost all $\mathbf{x}$ in $|\mathbf{x}| < r$.

Remark: This result will be applied to functions $f(u)$ of a solution u, where f is a nonnegative convex function of u. For $v = |u|$ one obtains a bound

$$|u|(\mathbf{x}) \le c\delta^{-n/2}\left(\int_\Omega u^2\right)^{1/2}$$

for all $\mathbf{x} \in \Omega' \subset\subset \Omega$ satisfying $\mathrm{dist}(\Omega', \partial\Omega) \ge \delta$.

证明. Since $v(\mathbf{x})$ is a subsolution, so is

$$w = |v(\mathbf{x})|^p = v(\mathbf{x})^p$$

for $p \ge 1$ because $f(v) = v^p$ is a nonnegative convex function with $f'(v) \ge 0$ for $v \ge 0$. Let $p = \kappa^\nu$ and

$$w_\nu = v^{\kappa^\nu}, \quad \nu = 0,1,2,\ldots,$$

where $\kappa = n/(n-1)$.

According to (A.4.11) we can estimate higher and higher norms of v. Let $2r \geq \rho_0 > \rho_1 > \cdots$ be a sequence of positive numbers satisfying $\rho_{\nu-1} \leq 2\rho_\nu$, then (A.4.11) applied to $w = w_{\nu-1}$, $\rho = \rho_\nu$, $\sigma = \rho_{\nu-1} - \rho_\nu \leq \rho_\nu$, yields

$$\phi_\nu = \rho_\nu^{-n} \int_{|\mathbf{x}|<\rho_\nu} w_\nu^2 = \rho_\nu^{-n} \int_{|\mathbf{x}|<\rho_\nu} w_{\nu-1}^{2\kappa} \leq c_2 \left| 1 + \left(\frac{\rho_\nu}{\rho_{\nu-1} - \rho_\nu} \right)^2 \right|^\kappa \phi_{\nu-1}^\kappa.$$

Choosing, for instance, $\rho_\nu = r(1 + 2^{-\nu})$ which implies $\rho_{\nu-1} \leq 2\rho_\nu$ and

$$\frac{\rho_\nu}{\rho_{\nu-1} - \rho_\nu} = 2^\nu + 1 \leq 3^\nu,$$

we find

$$\phi_\nu \leq c_2 10^{\kappa\nu} \phi_{\nu-1}^\kappa \leq c^\nu \phi_{\nu-1}^\kappa.$$

By previous arguments, we know that

$$\limsup_{\nu \to \infty} \phi_\nu^{1/\kappa^\nu} \leq c_1 \phi_0.$$

Since the left hand side converges to the essential maximum of v^2, the theorem is established.

□

The following theorem represents a Harnack type inequality which refers to nonnegative solutions u. The assumption that u is not identically 0 is expressed by the requirement that the set $|\mathbf{x}| < r$, where $u > 1$, has at least the measure $c_0^{-1} r^n$ with an appropriate constant $c_0 > 0$:

$$m(u > 1\,;\ |\mathbf{x}| < r) > c_0^{-1} r^n. \tag{A.4.15}$$

This assumption does not reduce generality according to the strong maximum principle for weak solutions (see Gilbarg-Trudinger section 8.7).

Theorem A.4.3. *Let $u \geq 0$ be a solution of (A.4.1) in $|\mathbf{x}| < 2r$ satisfying (A.4.15). Then there is a constant $c > 0$ depending on n and λ only such that*

$$u(\mathbf{x}) > c^{-1} \text{ in } |\mathbf{x}| < r/2.$$

证明. Using a similar approximation procedure by which we derived inequality (A.4.3) for all $v = f(u)$, when $f \geq 0$ is a convex function, we now derive

$$\int_{|\mathbf{x}|<r} |\nabla v|^2 \leq c r^{n-2} \tag{A.4.16}$$

for functions $v = f(u)$ for which also $h = -e^{-f}$ is a convex function. To prove this result we consider first functions f which are twice continuously differentiable. Then the convexity of h implies

$$f'' - f' = e^f h'' \geq 0. \tag{A.4.17}$$

Let $\phi(\mathbf{x}) = f'\psi(\mathbf{x})$, where $\psi \geq 0$ is of compact support in $|\mathbf{x}| < 2r$. For $f' \neq 0$, one has

$$a^{ij}\phi_i u_j = a^{ij}\psi_i v_j + f''\psi a^{ij} u_i u_j = a^{ij}\psi_i v_j + \frac{f''}{(f')^2}\psi a^{ij} v_i v_j.$$

This and (A.4.17) show that

$$0 \geq \int a^{ij}\psi_i v_j + \psi a^{ij} v_i v_j,$$

or with $\psi = \eta^2$, where η again is a function of compact support in $|\mathbf{x}| < 2r$, we find

$$\int \eta^2 |\nabla v|^2 \leq 2\lambda^2 \left(\int |\nabla \eta|^2\right)^{1/2} \left(\int \eta^2 |\nabla v|^2\right)^{1/2},$$

and with a new constant c

$$\int \eta^2 |\nabla v|^2 \leq c \int |\nabla \eta|^2.$$

Choosing for η a function which is piecewise linear in $|\mathbf{x}|$ and equal to 1 in $|\mathbf{x}| < r$, one obtains (A.4.16).

We apply (A.4.16) to

$$v = f(u) = \max(-\log(u + \epsilon), 0),\ 0 < \epsilon < 1.$$

Then

$$h = \max(-(u + \epsilon), -1)$$

is obviously convex and v is well-defined because $u > 0$. Since by (A.4.15) $v = 0$ on a set of measure $> c_0^{-1} r^n$, Lemma A.4.2 with $\kappa = 1$ and (A.4.16) yields

$$r^{-n} \int_{|\mathbf{x}|<r} v^2 \leq c r^{2-n} \int_{|\mathbf{x}|<r} |\nabla v|^2 < c_3.$$

On the other hand f is convex and nonnegative. Therefore Theorem A.4.2 gives, for $|\mathbf{x}| < r/2$,

$$v^2(\mathbf{x}) \leq c_4 r^{-n} \int_{|\mathbf{x}|<r} v^2 \leq c^2,$$

whence, by definition of v, $-\log(u + \epsilon) \leq c$, or

$$u + \epsilon \geq e^{-c} \text{ in } |\mathbf{x}| < r/2$$

for all $0 < \epsilon < 1$. Sending $\epsilon \to 0$ one obtains the theorem. □

A.4.3 Proof of Theorem A.4.1

By Theorem A.4.2, a solution u satisfying $\int_\Omega u^2 \leq 1$ is bounded in every compact subdomains Ω' by

$$|u(\mathbf{x})| \leq c\delta^{-n/2}$$

provided $\mathrm{dist}(\Omega', \partial\Omega) \geq \delta > 0$.

It is the aim to estimate the oscillation of $u(\mathbf{x})$ in $|\mathbf{x}| < \rho$ in dependence on ρ: Let

$$\omega(\rho) = \max_{|\mathbf{x}|<\rho} u(\mathbf{x}) - \min_{|\mathbf{x}|<\rho} u(\mathbf{x}),$$

assuming that the ball $|\mathbf{x}| < \rho$ lies in Ω'. Obviously

$$\omega(\rho) < 2c\delta^{-n/2}.$$

Fixing $\rho = 2r \leq \delta$ and adding an appropriate constant to u (which does not alter the oscillation) we can assume taht

$$\max_{|\mathbf{x}|<\rho} u(\mathbf{x}) = -\min_{|\mathbf{x}|<\rho} u(\mathbf{x}) = \frac{1}{2}\omega(2r) = M.$$

Then

$$\frac{M+u}{M} = 1 + \frac{u}{M}, \quad \frac{M-u}{M} = 1 - \frac{u}{M}$$

are also solutions of (A.4.1). They are both nonnegative and at least one of them satisfies condition (A.4.15) (the constant c_0^{-1} being half the volume of the ball $|\mathbf{x}| < 1$) depending on whether $u \geq 0$ or $u \leq 0$ occurs more frequently. Taking the first case we obtain by Theorem A.4.3

$$\frac{u+M}{M} > c^{-1} \text{ in } |\mathbf{x}| < \frac{r}{2} = \frac{\rho}{4},$$

or

$$-M(1-c^{-1}) \leq u(\mathbf{x}) \leq M \text{ in } |\mathbf{x}| < \frac{\rho}{4}.$$

In any case we arrive at

$$\omega\left(\frac{\rho}{4}\right) \leq M(2-c^{-1}) = \omega(\rho)(1-(2c)^{-1})^{\dagger}$$

for $\rho \leq \delta$. Applying this inequality repeatedly we find for $r = 4^{-m}\rho$

$$\omega(r) \leq \omega(4^m r)(1-(2r)^{-1})^m = \omega(\rho)\left(\frac{r}{\rho}\right)^{\alpha},$$

where $\alpha = -\log_4(1-(2c)^{-1}) = c_5^{-1}$. For every $r \leq \delta$ one can find an integer $m \geq 0$ such that $\rho = 4^m r$ lies in

$$\frac{\delta}{4} < \rho \leq \delta,$$

which gives

$$\omega(r) \leq \omega(\rho)\left(\frac{4}{\delta}\right)^{\alpha} r^{\alpha} \leq c\delta^{-n/2-\alpha} r^{\alpha}$$

if $|\mathbf{x}| < \delta$ lies in Ω'. Now let $\Omega'' \subset\subset \Omega$ such that $\operatorname{dist}(\Omega'', \partial\Omega) \geq 2\delta$. Then the ball of radius δ about $\mathbf{x}$ lies in Ω' and we have for any two points $\mathbf{x}, \mathbf{y} \in \Omega''$,

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq \omega(|\mathbf{x}-\mathbf{y}|) \leq c\delta^{-n/2-\alpha}|\mathbf{x}-\mathbf{y}|^{\alpha}$$

if $|\mathbf{x}-\mathbf{y}| \leq \delta$. On the other hand if $|\mathbf{x}-\mathbf{y}| > \delta$, one has trivially

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq 2c\delta^{-n/2} \leq 2c\delta^{-n/2}\frac{|\mathbf{x}-\mathbf{y}|^{\alpha}}{\delta^{\alpha}}.$$

This proves the theorem with $\alpha = c_5^{-1}$ and $\beta = c_6\delta^{-n/2-\alpha}$.

†This inequality can be used to establish a Liouville type theorem: with $\omega = \lim_{\rho\to\infty}\omega(\rho) < \infty$, one obtains $0 \leq \omega \leq (1-(2c)^{-1})\omega$, and hence $\omega = 0$.