HOMOGENIZATION IN COMPLEX CONDUCTIVE NETWORKS AND INTERFACE MOTIONS

by

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Abstract

This dissertation consists of three projects in the analysis of partial differential equations and geometric variational problems, motivated by questions arising in homogenization theory, free boundary problems, shape optimization and geometric analysis. The unifying theme is the rigorous characterization of singular limits and extremal structures from scientific or engineering problems, where classical notions and techniques break down. We will also discuss the results in this dissertation in a slightly broader context in Chapter 1 and raise some challenging problems.

The first project, see Chapter 2, studies optimal effective conductivity bounds for composite materials whose conductive phase is supported on lower-dimensional sets such as networks. We address the attainability of the classical Wiener (Voigt–Reuss–Hill) bounds in the singular regime. For the lower Wiener bound, we establish a sharp topological characterization: under a mild coercivity condition, the kernel of the effective conductivity tensor coincides with the orthogonal complement of the homotopy classes of loops in the supporting network. In particular, for periodic planar networks, positivity of the effective tensor is equivalent to a reticulation condition. For the upper Wiener bound, we introduce a correspondence between varifolds and matrix-valued measures, allowing us to reinterpret conductance maximality as a form of area criticality. Using a fractional monotonicity formula for stationary varifolds, we derive a pointwise dimensional bound for conductance maximizers. These results give a complete characterization of when extremal effective properties can be attained and have implications for models of complex conductive networks such as leaf venation patterns.

The second project, see Chapter 3 and Chapter 4, is a joint work with my advisor William Feldman and consists of two closely related works on elliptic and

parabolic equations, in which the Neumann boundary data oscillates rapidly in the vertical u-variable and the equations show singular behaviors after homogenization. In the homogenization limit, we prove the emergence of a novel singularly pinned rate-independent boundary condition: at zero tangential slope, the effective Neumann condition becomes singular and anisotropic, producing an unconstrained contact set reminiscent of the parabolic Signorini problems. A new comparison principle for the heat equation with this singular boundary condition is established, enabling a viscosity-solution-based convergence argument. This provides the first multi-dimensional PDE example in which rate-independent pinning behavior arises as the limit of a gradient flow with highly oscillatory energy. Motivated by this limit problem, we then develop a regularity theory for a gradient-degenerate Neumann problem. Such gradient-degenerate problems correspond to the stationary case of the limit parabolic problem. They also generalize the classical Signorini problem and feature new mathematical challenges. We prove optimal $C^{1,\frac{1}{2}}$ regularity in two dimensions and obtain higher-dimensional regularity under a non-accumulation condition on degenerate values. We also prove an alternative characterization of the minimal supersolutions.

The third project, see Chapter 5, focuses on a foundational question in geometric analysis: whether mean curvature flow can be realized as a genuine gradient flow with respect to a nondegenerate Riemannian metric on the space of curves / surfaces. While mean curvature flow is formally the L^2–gradient flow of perimeter, the associated metric structure is degenerate. We rigorously analyze two natural nondegenerate candidates on the space of planar curves: the uniformness-preserving metric and the curvature-weighted metric. In both cases, we prove that mean curvature flow cannot be expressed as a gradient flow of any energy functional with respect to these metrics. This yields a definitive negative result for these natural geometric structures and clarifies intrinsic obstructions to interpreting mean curvature flow as a true metric gradient flow.

For my mom, Anning.

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Acknowledgements

To have a period in life when one moves forward without hesitation is something to cherish — and that describes my five years as a PhD student. As I graduate and face a future with growing uncertainty, I want to express my gratitude to those who shaped these years.

The most important person in my life is my mother. She raised me to be optimistic and resilient, qualities that sustained me throughout the PhD. Her encouragement and praise gave me strength in difficult moments. No words are sufficient to express my gratitude; I wish her health and long life.

I was fortunate to encounter remarkable teachers on my path into mathematics. My high school teacher, Xianxiang Zhang, carefully explained concepts rather than focusing solely on techniques. His teaching helped me, once confused by mathematics, learn to think rigorously. That foundation later supported my undergraduate study.

At SUSTech, choosing mathematics was not immediate. Encouraged by my roommates Runxiong Wu and Xuanyu Liu, I took Mathematical Analysis, which is the first course in a math major, instead of standard calculus. The course, taught by Professor Bingsheng He, introduced me to rigorous analysis and left a lasting impression. His clarity guided me into analysis.

Another pivotal influence was Professor Xuefeng Wang. Professor Wang not only taught me analysis and PDE techniques but also influenced how I think about mathematics. In his PDE class, he wrote Picasso's line: "Art is a lie that makes us realize the truth." Though philosophically familiar, internalizing this idea in research is nontrivial. It shaped my academic taste and my view of the relationship between theory and reality.

During my time at SUSTech, I benefited from many mentors and peers, including Professor Linlin Su and numerous other professors whose work I deeply admire. Study sessions with Qiyun Yang and Hongze Tan were formative; we learned and grew together. I remain grateful to all the friends and classmates who made those years intellectually vibrant.

During my masters, I became close friends with Lei Yan. We studied and exercised together, and he introduced me to a disciplined lifestyle that proved essential during the intense year when I wrote my first paper. Later, during our years in the United States, we spent many evenings playing CSGO together—though our plan of 1,000 hours stalled somewhere past six hundred.

I was fortunate to receive a PhD offer from the University of Utah and become Will's student. I later learned that Will found my masters thesis interesting, which led him to admit me as his first PhD student. His support gave me the freedom to explore ambitious ideas. When I proposed using homogenization theory to describe leaf venation patterns, he treated the idea seriously even though it seemed speculative. His confidence allowed me to pursue it fully.

Conversations with Will were initially overwhelming; he seemed to command every deep theory I encountered. Over time, as I caught up, I refined my earlier ideas and eventually completed the leaf-venation project. The process was demanding but deeply rewarding.

The idea of studying leaf venation originated during my master's work on the homogenization of networks. After discussions with Professor Sean Lawley, who works on mathematical biology here in Utah, I became convinced the idea was viable, though writing the theory required significant reinterpretation. A foundational question is whether the assumptions in homogenization theory are indeed reliable in analyzing the geometry of leaf venations. This is beyond mathematical discussions. This motivated me to read papers in other scientific fields, not as people usually do in pure math, to find a good interpretation of the ideas. This is indeed an important and exciting experience in my life as I no longer simply follow the existing models but propose a new one by myself. The work is also mathematically interesting as I eventually reformulated homogenization theory in the language of geometric measure theory—a perspective shaped in part by my later work on free-boundary problems.

My work on leaf venation patterns is also an extension of the Wiener bound on the effective conductance of composite materials, a nearly century-old theory. Due to the limitations of mathematical tools in the past, it was difficult to discuss the attainability issues in singular cases. Professor Graeme Milton highly recognized my efforts in this new direction and wrote a recommendation letter for me. I truly appreciate his recognition, which serves as significant encouragement for me.

During my PhD, I also collaborated with Will on a class of free boundary problems requiring extensive application of viscosity solution theory and geometric measure theory. Two papers from this research will be published in the prestigious journals JMPA and ARMA, respectively. In these works, we rigorously derived in a reasonable linearized setting the microscopic origin of the rate-independent motion of droplets on rough surfaces using homogenization theory. This is unprecedented in partial differential equations, especially in high-dimensional cases. The homogenized equations possess some peculiar properties; for example, a free boundary condition resembling the well-studied thin obstacle problem unexpectedly emerges. We discovered and solved some new challenges in the regularity problem of this type of equation, although many important unsolved issues remain. I would also like to thank Professor Inwon Kim, Will's doctoral advisor, for her appreciation of this work. Her recognition encourages me a lot. I also want to thank her for her many suggestions during our meetings in several conferences.

Conferences and summer schools significantly expanded my horizons. At an optimal transport meeting in Seattle, I met Francesco, whose work at the intersection of probability and PDE impressed me deeply. His results on stationary optimal matching for random point clouds exposed me to new questions and styles of research. Later, Francesco invited me to a conference on random PDEs in Leipzig, where I encountered a different mathematical atmosphere and many outstanding young researchers. That experience broadened my perspective considerably.

In subsequent meetings on geometric measure theory and harmonic analysis, I continued to learn from many generous mathematicians. I am especially grateful to Ben Foster, whose work on critical set estimates in harmonic analysis

I greatly admire. His invitation to speak at WashU led to valuable mathematical conversations that further shaped my thinking.

These conferences consistently reminded me of both my limitations and the richness of modern analysis. I remain thankful to the many colleagues who shared their ideas with openness and generosity.

Although few peers worked directly in my area, friendships with fellow graduate students were invaluable. Informal reading groups, shared offices, and countless board games—especially Catan—created lasting memories. I am particularly grateful to Yibo, Zijie and the rest of the cohort for their companionship during these years.

Looking back, I was fortunate to complete the goals I set at the beginning of my PhD. None of this was achieved alone. It was shaped by conversations, collaborations, and friendships with many generous and talented people. I am deeply grateful to all of you. Unfortunately, due to space constraints, I cannot inscribe all of your names within these pages. I hope our paths will continue to intersect and that we may create many more meaningful results together.

致谢

在人生当中获得一段毫不迟疑，径直行动的经历是一件值得珍惜的事情，而这正是我五年博士阶段的真实写照。在我临将毕业、未来重归模糊的这一刻，我想记下一笔我的思索、生活以及遇见的那么多的值得我学习的人。

对我人生最重要的无疑是我的妈妈。她培育了我乐观和坚韧的品格。这是我能够在博士阶段坚定走下去的主要依靠。她的鼓励和赞扬也深深地激励了我。这不仅仅使我能够面对博士期间的困难，也使我能够有勇气面对生活中的困难和未来更大的未知的挑战。感谢已经不足以表达我的感情，祝我的妈妈健康长寿。

在我前往数学的道路上遇到了不少良师。首先是我的高中数学老师张先祥老师。在他的课上我第一次真正理解数学的一些概念和命题。我的印象里他不属于通常那种高深莫测的老师。他会细心地讲好概念而不是直接教我们做题。这使得我从一个面对数学无所适从的人慢慢学会适应用数学思考。有了这种基础，在高考那种高强度的复习压力下，我学会了自己训练自己。这也为大学时候进修数学专业打下了深厚的基础。

说起来我在大学的时候选择数学专业其实具有一定的偶然性。由于南科大本科生在第一年和第二年的时候是允许自由选专业的，一开始我并没有急着选专业。如果说有一点倾向的话，那一定是化学专业，因为我高中的时候还参加过一点化学竞赛。正是在室友伍润雄和刘轩与的引诱下我选择了数学专业必修的数学分析而不是通常的高等数学也就是微积分。数学分析与微积分类似但是以严格的实数理论开始的，因而比微积分更抽象也更难学。有趣的是，趁着高中时学习的冲劲，我顺利地取得了高分。这为我提供了巨大的信心。当时教课的老师是何炳生老师。他教课十分细致。课上演算步步推来，不觉称奇，但课后总有体悟，时常感叹。何老师引我入门分析学，是我人生路上重要的领路人。

另一个在我人生路上举足轻重的人便是王学锋老师了。我仍然记得他在偏微分方程课上写下Picasso的名句“Art is a lie that makes us realize the truth”。这使我第一次真正意识到科学和数学研究并不直接表示现实，现实

与理论是既冲突又包涵的。这种认识在哲学中当然是老生常谈，而真正体会到并应用到研究当中却又不太一样。总之，王学锋老师不仅教会了我大量的偏微分方程里的分析学理论和技巧，引领我进入这个领域，他还深刻影响了我学术上的喜好。未来我还将在他的影响下继续前进。

在南科大的时候我遇到了许多良师益友，包括苏琳琳老师以及许多我深深敬仰的老师。与杨绮云，谭洪泽组成的讨论小组对我而言意义重大，我们在交流中共同成长。我也由衷感谢所有使那段时光充满活力的朋友与同学。硕士期间，我与闫磊成为挚友。我们一起学习、锻炼，这期间我学会了规律地生活，这在我撰写第一篇论文的高强度一年中至关重要。是这样规律的生活支撑着我，使我最终在硕士阶段写下了人生中第一篇真正意义上的数学论文。后来在美国的几年里，由于没有多少夜生活，我们也常常一起打CSGO。记得当时信心满满要一起打够一千个小时的CSGO，不过到现在为止也不过六百多个小时。

申请博士的时候我收到了犹他大学的一份录取通知，这让我有幸成为 Will 的学生。他改变了我的人生。后来我得知，Will 对我的硕士论文很感兴趣，因此决定招收我为他的第一位博士生。他的支持让我有勇气去探索大胆的想法。当我提出用均匀化理论来描述叶脉结构时，尽管这个想法听起来颇为大胆，他仍然认真对待。正是他的信任，使我能够全力投入这一方向。起初与 Will 的讨论常让我深感震撼；他似乎对我所接触的每一个艰深的理论都驾轻就熟。后面渐渐地能够跟得上他的话了，便对自己之前的想法又有了新的认识。正是在这样对过去思想的不断打磨下，我完成了我关于用均匀化理论描述叶脉图案的工作。这一过程痛苦却又是值得的。

研究叶脉结构的想法源于我硕士阶段关于网络均匀化的工作。在与犹他大学从事生物数学研究的 Sean Lawley 教授讨论之后，我更加确信这一方向是可行的，尽管书写理论需要大量重新诠释。一个根本性的问题在于：均匀化理论中的假设，是否真的适用于分析叶脉的几何结构？这已超越了纯粹数学的讨论。正因如此，为了寻找更合理的解释框架，我开始阅读其他科学领域的论文，而不是像纯数学中那样仅关注数学证明。这是我人生中一次重要又激动的经历——我不再只是沿用既有模型，而是尝试提出属于自己的模型。这项工作在数学上也颇具意义，因为我最终用几何测度论的语言重新表述了均

匀化理论并得到了一些全新的结果，这一视角部分得益于我后来在自由边界问题上的研究。

我的这项工作实际上也是对Wiener bound, 一个有将近一百年历史的关于复合材料传导率估计，可达性问题的一个扩展。过去的理论由于数学工具的局限，难以讨论奇异情形下的可达性问题，Graeme Milton教授非常认可我在这一全新的方向所做的努力并为我写了推荐信。我非常感谢他的认可，这是对我的一个重要的鼓励。

在博士期间，我与 Will 合作研究了一类需要使用大量粘性解理论和几何测度论的自由边界问题。这项研究中的两篇文章将分别发表在著名的杂志JMPA和ARMA上面。在这些工作里我们严格地用均匀化理论在一种合理的线性化后的情形下推导了粗糙表面上液滴的rate independent motion的微观起源。这在偏微分方程里，特别是高维的情形下，尚属首次。均匀化后的方程具有一些奇特的性质，比如，一种类似于thin obstacle problem的自由边界意外地显现出来。我们发现并解决了这类方程的正则性问题中一部分新出现的挑战，尽管其中仍然有很多重要而待解决的问题。我特别想感谢Inwon对我们这项工作的赞赏。她是Will的导师，并且是自由边界问题的知名专家。她的认可给了我很大的鼓励。

这个研究似乎与之前关于叶脉的研究毫不相关，但事实上，对自由边界问题的思考极大地拓宽了我的视野，使我逐渐理解很多几何测度论里的思想和技术并最后用于叶脉图案的理论建设。正是通过对自由边界理论的深入研究使我能够自由地掌握这些理论并随心运用。

会议和暑期学校极大地开阔了我的眼界。在西雅图的一次最优传输会议上，我结识了 Francesco，他在概率与偏微分方程交叉领域的工作令我印象深刻。他关于最优匹配中平移不变稳态的研究，使我接触到新的问题与思考方式。后来，他邀请我参加在莱比锡举行的随机偏微分方程会议，在那里我感受到一种不同的数学氛围，也结识了许多杰出的青年学者。这段经历极大拓宽了我的视野。在后面参加的几何测度论与调和分析会议上，我也从许多很厉害的数学家那里获益良多。我特别感谢 Ben Foster，他在调和分析中关于临界集估计的工作让我十分敬佩。他邀请我到华盛顿大学做报告，使我有机

会进行富有价值的数学交流，也进一步了解了他们领域的内容。这些会议不断提醒我自身的不足，以及现代分析的广阔与丰富。我始终感谢那些以开放与慷慨态度分享思想的同行。

虽然直接从事我研究方向的博士生同学并不多，但与其他博士生建立的友谊同样弥足珍贵。非正式的读书小组、共享办公室的时光，以及无数次桌游——尤其是Catan——都成为难忘的回忆。我尤其感谢毅博和子杰还有很多其他同学在这些年里的陪伴。

回顾过去，我很幸运完成了博士之初设定的目标。这一切从来不是独自完成的，而是在无数交流、合作与友谊中逐渐成形。我由衷感谢你们每一个人。可惜的是由于篇幅有限，我无法将你们每个人的名字刻写再此。但愿未来的道路上，我们仍能相遇，并共同创造更多有意义的成果。

Chapter 1

Background of the dissertation

This chapter is meant to present the materials in this dissertation in a slightly broader context and raise some new problems. That is to say, we will not cover the materials in the dissertation in detail, so please read the introduction of each chapter for the detailed descriptions of the results.

1.1 Loops in conductive networks

Networks can exhibit complex geometric structures at certain intermediate scales as we vary their apparent conductivity properties. A particular example is the appearance of loops in networks that are under random fluctuations in the pressure or damages in the links. In 2007, Bohn and Magnasco [34] studied a type of conductive network with power sum constraints, which primarily models biological networks such as leaf venation patterns. Specifically, given a finite connected graph $G = (V, E)$, assign to each node $\mathbf{i} \in V$ a sink-source $I(\mathbf{i}) \in \mathbb{R}$ such that $\sum_{\mathbf{i}\in V} I(\mathbf{i}) = 0$, and to each edge $\mathbf{ij} \in E$ there is a positive number $\kappa_{\mathbf{ij}} = \kappa_{\mathbf{ji}}$ that represents the conductance and a number $J_{\mathbf{ij}} = -J_{\mathbf{ji}} \in \mathbb{R}$ that represents the flux from $\mathbf{i}$ to $\mathbf{j}$, and all these quantities satisfy: the Kirkhoff current law: for fixed $\mathbf{i} \in V$, denote $V_{\mathbf{i}}$ as the set of nodes so that there is an edge in E connecting $\mathbf{i}$ and $\mathbf{j}$

$$\sum_{\mathbf{j}\in V_{\mathbf{i}}} J_{\mathbf{ij}} = I(\mathbf{i}), \tag{1.1.1}$$

and a power sum constraint

$$K^{\gamma} = \sum_{\mathbf{ij}\in E} \kappa_{\mathbf{ij}}^{\gamma} \tag{1.1.2}$$

for some $K > 0$ and $\gamma \in (0,1)$. A good network (G, κ, J) minimizes the following total dissipation

$$\min_{\kappa} \min_{J} \sum_{\mathbf{ij} \in E} \frac{|J_{\mathbf{ij}}|^2}{\kappa_{\mathbf{ij}}}, \tag{1.1.3}$$

over all configurations of fluxes J and conductance κ that satisfy (1.1.1) and (1.1.2) respectively.

Numerically it was shown that an optimal configuration $(\kappa_{\text{opt}}, J_{\text{opt}})$ will exhibit tree-like patterns. That is, there are no sequences of edges of the form $\mathbf{i}_1\mathbf{i}_2, \mathbf{i}_2\mathbf{i}_3, \ldots, \mathbf{i}_n\mathbf{i}_1$ in G on which the conductances are all positive.

However if one takes a closer look at tree leaves, loops will show up at small scales. This mystery led to the works of Katifori, Szöllősi and Magnasco in [104] and Corson in [58]. In these works, it was observed that the appearance of loops is a result of the random opening and closing of the sink-source $I(\mathbf{i})$ and possible damage of edges. Specifically in the setting of Corson, we consider on a chess board $G = (V, E)$, where $V = \mathbb{Z}^2 \cap [0, N]^2$ for some $N \in \mathbb{N}_+$ and E is the set of unit intervals that connect nodes in V, the source at the origin $I(0) > 0$ and the sinks elsewhere $I(\mathbf{j})$ are random variables with expectation -1 and $\sum_{\mathbf{i} \in V} I(\mathbf{i}) = 0$ in all instances. In this scenario, a good network will minimize the *expected* total dissipation

$$\min_{\kappa} \mathbb{E}\left[\min_{J} \sum_{\mathbf{ij} \in E} \frac{|J_{\mathbf{ij}}|^2}{\kappa_{\mathbf{ij}}}\right], \tag{1.1.4}$$

where the expectation $\mathbb{E}$ is taken with respect to the random sink-source $I(\mathbf{i})$ and the minimization is again taken over all the possible configurations of J and κ. Under appropriate assumptions on the randomness of $I(\mathbf{i})$, they did observe the appearance of loops in the optimal networks, see Figure 1 in [58] for example.

A natural and yet difficult question is

Can we mathematically illustrate the transition from tree-like
to loopy as we increase the fluctuation of the sink-source? **(P1)**

Until now the author is only aware of a trial in this direction in the work of De Masi, Marchese and Massaccesi in [118], which gives a well-posedness result of a relaxed form of the minimization problem (1.1.4). In Chapter 2 of this dissertation, the author derived loopiness under an additional periodicity assumption. Please see the chapter for more details of the result.

1.1.1 Mathematical formulation

Let us now give a precise description of the problem (**P1**). To make a better connection with geometric measure theory and especially the branched optimal transport theory, see [29, 57, 176], we will reformulate the problems (1.1.3) and (1.1.4) in a continuum setting. We emphasize here that the mathematical formulation in this subsection is meant to present a possible framework to work with, and it is possible that the actual answer deviates from the expectation made here.

Let ν_+ and ν_- be two probability measures supported on a bounded open domain $\Omega \subset \mathbb{R}^d$ that represent sources and sinks respectively, and μ a 1-rectifiable Radon measure (for the definition see Section 2.2.3.4) supported on $\overline{\Omega}$ that represent a conductive network that transfers heat / water / electricity from ν_+ to ν_-. The total power of dissipation of the configuration

$$\mu \text{ and } \nu = \nu_+ - \nu_-$$

can be computed via

$$H(\mu;\nu) := \inf\left\{ \int_{\mathbb{R}^d} |\tau|^2 d\mu \, ; \, \tau \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d), \tau(x) \in T_x(\mu) \text{ and } \nu + \operatorname{div}(\tau\mu) = 0 \right\}, \quad (1.1.5)$$

where $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ is the L^2 space weighted by the measure μ, $T_x(\mu)$ is the tangent line of μ at μ-a.e. point x and the divergence "div" is defined in the distributional sense. Given a parameter $\alpha \in (0,\infty)$, the α-mass of the network μ is defined as

$$N_\alpha(\mu) := \int_E w^\alpha d\mathcal{H}^1, \quad (1.1.6)$$

where $d\mu = w\, d\mathcal{H}^1\big|_E$ with $\mathcal{H}^1$ the 1-dimensional Hausdorff measure. The α-*branched shape optimization* aims to solve the following minimization problem

$$\mathcal{C}_\alpha(\nu) := \inf_{N_\alpha(\mu) \le K^\alpha} H(\mu;\nu), \quad (1.1.7)$$

where $K > 0$ is a constant and the infimum is taken over all 1-rectifiable Radon measures μ that are supported on $\overline{\Omega}$. We shall discuss the well-posedness of the problem $\mathcal{C}_\alpha$ in Theorem 1.1.1 by establishing a connection of the problem $\mathcal{C}_\alpha$ with a β-branched optimal transport problem for $\beta = \frac{2\alpha}{1+\alpha}$.

In the random scenarios, the sink-source $\nu = \nu_+ - \nu_-$ is the subtraction of two random probability measures ν_+ and ν_- with law $\mathbb{P}$, and the minimization problem becomes

$$\hat{C}_\alpha(\mathbb{P}) := \inf_{N_\alpha(\mu)\le K^\alpha} \mathbb{E}_\nu \left[H(\mu;\nu)\right]. \tag{1.1.8}$$

The problem (**P1**) can now be formulated as the following problems

1. Is the problem (1.1.8) well-posed for general $\mathbb{P}$? Note that the problem is indeed well-posed in the deterministic setting, see Theorem 1.1.1. The well-posedness issue has been carefully analyzed in a reasonable variant of (1.1.8) in [118].

2. Under what law $\mathbb{P}$ will the connected components of the support of an optimizing μ in the minimization problem (1.1.8) admit nontrivial first fundamental group?

The above problems can be made even more precise and quantitative if we make a specific choice of sink-source. Particularly we follow the ideas in [58] and assume that ν_- and ν_+ support on the grid points $\mathbf{i} \in V := \mathbb{Z}^2 \cap [0,N]^2$ for some $N \in \mathbb{N}_+$ and $\overline{\Omega} = [-2N,2N]^2$. The source ν_+ supports at the origin and the sink ν_- support on $V \setminus \{0\}$ and satisfies

$$\mathbb{E}[\nu_-(\mathbf{i})] = -1 \text{ and } \mathbb{E}[\nu_-(\mathbf{i})\nu_-(\mathbf{j})] = 1 + \sigma^2\delta_{\mathbf{ij}} \tag{1.1.9}$$

for $\mathbf{i},\mathbf{j} \in V \setminus \{0\}$ and some $\sigma > 0$. Note that ν_- becomes deterministic when $\sigma = 0$. As suggested by the works in [95] we ask the following question of the transition of phase from loopless to loopy:

3. Is there a critical $\sigma_c > 0$ such that the connected components of the support of a minimizing μ of (1.1.8) with ν as described in (1.1.9) will have trivial fundamental group if $0 < \sigma < \sigma_c$ and nontrivial ones if $\sigma > \sigma_c$?

1.1.2 Branched optimal transportation and well-posedness in the deterministic case

There is another highly relevant and widely known variational problem called *branched optimal transportation*. We refer to [29, 56, 57, 176] for more details about this problem. Specifically, we start with a 1-rectifiable current T, denoted as $[F,\theta,j]$, such that when viewed as a vector-valued measure

$$dT = j\theta\, d\mathcal{H}^1\big|_F,$$

where $F \subset \overline{\Omega}$ is 1-rectifiable, θ is a Borel nonnegative function and j is a unit vector field such that $j(x)$ belongs to the tangent line of F at $\mathcal{H}^1$-almost every $x \in F$. Given $\beta \in (0, \infty)$, the β-mass of the current T is defined as

$$M_\beta(T) := \int_F \theta^\beta d\mathcal{H}^1, \text{ where } d\mu = j\theta\, d\mathcal{H}^1\big|_F. \tag{1.1.10}$$

The β-branched optimal transportation aims to solve the following problem

$$\mathcal{E}_\beta(\nu) = \inf_{\partial T = \nu} M_\beta(T), \tag{1.1.11}$$

where the infimum is taken over 1-rectifiable currents $T = [F, \theta, j]$ such that $F \subset \overline{\Omega}$ and ∂T is the boundary of T that satisfies

$$\partial T(\phi) = T(D\phi)$$

for any $\phi \in C_0^1(\Omega)$.

Theorem 1.1.1. *For any $\alpha \in (0, \infty)$ and $\nu = \nu_+ - \nu_-$ a difference of probability measures $\nu_\pm$ supported on Ω, there is*

$$\mathcal{C}_\alpha(\nu) = \frac{1}{K} \left(\mathcal{E}_{\frac{2\alpha}{1+\alpha}}(\nu) \right)^{\frac{1+\alpha}{\alpha}}, \tag{1.1.12}$$

where $\mathcal{C}_\alpha$ is defined in (1.1.7)*, $K > 0$ is a constant and $\mathcal{E}_{\frac{2\alpha}{1+\alpha}}$ is defined in* (1.1.11)*. In particular, the problem $\mathcal{C}_\alpha(\nu)$ is well-posed if and only if $\mathcal{E}_{\frac{2\alpha}{1+\alpha}}(\nu)$ is well-posed.*

Remark 1.1.2. The transition from the exponent $\alpha > 0$ in the problem $\mathcal{C}_\alpha$ to $\beta = \frac{2\alpha}{1+\alpha}$ in the problem $\mathcal{E}_\beta$ was discovered in [34] in the discrete setting. Here we check this relation in the continuum setting.

Remark 1.1.3. The result was obtained by Bouchitté and Buttazzo in [37] in 2001 in the case when $\alpha = \frac{2\alpha}{1+\alpha} = 1$, also see [42]. In this case, one obtains the equivalence of the heat conduction optimization problem, the 1-D Plateau problem and the Monge-Kantorovich problem with distance cost. The proof of Theorem 1.1.1 is distinct from that in [37] as here we have a non-convex mass constraint.

Proof. Let us start with the total dissipation

$$H(\mu; \nu) = \inf \left\{ \int_{\mathbb{R}^d} |\tau|^2 d\mu \,;\, \nu + \operatorname{div}(\tau\mu) = 0,\ \tau \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d) \text{ and } \tau(x) \in T_x(\mu) \right\}.$$

Note that $H(\lambda\mu;\nu) = \frac{1}{\lambda}H(\mu;\nu)$ for any $\lambda > 0$ and

$$\mu = \frac{K\tilde{\mu}}{N_\alpha(\tilde{\mu})^{1/\alpha}}$$

satisfies $N_\alpha(\mu) \le K^\alpha$ for arbitrary $\hat{\mu}$ without mass constraint. This implies that

$$\begin{aligned}
\mathcal{C}_\alpha(\nu) &= \inf_{N_\alpha(\mu)\le K^\alpha} H(\mu;\nu) \\
&= \inf_{\tilde{\mu}} \frac{1}{K} H(\tilde{\mu};\nu) N_\alpha(\tilde{\mu})^{1/\alpha} \\
&= \frac{1}{K} \inf_{d\tilde{\mu}=\tilde{w}d\mathcal{H}^1|_{\tilde{E}}} \inf_{\nu+\mathrm{div}(\tau\tilde{\mu})=0} \left\{ \left[\int_{\tilde{E}} |\tau|^2 \tilde{w} d\mathcal{H}^1 \right]^{\frac{\alpha}{1+\alpha}} \left[\int_{\tilde{E}} \tilde{w}^\alpha d\mathcal{H}^1 \right]^{\frac{1}{1+\alpha}} \right\}^{\frac{1+\alpha}{\alpha}} \qquad (1.1.13) \\
&\ge \frac{1}{K} \inf_{d\tilde{\mu}=\tilde{w}d\mathcal{H}^1|_{\tilde{E}}} \inf_{\nu+\mathrm{div}(\tau\tilde{\mu})=0} \left(\int_{\tilde{E}} |\tilde{w}\tau|^{\frac{2\alpha}{1+\alpha}} d\mathcal{H}^1 \right)^{\frac{1+\alpha}{\alpha}} \\
&\ge \frac{1}{K} \left(\mathcal{E}_{\frac{2\alpha}{1+\alpha}}(\nu) \right)^{\frac{1+\alpha}{\alpha}}.
\end{aligned}$$

On the other hand, for any 1-rectifiable optimal current $T_o = [F_o, \theta_o, j_o]$ of $\mathcal{E}_{\frac{2\alpha}{1+\alpha}}$ such that $\partial T_o = \nu$, we can construct an optimal $d\mu_o = w_o d\mathcal{H}^1|_{E_o}$ and $\tau_o \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ as follows

$$E_o := F_o, \; w_o := \frac{K}{\left(\mathcal{E}_{\frac{2\alpha}{1+\alpha}}(\nu)\right)^{1/\alpha}} \theta_o^{\frac{2}{1+\alpha}} \text{ and } \tau_o := \frac{\left(\mathcal{E}_{\frac{2\alpha}{1+\alpha}}(\nu)\right)^{1/\alpha}}{K} \theta_o^{\frac{\alpha-1}{\alpha+1}} j_o. \qquad (1.1.14)$$

Note that $N_\alpha(\mu_o) = K^\alpha$ whenever $\mathcal{E}_{\frac{2\alpha}{1+\alpha}}(\nu)$ is positive and finite, and the boundary condition $\nu + \mathrm{div}(\tau_o\mu_o) = 0$ is also satisfied. This shows that the inequality (1.1.13) is an actual equality whenever $\mathcal{E}_{\frac{2\alpha}{1+\alpha}}$ is well-posed, and the proof is done.

□

Remark 1.1.4. From the proof we can observe that even in the random case, i.e., ν is a random signed measure with some law $\mathbb{P}$, a slight modification of the inequality (1.1.13) gives a lower bound on the problem $\hat{C}_\alpha$ whenever the branched optimal transport problem $\mathcal{E}_{\frac{2\alpha}{1+\alpha}}$ is well-posed. However, we should not expect that this inequality is sharp as we saw them in the deterministic case in (1.1.14). The main reason is that the optimizers of the branched optimal transport (in the deterministic case) is known to be tree-like, see [29], and we expect the optimizers in the random case to be loopy.

1.2 Homogenization of the parabolic Bernoulli problem

The homogenization results in Chapter 3 and the gradient degenerate equations in Chapter 4 are in close relation to the homogenization of the following parabolic one-phase Bernoulli problem in dimensions $d \geq 2$

$$\begin{cases} u^\varepsilon_t = \Delta u^\varepsilon & \text{in } \{u^\varepsilon > 0\} \\ |\nabla u^\varepsilon| = f(x/\varepsilon) & \text{on } \partial\{u^\varepsilon > 0\}, \end{cases} \tag{1.2.1}$$

where f is a continuous positive periodic function on $\mathbb{R}^d$ and $\varepsilon \to 0^+$. In the stationary case (i.e. $u^\varepsilon_t = 0$) the homogenization is well-studied [45, 50, 80, 81, 108]. The motivation of the problem comes from the homogenization of flame propagation at certain scales [47, 48, 107] and mean curvature flows with rough capillary boundary conditions [5, 172].

For the stationary case it is known that there are pinning intervals

$$I_f[e] = [Q_f(e), Q^f(e)] \subset [\min f, \max f]$$

for each $e \in \mathbb{R}^d$, where

$$I_f[e] = I_f[-e] \text{ and } I_f[\lambda e] = I_f[e] \text{ for all } \lambda > 0.$$

The limit solution u solves the following anisotropic one-phase Bernoulli problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| \in I_f[\nabla u] & \text{on } \partial\{u > 0\}. \end{cases}$$

The homogenized coefficients $Q_f(e)$ and $Q^f(e)$ are obtained via the following plane-like solutions, or in homogenization terms, correctors

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| = f(x) & \text{on } \partial\{u > 0\} \\ \sup_{x \in \{u>0\}} |u(x) - \alpha e \cdot x| < \infty. \end{cases} \tag{1.2.2}$$

The slopes $Q_f(e)$ and $Q^f(e)$ are defined as the smallest and largest such α in (1.2.2) respectively.

Define for each $e \in \mathbb{R}^d$ and $\tau \in \mathbb{R}$

$$R_f[\tau; e] := Q^f(e)\tau_+ - Q_f(e)(-\tau)_+.$$

We believe that the homogenized problem takes the following form that admits a rate-independent boundary motion law

$$\begin{cases} u_t = \Delta u & \text{in } \{u > 0\} \\ |\nabla u| \in \partial_\tau R_f[u_t; \nabla u] & \text{on } \partial\{u > 0\}, \end{cases} \tag{1.2.3}$$

where $\partial_\tau R_f[\tau; e]$ is the subdifferential of $R_f[\tau; e]$ in τ.

In Chapter 3, we characterize the homogenized limit in a linearized version of the above parabolic equation under the assumption that $f(x) = f(x_1)$ depends on only one variable. See Chapter 4 for the linearization near the free boundary points in the stationary case. Specifically we derive the homogenized equation of the following problem

$$\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon & \text{in } B_1 \cap \{x_1 > 0\} \times (0, \infty) \\ u_{x_1}^\varepsilon = g(u^\varepsilon/\varepsilon) & \text{on } B_1 \cap \{x_1 = 0\} \times (0, \infty), \end{cases}$$

where g is periodic. For convenience let's take $g(v) = \cos(v)$, and then the homogenized equation is the following parabolic problem with a rate-independent gradient-degenerate boundary condition

$$\begin{cases} u_t = \Delta u & \text{in } B_1 \cap \{x_1 > 0\} \times (0, \infty) \\ u_{x_1} \in \partial |u_t| 1_{\{\nabla' u = 0\}} & \text{on } B_1 \cap \{x_1 = 0\} \times (0, \infty), \end{cases} \tag{1.2.4}$$

where $\partial|\cdot|$ is the subdifferential of the absolute value function and $\nabla' u$ is the (spatial) tangential gradient of u on $B_1 \cap \{x_1 = 0\} \times (0, \infty)$.

There are two main interests in the problem (1.2.3). The first interest is whether we have uniqueness (or a comparison principle) for this limit parabolic problem. It can be shown that the linearized problem (1.2.4) admits a comparison principle after some effort of deriving the viscosity solutions. This phenomenon is definitely not obvious even in the linear case due to the differential inclusion condition and the uniqueness actually does not appear in the stationary case. The second interest lies in the occurrence of free facets (portions that are flat) on the free boundary $\partial\{u > 0\}$ and their evolutions. These free facets have been discovered in the stationary case in general, see [80, 81], but the temporal evolutions are still mysterious. In Chapter 3, the viscosity solution notions we derive provide a description of the rate-independent motion of the free facets in the linearized case.

It will be interesting to generalize the results to the case of Bernoulli problems as in (1.2.1).

It is also interesting to note that the stationary problem of (1.2.4) takes the following form

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \cap \{x_1 > 0\} \\ u_{x_1} |\nabla' u| = 0 & \text{on } B_1 \cap \{x_1 = 0\}, \end{cases}$$

which generalizes the well-studied thin obstacle problem, see [19, 20, 84, 163]. Whether such a solution u belongs to $C^{1,1/2}$ up to the boundary $B_1 \cap \{x_1 = 0\}$ is still unknown. This regularity issue is also closely related to the flat asymptotic regularity of some anisotropic one-phase Bernoulli problems, which we do not specify here.

Chapter 2

Attainability of the singular Wiener bound and complex conductive networks

2.1 Introduction

What are the appropriate geometric objects to describe leaf vein patterns? While they are common planar networks, it is difficult to effectively characterize the geometry energetically. In this paper, we approach this issue through a variational problem on the hydraulic conductivity properties of leaves. More specifically, we study the lower and upper attainability of a singular version of the Wiener bound [174] for mixtures of conductive materials. This bound is also widely known as the conductive analogue of the Voigt-Reuss-Hill bound [94, 149, 171] in elasticity literature. In this new singular setting, high-conductance materials concentrate on lower dimensional sets. Here by "attainability" we mean the characterization of the mixtures that attain or do not attain the lower or the upper bound.

In particular, our work on the lower attainability provides a potential mathematical support to an existing biological theory on the reticulation phenomenon in higher-order leaf vein patterns, which states that random hydraulic fluctuations imply reticulation [58, 104].

Let us recall the original version of the Wiener bound. By a standard homogenization argument [100, 134, 168] (also see Appendix 2.7.2), a material mixture is represented by a positive definite matrix field $A(x)$ on the torus $\mathbb{T}^n$ (or equivalently $A(x)$ is a periodic matrix field on $\mathbb{R}^n$). Such a matrix field often satisfies the following ellipticity condition

$$\lambda^{-1} \le A(x) \le \lambda, \text{ for all } x \in \mathbb{T}^n, \tag{2.1.1}$$

for some positive constant λ. The *effective conductance tensor* of the mixture $A(x)$ is defined as the positive definite matrix $Q(A)$ that satisfies the following variational problem

$$p \cdot Q(A)p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\nabla\varphi(x) + p) \cdot A(x)(\nabla\varphi(x) + p) d\mathcal{L}^n, \tag{2.1.2}$$

where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{T}^n$ and $p \in \mathbb{R}^n$ is an arbitrary vector that represents the effective pressure gradient from the exterior that applies to the mixture $A(x)$.

The *classical Wiener bound* provides an explicit range of the effective tensor $Q(A)$.

$$\left(\int_{\mathbb{T}^n} A^{-1}(x) d\mathcal{L}^n\right)^{-1} \le Q(A) \le \int_{\mathbb{T}^n} A(x) d\mathcal{L}^n. \tag{2.1.3}$$

The upper bound is attained if and only if $\operatorname{div} A = 0$, and the lower bound is attained if and only if $\operatorname{curl} A^{-1} = 0$. See [100, Section 1.6] for more details and literature.

Although the attainability for the classical Wiener bound (2.1.3) can be characterized explicitly by the equations $\operatorname{div} A = 0$ and $\operatorname{curl} A^{-1} = 0$ respectively, the problem becomes much harder as one looks at A's that concentrate on lower dimensional sets.

Indeed, to model leaf vein patterns, it is natural to consider the matrix fields A on $\mathbb{T}^2$ that have high magnitude near a 1-dimensional network $\Gamma \subset \mathbb{T}^2$. To give a heuristic discussion we consider the 2×2 matrix fields A_δ of the following form

$$A_\delta(x) := \begin{cases} \frac{1}{\delta} I_{2\times 2} & \text{for } \operatorname{dist}(x, \Gamma) < \delta/2 \\ \delta I_{2\times 2} & \text{elsewhere,} \end{cases}$$

where $\delta > 0$ is a small parameter. Note that A_δ does not satisfy the ellipticity condition (2.1.1) uniformly. To sharply characterize the conductivity property of Γ we send the small parameter $\delta \to 0^+$. The limit of A_δ turns out to be a matrix-valued measure A_0 of the form

$$dA_0(x) = I_{2\times 2}\, d\mathcal{H}^1\big|_\Gamma(x), \tag{2.1.4}$$

where $\mathcal{H}^1|_\Gamma$ is the one-dimensional Hausdorff measure restricted to the 1-D set Γ. Note that at least heuristically the classical Wiener bound (2.1.3) degenerates to

$$0 \le Q(A_0) \le \int_{\mathbb{T}^2} dA_0(x), \tag{2.1.5}$$

where the lower bound becomes zero because A_0^{-1} is infinite for Lebesgue almost every point.

Now the difficulty immediately occurs as one looks back on the attainability equations for the upper and lower bound in (2.1.5)

$$\operatorname{div} A_0 = 0 \;\text{ and }\; \operatorname{curl} A_0^{-1} = 0.$$

Notice that A_0 is a singular measure supported on a lower dimensional set Γ. The first equation $\operatorname{div} A_0 = 0$ can still be interpreted in the sense of distributional derivatives. Unfortunately the second equation $\operatorname{curl} A_0^{-1} = 0$ is ill-posed even in the distributional sense. This is because A_0 is a singular measure and A_0^{-1} is supposed to be infinity for Lebesgue almost every point.

In this paper, we develop a theory to deal with both the lower and upper attainability problem in the singular case (2.1.5), by using techniques from geometric measure theory, calculus of variations, homotopy groups and etc. The main results are the two theorems:

Theorem A on the lower attainability and Theorem B on the upper attainability.

To make a more precise presentation, we will model singular matrix fields like A_0 in (2.1.4) by positive semi-definite matrix-valued Radon measures, which we call *medium* for convenience (see Definition 2.3.1). More precisely we call θ a medium if it is a matrix-valued measure on $\mathbb{T}^n$ that takes the form

$$d\theta = \sigma \, d\,\|\theta\| \tag{2.1.6}$$

for some Radon measure $\|\theta\|$ and some positive semi-definite matrix field σ such that the trace $\operatorname{Tr}(\sigma(x)) = 1$ for $\|\theta\|$-almost all $x \in \mathbb{T}^n$. Here σ represents the local anisotropy and $\|\theta\|$ represents the local magnitude of conductance. Note that the effective tensor $Q(\theta)$ can be defined similarly as in (2.1.2) by replacing $A(x)$ by

$\sigma(x)$ and $d\mathcal{L}^n$ by $d\|\theta\|$. In this general setting, the singular Wiener bound (2.1.5) becomes

$$0 \le Q(\theta) \le \theta(\mathbb{T}^n). \tag{2.1.7}$$

Our first main result, Theorem A, focuses on the lower attainability of isotropic 1-D media θ on $\mathbb{T}^n$. By 1-D we mean that the medium θ takes the form

$$d\theta = I_{n\times n}\, dw = \frac{1}{n} I_{n\times n}\, d(nw) \tag{2.1.8}$$

with the Radon measure w satisfying

$$0 < \limsup_{r\to 0^+} \frac{w(B_r(x))}{r} < \infty \tag{2.1.9}$$

for w-almost every $x \in \mathbb{T}^n$. We also require the following *coercivity* condition

$$\limsup_{r\to 0^+} \frac{w(B_r(x))}{r} > c > 0 \tag{2.1.10}$$

for some positive constant c and $\mathcal{H}^1$-almost every x in the support Spt w = Spt θ. An immediate consequence of the coercivity condition is that $\mathcal{H}^1(\text{Spt }\theta) < \infty$ (see Section 2.2.3.5). Also note that whenever E is a closed subset of $\mathbb{T}^n$ such that $\mathcal{H}^1(E) < \infty$, the restriction $w = \mathcal{H}^1\big|_E$ satisfies (2.1.9) and (2.1.10).

For such 1-D media θ, we characterize the relation between the kernel of the effective tensor $Q(\theta)$ and the collection of *homotopy classes* of closed paths in the support Spt $\theta \subset \mathbb{T}^n$ of θ. As we shall see later, the kernel of the effective tensor represents the directions in which there is zero conductance in large scales. The interest of this characterization lies in the interaction between the large-scale behavior in conductance and the microscale topology of the support of the medium.

The notion of homotopy classes is a standard tool to classify closed paths in a topological space (see Section 2.2.4 for the preliminaries and references). However, to describe the corresponding topological properties of the support Spt θ for different $\ker Q(\theta)$, we need to introduce several new notions that apply the homotopy classes in a slightly unusual way. First of all, we need to define a notion that characterizes the abundance of loops in a given subset E of $\mathbb{T}^n$ in terms of the closed paths in E. Specifically for each subset $E \subset \mathbb{T}^n$, we define $H_E \subset \mathbb{Z}^n$ the collection of all the homotopy classes of closed paths in $\mathbb{T}^n$ that have image contained in E, see Definition 2.6.1 for a precise definition. Next, we call E

loopy if H_E spans the whole $\mathbb{R}^n$ (not $\mathbb{Z}^n$; see Remark 2.6.4), and call E **reticulate** if there is a loopy connected component in E, see Definition 2.6.2 for more details.

We remark here that loopiness is equivalent to reticulation in dimension $n = 2$, but they are not necessarily the same in higher dimensions. Please see Remark 2.6.3 and Lemma 2.6.15 for more details.

Theorem A (see Theorem 2.6.5). *Suppose a medium θ takes the form* (2.1.8) *with w satisfying* (2.1.9) *and the coercivity condition* (2.1.10). *Then the following identity holds*

$$\ker Q(\theta) = H_\Gamma^\perp,$$

where $H_\Gamma^\perp$ is the orthogonal space of the homotopy classes of closed paths in the support $\Gamma = \mathrm{Spt}\, w = \mathrm{Spt}\, \theta$. In particular, $Q(\theta)$ is positive definite if and only if $\mathrm{Spt}\, \theta$ *is loopy.*

As loopiness is equivalent to reticulation in dimension $n = 2$, the theorem immediately implies the following corollary.

Corollary 2.1.1 (see Theorem 2.6.6). *Under the same assumptions in Theorem A, and in dimension $n = 2$, the effective tensor $Q(\theta)$ is positive definite if and only if* $\mathrm{Spt}\, \theta$ *is reticulate.*

We refer to Section 2.1.1.1 for an outline of the proof of both Theorem A and the above corollary.

Both the positive definiteness of $Q(\theta)$ and the term "reticulate" are closely related to the modeling of leaf vein patterns. To give an intuition on the term "reticulate", we refer to Figure 2.1 for a graphical comparison among a real leaf pattern, a reticulate network and two non-reticulate networks.

The positive-definiteness of $Q(\theta)$ is closely related to the resilience of a conductive network to random fluctuations. According to the biological literature [58, 104] (also see Section 2.1.2.4 below), a leaf will experience random fluctuations of hydraulic pressure due to various factors like sunlight, humidity, insect bites and etc. In terms of mathematics, the vector p in the definition of effective tensor (2.1.2) is a random vector with no preference in directions. It is then natural to define a "resilient network" to be those having positive effective conductance

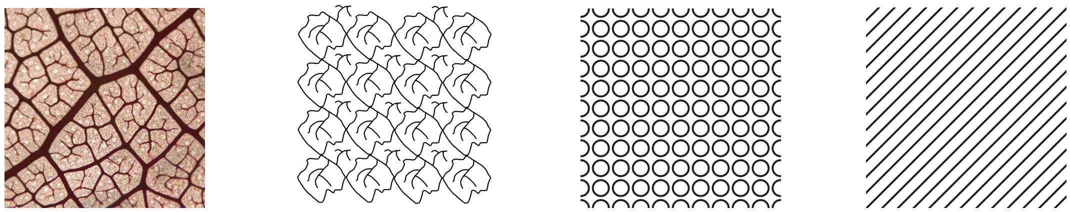

Figure 2.1: Left: a picture of higher-order veins in a tropical forest tree, *Ampelocera ruizii*. The picture is reproduced from [156], with permission from Wiley; Middle left: a reticulate $\mathbb{Z}^2$-periodic network that has full rank effective tensor; Middle right: a non-reticulate $\mathbb{Z}^2$-periodic network that has zero effective tensor; Right: a non-reticulate $\mathbb{Z}^2$-periodic network that has rank 1 effective tensor.

in all directions. Equivalently, this is saying that the effective tensor of this network is positive definite. See Definition 2.3.4 for a precise definition of resilience in terms of media.

We remark here that there is a gap between the original setting in [58, 104] and the explanation above about the term "resilience". We will make a clearer connection between these ideas in Appendix 2.7.1 by presenting a formal derivation of the positivity of $Q(\theta)$ from the minimization of the expected total dissipation of a piece of leaf under random fluctuations.

Our second main result concerns the upper attainability of the singular Wiener bound (2.1.7) for general media. We use a transformation, first introduced in [13, Remark 3.2] and also see [62], to show that all media can be equivalently represented by varifolds. Varifolds are measure-theoretic extensions of surfaces with nice compactness properties, which has been widely used in the field of minimal surfaces (see Section 2.2.3.7 for more details and references).

By using the notion of varifolds, we have the following formal identity

$$\textit{Conductance Maximality} = \textit{Area Criticality}. \tag{2.1.11}$$

On the left of the above identity, we mean media that attain the singular upper Wiener bound (2.1.7). On the right we mean stationary varifolds, which is a weak notion of the critical points to the area functional of surfaces [6]. More precisely, we have the second main theorem.

Theorem B (see Theorem 2.4.1)**.** *There is a continuous surjective map $\mathcal{T}$ from the space of all varifolds on $\mathbb{T}^n$ to the space of media. The following statements are* *equivalent for a fixed medium θ:*

(a) *The medium θ attains the upper Wiener bound* (2.1.7)*.*

(b) *The medium θ satisfies $\nabla \cdot \theta = 0$ in the distributional sense (see Lemma 2.3.16).*

(c) *All varifold realizations $\mu \in \mathcal{T}^{-1}(\theta)$ are stationary.*

(d) *There exists a stationary varifold $\mu \in \mathcal{T}^{-1}(\theta)$.*

To give a quick explanation of Theorem B we apply the well-known Allard-Almgren characterization of 1-D stationary varifolds that have positive densities (see [8]; also see Theorem 2.4.13 for a presentation of this theorem), and obtain the following corollary. We refer to Figure 2.2 for a graphical illustration of 1-D stationary varifolds.

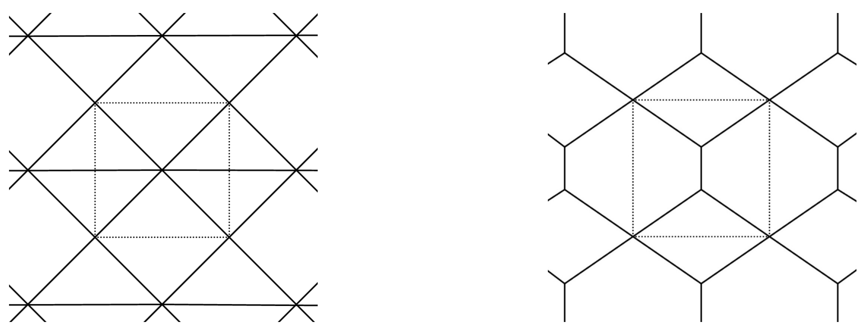

Figure 2.2: Both networks are $\mathbb{Z}^2$-periodic stationary 1-varifolds, with one period indicated in the box enclosed by dotted lines. Theorem B indicates that these are all appropriate models for leaf vein patterns, as leaves tend to maximize its hydraulic conductance to maintain its functions.

Corollary 2.1.2. *Let θ be a medium that attains the upper Wiener bound* (2.1.7) *and satisfies the 1-dimensional lower density bound*

$$\liminf_{r \to 0^+} \frac{\|\theta\| (B_r(x))}{2r} \geq \delta > 0 \tag{2.1.12}$$

for $\|\theta\|$-almost every $x \in \mathbb{T}^n$. Then $\mathcal{T}^{-1}(\theta) = \{\mu\}$ contains a unique 1-rectifiable stationary varifold μ that satisfies the following properties (we denote $\|\mu\| = \|\theta\|$ the area/mass distribution)

1. *the support* $\operatorname{Spt} \|\mu\| = \operatorname{Spt} \theta$ *is, up to an $\mathcal{H}^1$-null closed set S, a countable union of straight line segments, which are open relative to* $\operatorname{Spt} \|\mu\|$*;*

2. *on each line segment there is a constant* $c \geq \delta$ *such that the density*

$$\xi(x) := \lim_{r \to 0^+} \frac{\|\mu\| (B_r(x))}{2r} \equiv c,$$

for x *on the line segment;*

3. *at every* $x \in S$*, there exists a unique stationary tangent cone consisting of finitely many half lines with constant densities. See Figure 2.3 for the illustration of a stationary tangent cone. In such a cone, the tangent vectors* T_j *start from the origin and the densities* w_j *satisfy*

$$\sum_{j=1}^{m} w_j T_j = 0; \tag{2.1.13}$$

4. *if the density* ξ *is discretely valued, then the number of line segments is finite.*

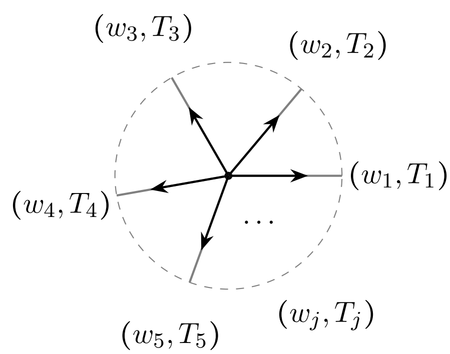

Figure 2.3: For some integer $m \geq 2$, there are m half lines with the unit outward tangent vectors denoted by T_j and densities $w_j > 0$ for $1 \leq j \leq m$. In a stationary tangent cone the vectors T_j and the weights w_j satisfy (2.1.13).

In 1977, Schulgasser [160] proved that the upper Wiener bound can be attained by sequentially laminated isotropic materials, in which the notion of attainability is slight weaker than what we mean here. Similar attainability result of maximizing media to Theorem B was first addressed in the case of networks in 2003 by Zhikov and Pastukhova [179, 180]. In 2020 the author [96] also addressed the network case. It was also discussed in [40] that the maximality is attained by the superpositions of thin laminates. We refer to Section 2.4 for a more precise discussion on Theorem B.

The fact that the conductance maximality is equivalent to the divergence free equation $\nabla \cdot \theta = 0$ is well-known [63, 90, 140, 146] in the composite-theory community (also see Section 2.3.5 for a rigorous derivation). On the other hand, it was discovered in [13, 62] that area criticality implies the divergence free equation $\nabla \cdot \theta = 0$. This implicitly proves that area criticality implies conductance maximality, although the transformation they proposed from varifolds to matrix-valued measures was initially designed to study surface energies.

Based on the identity in Theorem B, we establish a pointwise dimension bound similar to [18, Corollary 1.4] and [17] by introducing a *fractional* version of the monotonicity formula of stationary varifolds. See Section 2.1.1.2 for a more rigorous discussion. Also see Theorem 2.4.5 for more details.

Interestingly, Theorem B also allows us to access certain questions on leaf vein patterns in rigorous mathematical terms. Specifically, we can safely transfer the questions on the geometry of leaf vein patterns to the questions on 1-D stationary varifolds on $\mathbb{T}^2$, which has a more precise definition in mathematics and potentially allows a more careful analysis. For example we can ask the question of maximal valency of leaf vein patterns by asking the same question on 1-D stationary varifolds. We define a stationary network to be a stationary 1-varifold on $\mathbb{T}^2$ that has a.e. density 1. The maximal valency of such a network is defined as the maximal number of edges joining at some node (see Problem 2.4.18 for more details).

Problem 2.1.3 (see Problem 2.4.18)**.** Is the maximal valency of irreducible (see Definition 2.4.17) stationary networks on $\mathbb{T}^2$ bounded? Is it bounded by 4, 5 or 6?

The interest of this question is that the maximal valency bound 4 can be intuitively discovered from real leaves (see for example Figure 2.1), but the math is not so clear. We refer to Section 2.4.4 for the rigorous formulation of this question and some basic discussions. We also explain the notion of irreducibility there.

2.1.1 Rigorous discussion on the mathematical results

In the following we present some new techniques and consequences in Theorem A and Theorem B.

- **An outline for the lower attainability characterization**

Let us briefly outline the new notions and techniques in the proof of Theorem A. There are three main ingredients in proving Theorem A: countable decompositions, Ważewski parametrizations (see Section 2.2.3.8) and some covering space arguments (see Section 2.2.4). We refer to Section 2.6 for a complete presentation.

First, for a medium with its support having finite $\mathcal{H}^1$ measure, we have the following countable decomposition.

Theorem 2.1.4 (see Theorem 2.5.3). *Let θ be a medium such that $Q(\theta) \neq 0$ and $\mathcal{H}^1(\text{Spt }\theta) < \infty$. Then there exist countably many 1-rectifiable mutually disjoint connected components $E_i \subset \text{Spt }\theta$ such that the submedia $\theta_i := \theta\big|_{E_i}$ satisfy*

$$Q(\theta) = \sum_{i=1}^{\infty} Q(\theta_i). \tag{2.1.14}$$

The condition $\mathcal{H}^1(\text{Spt }\theta) < \infty$ is necessary as there is a counterexample in Example 2.5.7.

Theorem 2.1.4 is proved by an induction argument to select “nice” connected components, which hinges on the following characterization of nontrivial medium.

Theorem 2.1.5 (see Theorem 2.5.1). *Let θ be a medium with $Q(\theta) \neq 0$, then the support* Spt θ *is not totally disconnected, that is, there is a nonsingleton component in* Spt θ. *This implies that the 1-D Hausdorff measure*

$$\mathcal{H}^1(\text{Spt }\theta) > 0.$$

In particular, the Hausdorff dimension $\dim_{\mathcal{H}}(\text{Spt }\theta) \geq 1$.

The decomposition in Theorem 2.1.4 reduces the problem in Theorem A to the case where the support $\Gamma = \text{Spt }\theta$ is further connected and hence 1-rectifiable.

The second main step is the Ważewski parametrization, which states that there is a surjective constant speed reparametrization $\gamma : [0,1] \to \Gamma$, with constant multiplicity along with many other fine properties (see Theorem 2.2.9 for more details and references). Using this we prove the inclusion $H_\Gamma^\perp \subset \ker Q(\theta)$ (see Section 2.6.2). The key is to construct in a neighborhood of Γ the linear function $p \cdot x$ for

$p \in H_\Gamma^\perp$. This is accomplished by analyzing the path lifting in the covering space $\mathbb{R}^n/\mathbb{Z}H_\Gamma$ of a surjective Ważewski parametrization γ as described above.

In our last step (Section 2.6.3 and 2.6.4), we finish the proof by showing the following inequality

$$p \cdot Q(\theta)p \geq C|p|^2,$$

for all $p \in \mathbb{Z}H_\Gamma$ (which is equal to H_Γ because Γ is connected in this step; see Lemma 2.6.12), where $C > 0$ depends only on n, the geometry of Γ and the coercivity constant in (2.1.10). We establish this lower bound by a modification procedure that improves the estimation of lengths and multiplicities of Lipschitz closed paths in Γ without changing the homotopy classes via the techniques in [4].

The Corollary 2.1.1 follows from Theorem A and Lemma 2.6.15, which characterizes the equivalence relation between loopiness and reticulation in dimension $n = 2$.

- **Fractional monotonicity formula and pointwise dimension bound**

We will be using notations from the theory of varifolds. We refer to Section 2.2.3.7 for the preliminaries. It is known that a stationary k-varifold $\mu(x,\tau)$ satisfies that $\frac{1}{r^k}\|\mu\|(B_r(x))$ is monotone nondecreasing in $r > 0$, where $\|\mu\|$ is the area distribution. See [6, 165] for references, and also see Section 2.2.3.7. In Lemma 2.4.11 we extend this monotonicity result to a fractional version for media. To explain the term "fractional" we introduce the following new version of "rank" (see Definition 2.4.2) for a positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$:

$$1 \leq \dim_{\mathrm{r}}(A) := \frac{\mathrm{Tr}\,(A)}{\lambda_{\max}(A)} \leq \mathrm{rank}(A),$$

where $\lambda_{\max}(A)$ is the maximal eigenvalue of A. This new "rank" is useful as it provides a sharp characterization for media that can be realized as k-varifolds through the transformation in [13, 62]. Indeed, in Lemma 2.4.8, we show that a positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$ can be written as

$$A = \int_{G(k,n)} P_\tau \, d\rho(\tau)$$

for some probability measure ρ on the Grassmannian manifold $G(k,n)$ (see Section 2.2.3.7) if and only if $\dim_{\mathrm{r}}(A) \geq k$ and $\mathrm{Tr}\,(A) = k$, where P_τ is the orthogonal projection matrix from $\mathbb{R}^n$ to the k-dimensional subspace τ. Based on this observation we obtain that all media can be regarded at least as 1-varifolds. This is one of the key observations in proving Theorem B. We refer to Theorem 2.4.1 for more details.

For a medium $d\theta := \sigma\, d\,\|\theta\|$ as introduced in (2.1.6) we consider the lower semicontinuous envelope of the "rank" $\dim_{\mathrm{r}}(\sigma(x))$ of the local anisotropy σ

$$\underline{\dim}_{\mathrm{r}}(\theta)(x) := \sup_{\delta>0} \operatorname{ess\,inf}\{\dim_{\mathrm{r}}(\sigma(y))\ ;\ |y-x| \le \delta,\, y \in \operatorname{Spt}\theta\},$$

where Spt θ is the support of $\|\theta\|$ and "ess inf" is taken with respect to the Radon measure $\|\theta\|$. Based on these new notions we show a fractional monotonicity formula in Lemma 2.4.11 for media that attain the upper Wiener bound (2.1.7) (or equivalently divergence free media $\nabla \cdot \theta = 0$ by Lemma 2.3.16). Specifically we show that

$$\text{for } \alpha \in [1, \underline{\dim}_{\mathrm{r}}(\theta)(x)), \quad \frac{1}{r^\alpha}\|\theta\|(B_r(x)) \text{ is monotone nondecreasing}$$

in $0 < r < r_{\alpha,x}$ for some $r_{\alpha,x} > 0$. Moreover, if $\underline{\dim}_{\mathrm{r}}(\theta)(y) \ge m$ for $y \in U \cap \operatorname{Spt}\theta$ in an open neighborhood U of x, then α can be chosen as the constant m. This result slightly improves the standard monotonicity formula for k-varifolds, as the corresponding medium θ_μ of a k-varifold μ always satisfies

$$\underline{\dim}_{\mathrm{r}}(\theta_\mu)(x) \ge k \text{ for } x \in \operatorname{Spt}\theta$$

with equality holds if and only if for $\|\mu\|$-(or equivalently $\|\theta_\mu\|$-)almost every x the matrix field $\sigma_\mu(x) := \frac{d\theta_\mu}{d\|\theta_\mu\|}$ is an orthogonal projection to a k-dimensional subspace of $\mathbb{R}^n$.

Based on this fractional monotonicity result, we can show a pointwise dimension bound similar to [18, Corollary 1.4]. We also refer to [17] for more references on the problems of dimension bound.

Theorem 2.1.6 (see Theorem 2.4.5). *Suppose a medium θ attains the upper Wiener bound* (2.1.7). *Then for every point x in its support, one has the inequality*

$$\underline{\dim}_{\mathrm{r}}(\theta)(x) \le \underline{\dim}_{\mathrm{loc}}(\theta)(x), \tag{2.1.15}$$

where $\underline{\dim}_{\mathrm{loc}}(\theta)(x)$ is the standard lower local dimension of $\|\theta\|$ (see Definition 2.4.4). In particular, the lower local dimension always satisfies $\underline{\dim}_{\mathrm{loc}}(\theta)(x) \ge 1$.

The basic spirit of this result is that, when the maximality is attained, the "rank" of the matrix field does not exceed the lowest dimension of the local mass distribution. To be more specific, we consider a medium in $\mathbb{T}^3$ of the form

$$d\theta := \sigma \, d\mathcal{H}^2\big|_{\mathbb{T}^2\times\{0\}},$$

where σ is a constant positive semi-definite matrix. Notice that θ attains the upper Wiener bound (2.1.7) if and only if $\sigma p = 0$ for all $p \in \{0\}^2 \times \mathbb{R}$. An immediate consequence of this observation is that

$$\mathrm{rank}(\sigma) \le \dim_{\mathcal{H}}\left(\mathbb{T}^2 \times \{0\}\right) = 2.$$

This shows that, at least heuristically, the conductance-maximizing media tend to avoid the waste of material to put anisotropy off the plane $\mathbb{T}^2 \times \{0\}$. The bound (2.1.15) provides an extension of this observation in general maximizing media. We refer to Example 2.4.9 and Example 2.4.10 for more examples around this idea.

2.1.2 Literature: on leaves

- **The cohesion-tension theory**

Let us begin with a short presentation of the hydraulic behavior within a tree. The water transportation within trees can be explained by the *Cohesion-Tension* theory (CTT): the water molecules are cohesive and hence the water forms a continuous flow within the trees; because of the air/water surface tension caused by the effects of transpiration in the open pores of the stomata, the pressure of the water within the transport system is negative; the negative pressure then drives the water within the xylems upward against gravity to the leaves. After being driven through the xylem to the petiole of a leaf [123], the water will be transported through different hierarchies of the leaf veins to the opening sites of the stomata and evaporate there [124, 154, 156]. CTT was first proposed by Boehm [33] in 1893 and was soon summarized by Dixon and Joly [70] in 1894. The theory has been thoroughly studied by experiments and remains the most accepted explanation of water transport within plants [52, 106, 181, 182].

- **Darcy's law**

Darcy's law is a widely used model for porous media, including the hydraulic behavior in trees [122, 148]. It states that the flow velocity field j is related to the pressure gradient $\nabla\phi$ in the following form

$$j = -A\nabla\phi,$$

where A is a positive definite matrix field that represents the local conductance of the tree/leaf. We will give a more precise description of our model based on Darcy's law on leaf vein patterns in Appendix 2.7.1.

- **Hierarchy, fractal structure, and power law**

The hydraulic properties of leaf venations have received much attention for decades [55, 153, 154, 156, 157]. It is commonly accepted that lower-order veins are typically composed of a midrib and two orders of lateral veins and provide for bulk and far-reaching water transport, while higher-order veins often show smaller and reticulate patterns that provide "diffusive" local water distribution [154, 156]. This hierarchical order can also be observed in the models proposed in [34, 58, 104], where Bohn et al., Corson, and Katifori et al. studied the optimization of hydraulic dissipation of networks, based on some power law constraint [23], which naturally leads to a fractal geometric viewpoints on the venation patterns. A well-known power law, *Murray*'s law [161], originally proposed for modeling blood vessels, is also used in the study of leaf venations. The law was discovered by finding the balance between the efficiency of blood flow and the maintenance cost of vessel construction. Murray's law is shown in [121] to hold also for leaf venations under some assumptions, but it was also pointed out in [154] that this law may not hold for plants because plants are sun-powered and the plant vessels are often leaky.

In this paper, instead of looking at the local efficiency of veins, which leads to Murray type laws, we look for appropriate geometric shapes of networks from a larger scale by using homogenization method [100, 168]. We model leaf vein patterns, especially the higher-order ones, with these networks. This method no longer leads to a fractal geometry but it still satisfies the Darcy's law for porous medium, which is appropriate for modeling the “diffusive” water transportation in the higher order vein structures.

- **The random fluctuations and reticulation**

The random fluctuations of the pressure gradient p in (2.1.3) can be justified by the experimental results in [30, 31, 147], where it is observed that the closing and opening of the stomata pores (sites of transpiration) form a random patchy distribution (see Figure 2.4). The reticulation of veins is regarded as a redundancy to deal with such random fluctuations and possible exterior damages [58, 104]. In these works, the random opening and closing of sinks are shown to be sufficient for producing loops in the optimal structure. Without random fluctuations, the computation indicates that the optimal patterns are tree-like [24, 29, 34, 74].

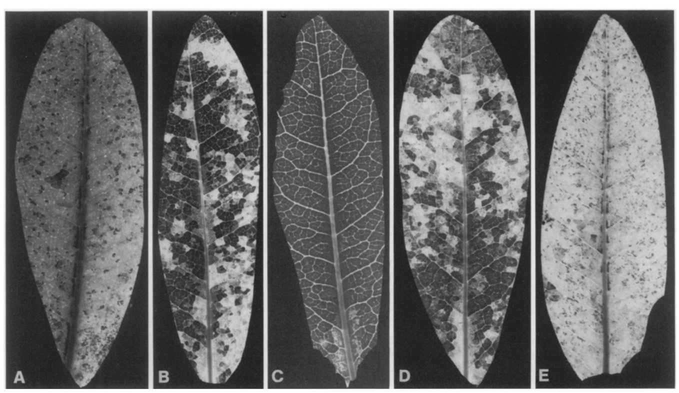

Figure 2.4: Backlit lower leaf surface of *Arbutus unedo* of different times of a day. (Light area: infiltrated area; dark area: non-infiltrated area. A: 9 a.m.; B: 10 a.m.; C: 12p.m.; D: 4 p.m.; E: 5 p.m.) This picture is reproduced from [30], with permission from Springer Nature.

- **Other models on the venation patterns**

Models on many other aspects of veins have also been proposed. In [66], Dimitrov and Zucker studied auxin concentration using reaction-diffusion equations and used this to model leaf venation formation in the early stage of a leaf. In [103, 151, 152], Ronellenfitsch and Katifori also proposed a model on auxin concentration and obtained the reticulation result if one assumes fluctuations in auxin production. In [150], Ronellenfitsch justified the optimality of reticulation in the elasticity aspect of leaf vein structures.

2.1.3 Literature: mathematics

- **Optimal design and Hashin-Shtrikman bound**

The upper and lower attainability problem for the singular version of the Wiener bound (2.1.3) lies within the more general realm of optimal design [37, 39, 43, 113]. We also refer to [54, 134] for a detailed introduction in this field. In general, an optimal design problem concerns about minimizing a cost function under some mass constraints. In the theory of conductive composites, the most famous one, other than the Wiener bound, could be the Hashin-Shtrikman bound [92, 114]. In this problem, one can derive a bound sharper than the classical Wiener bound (2.1.3) for two-phase isotropic mixtures [92, 114–116, 167]. Specifically in the case that the matrix field $A = [\alpha 1_{U_\alpha} + \beta(1 - 1_{U_\alpha})] I_{n\times n}$ for two positive numbers $\alpha > \beta > 0$ and $|U_\alpha| = h \in (0,1)$, the effective tensor $Q(A)$ satisfies

$$\beta\frac{\alpha + (n-1)\beta + (n-1)h(\alpha-\beta)}{\alpha + (n-1)\beta - (\alpha-\beta)h} \le \frac{\mathrm{Tr}(Q(A))}{n} \le \frac{n\alpha\beta + (n-1)\alpha^2 h - (n-1)\alpha\beta h}{n\alpha - \alpha h + h\beta}. \quad (2.1.16)$$

Interestingly, in the case that high conductive material concentrate on lower dimensional sets, i.e. $\alpha \to \infty$, $\alpha h \to m \in (0,\infty)$ and $\beta \to 0$, the upper H-S bound becomes $\frac{n-1}{n}m$. This is exactly the upper singular Wiener bound (in the form of average eigenvalues) for $(n-1)$-dimensional medium θ of the form $d\theta(x) = P_{\tau_x}\, dw(x)$, where w is an $(n-1)$-rectifiable Radon measure and P_{τ_x} is the orthogonal projection matrix from $\mathbb{R}^n$ to the $(n-1)$-dimensional tangent space τ_x of w at x. Note that there is a formal dimension reduction from $I_{n\times n}$ to P_{τ_x}. One should notice that the classical sequentially laminated optimizers of the H-S bound no longer capture the attainability property of this singular H-S bound as they simply homogenize in the limit. The author studied the attainability of this singular bound in dimension 2 in [96], and this became one of the motivation for studying the singular Wiener bound in a more general setting.

- **Singular structures, metric Sobolev spaces and $\mathcal{A}$-free measures**

The modeling of singular structures is one of the most challenging problems in calculus of variations. This topic has attracted attention for decades [22, 36, 38, 155, 180]. It also motivated the huge development of Sobolev spaces on metric measure spaces in recent years [11, 12, 117]. Interestingly the study of minimal surfaces also provides some good insight in some problems of singular structures. As we have pointed out, the transformation from stationary varifolds to divergence free matrix-valued measures made in [13, 62] is an important ingredient in proving Theorem B. In fact, such observations have led to an extensive study of linear distributional PDEs, specifically $\mathcal{A}$-free measures [17, 18, 62, 142].

- **Optimal transport as a model of leaf veins**

There are also some works on modeling leaf vein patterns by using the theory of optimal transportation [170]. We refer to [29, 176] for more details. In some sense the optimal transportation approach follows the same route of Murray type power laws. As we have alluded in Section 2.1.2.4, these models lead to tree-like patterns. To the best of the author's knowledge, there are no analytic results on the emergence of reticulation under the assumption of random fluctuations.

2.1.4 Structure of the paper

In Section 2.2 we present some preliminary results from various fields, including measure theory, geometric measure theory, calculus of variations and homotopy groups on the torus. In Section 2.3, we introduce the notion of medium and its effective conductance tensor. We also introduce some novel properties of the effective tensors in the singular case, which includes super-additivity, upper semi-continuity in weak* topology, discontinuity and faithful convergence, and efficient submedium. We also provide a rigorous derivation of the Euler-Lagrange equation $\nabla \cdot \theta = 0$ for maximal media. In Section 2.4 we discuss the transformation of varifolds and establish Theorem B rigorously. We provide a proof for the pointwise dimension bound in Theorem 2.1.6. We also provide a rigorous discussion on Problem 2.1.3. In Section 2.5 we provide the proofs of several useful theorems that characterize the supports of media θ that satisfy $Q(\theta) \neq 0$. In particular, we prove the countable decomposition Theorem 2.1.4. We also provide some examples to show the sharpness of these theorems. In Section 2.6, we prove Theorem A and Corollary 2.1.1.

2.1.5 Acknowledgments

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2.2 Preliminaries

2.2.1 Notations

- Let $n \geq 1$ be an integer that represents the dimension. We denote $\mathbb{R}^n$, $\mathbb{Z}^n$ and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the Euclidean space, integer lattice and the torus with dimension n.

- If not particularly mentioned, θ represent a medium and μ represent a varifold.

- Denote $\mathcal{L}^n$ as the Lebesgue measure on $\mathbb{R}^n$ or $\mathbb{T}^n$.

- Denote $\mathrm{Tr}(A)$ to be the trace of a square matrix A, that is, the sum of the diagonal entries of A.

- For any linear subspace $\tau \subset \mathbb{R}^n$, we denote P_τ to be the *orthogonal projection* from $\mathbb{R}^n$ to τ.

- Denote $G(k,n)$ the Grassmannian manifold consisting of k-dimensional subspaces of $\mathbb{R}^n$.

- For a measure w on a metric space X, we denote

$$\text{Spt } w = \text{support of } w,$$

which is the complement of the largest open subset of X that has zero w-measure. For a matrix-valued measure the support is defined as that of its total variation.

- For a real number $t \in \mathbb{R}$ we denote

$$\lfloor t \rfloor = \text{ largest integer below } t,$$

and $\lceil t \rceil = -\lfloor -t \rfloor$.

- Given a measure w on a measure space X and a measurable map $\Psi : X \to Y$, we denote the pushforward as

$$\Psi_{\#} w(A) := w(\Psi^{-1}(A)) \text{ for all measurable } A \subset Y. \tag{2.2.1}$$

Given a measurable subset $E \subset X$, define the restriction of w as

$$w_E(A) := w(E \cap A) \text{ for all measurable } A \subset X. \tag{2.2.2}$$

- Let $f : X \to Y$ be any mapping, then define $\text{Im} f$ as the image of f. If $f : X \to Y$ is further a linear mapping on linear spaces, then define $\ker f$ as the kernel of f.

- Let S be any finite set, define $\#S$ as the number of elements in S.

2.2.2 Measure theory

Let X be a metric space. In this subsection we define and present some basic properties of Radon measures and the matrix-valued versions.

- **Radon measure and Riesz representation**

A Radon measure w on a metric space X is a measure on the σ-algebra of Borel sets on X that satisfies

- $w(A)$ is finite for any compact $A \subset\subset X$

- w is outer regular on Borel sets and inner regular on open sets.

Denote by $\mathcal{R}(X)$ the space of all Radon measures on X. There is a functional analytic characterization of $\mathcal{R}(X)$.

Theorem 2.2.1 (Riesz representation)**.** *Let $C_0(X)$ be the space of compactly supported continuous functions, then there is a 1-1 correspondence between the positive linear functionals on $C_0(X)$ and $\mathcal{R}(X)$. Specifically a nonnegative linear functional ψ is related to a Radon measure w in the following form*

$$\psi(f) = \int_X f dw \text{ for all } f \in C_0(X).$$

In particular, the functional norm

$$\|\psi\|_{C_0^*(X)} = w(X)$$

if both sides are finite.

A convenient consequence of this is the following compactness result.

Lemma 2.2.2. *Suppose w_j is a sequence of Radon measures with bounded total mass*

$$\sup_j w_j(X) < \infty,$$

then up to a subsequence there is another Radon measure w_∞ such that $w_\infty(X) < \infty$ and

$$\int_X f dw_j \to \int_X f dw_\infty \tag{2.2.3}$$

for all $f \in C_0(X)$ as $j \to \infty$.

Proof. This is an immediate consequence of the Banach-Alaoglu theorem. □

Definition 2.2.3. Call the notion of convergence in (2.2.3) the weak* convergence and denote $w_j \overset{*}{\rightharpoonup} w_\infty$.

- **Matrix-valued measures and polar decomposition**

A matrix-valued measure on a metric space X is an $\mathbb{R}^{n\times n}$-valued set function θ such that for any countable sequence of pairwise disjoint Borel sets $U_j \subset X$ we have

$$\theta\left(\bigcup_{i=1}^{\infty} U_j\right) = \sum_{i=1}^{\infty} \theta(U_j).$$

In this paper, we focus on positive semi-definite matrix-valued Radon measures. More precisely we require

- $\theta(E)$ is positive semi-definite for all Borel $E \subset X$.
- The trace
$$\|\theta\|(E) := \mathrm{Tr}\,(\theta(E)) \tag{2.2.4}$$
is a Radon measure.

Lemma 2.2.4. *A matrix-valued measure θ is a positive semi-definite matrix-valued Radon measure if and only if θ decomposes as*

$$d\theta = \sigma\, d\,\|\theta\|$$

where $\|\theta\|$ the trace is a Radon measure and $\sigma \in L^1(\|\theta\|;\mathbb{R}^{n\times n})$ is the Radon-Nikodym derivative
$$\sigma(x) := \frac{d\theta}{d\,\|\theta\|}(x),\ \ x \in X \tag{2.2.5}$$
that defines a positive semi-definite matrix field and satisfies $\mathrm{Tr}\,(\sigma(x)) = 1$ for $\|\theta\|$-almost all $x \in X$.

Proof. By applying the polar decomposition theorem [9, Corollary 1.29]

$$d\theta = f\, dw$$

for some Radon measure w and a unique positive semi-definite matrix field f such that $|f|_* \equiv 1$ for some norm $|\cdot|_*$ on $\mathbb{R}^{n\times n}$. Observe that $d\,\|\theta\|(x) = \mathrm{Tr}\,(f(x))\,dw(x)$. We then define
$$\sigma(x) = \frac{f(x)}{\mathrm{Tr}\,(f(x))}.$$
Notice that σ is well-defined because $f(x)$ is positive semi-definite and then $f(x) \neq 0$ if and only if $\mathrm{Tr}\,(f(x)) \neq 0$. The proof is done by checking that for each $1 \le i, j \le n$ the component σ_{ij} is exactly the Radon-Nikodym derivative $d\theta_{ij}/d\,\|\theta\|$. □

- **Disintegration theorem**

The following theorem will be often referred to. For convenience we only present a specific version. See [10, Theorem 5.3.1] for a more general version and its proof.

Theorem 2.2.5 (Disintegration theorem)**.** *Let μ be a Radon measure on the product metric space $X \times Y$. Let π be the projection $\pi : X \times Y \to X$ that sends $(x, y) \in X \times Y$ to x. Then if we denote $\|\mu\| := \pi_{\#}\mu$, there is a family of probability measures ρ_x on Y such that*

- *for any Borel set $B \subset Y$ the function $x \to \rho_x(B)$ is Borel measurable on X.*
- *for $\|\mu\|$-almost all $x \in X$, we have $\rho_x(Y \setminus \pi^{-1}(x)) = 0$.*
- *for all Borel function f on $X \times Y$ we have*

$$\int_{X\times Y} f(x,y)d\mu(x,y) = \int_X \int_{\pi^{-1}(x)} f(x,y)d\rho_x(y)d\|\mu\|(x).$$

2.2.3 Geometric measure theory

- **Lipschitz map and function**

A Lipschitz map is a map $f : X \to Y$ from a metric space (X, d_X) to another metric space (Y, d_Y) satisfying that there is a constant $C > 0$ so that for all $x_1, x_2 \in X$

$$d_Y(f(x_1), f(x_2)) \le C d_X(x_1, x_2).$$

A Lipschitz function is a Lipschitz map from (X, d_X) to the real line.

- **Lipschitz path and length**

A path is a continuous map $\gamma : [0,1] \to X$. Define the multiplicity of γ at $x \in X$ as $m(\gamma, x) := \#\gamma^{-1}(x)$ (possibly ∞).

Define the *length* of a Lipschitz path γ in $(a_0, b_0) \subset [0,1]$ as

$$\ell(\gamma, (a_0, b_0)) := \sup \left\{ \sum_{i=1}^{N-1} d_X(\gamma(a_i), \gamma(a_{i+1})) \right\} < \infty$$

with the supremum taken over all possible choices of $a_0 \le a_1 \le a_2 \le \cdots \le a_N \le b_0$. We write $\ell(\gamma) := \ell(\gamma, [0,1])$ as the total length.

Call a Lipschitz path γ to have constant speed if $\ell(\gamma, (a_0, b_0))$ is proportional to $(b_0 - a_0)$ regardless of a_0, b_0. It is known that any Lipschitz path having finite length admits a constant speed reparametrization. Notice that if X is a Riemann manifold with Riemann metric g, then a constant speed Lipschitz path γ satisfies

$$\mathrm{Lip}(\gamma) = |\gamma'|_g = \ell(\gamma).$$

See [4, Section 3] for more details.

- **Hausdorff measure and dimension**

On a metric space (X,d), the diameter of a set $E \subset X$ is defined as

$$\mathrm{diam}(E) := \sup_{x,y\in E} d(x,y).$$

For $\alpha \geq 0$, the α-dimensional Hausdorff measure of a subset $E \subset X$ is defined as

$$\mathcal{H}^\alpha(E) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^\alpha \ ; \ \mathrm{diam}(U_i) \leq \delta \text{ and } E \subset \bigcup_{i=1}^{\infty} U_i \right\}.$$

The Hausdorff dimension of a set E is defined as

$$\dim_{\mathcal{H}}(E) = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(E) = \infty\},$$

where we take $\inf \varnothing = +\infty$ and $\sup \varnothing = 0$. Notice that whenever E has finite $\mathcal{H}^\alpha$ measure the restriction $\mathcal{H}^\alpha\big|_E$ always defines a Radon measure. See [9, Section 2.8] for more discussions on Hausdorff measures.

- **Rectifiable sets and measures**

On a metric space X, a subset $E \subset X$ is called k-*rectifiable* for some integer $1 \leq k \leq n$ if there are countably many Lipschitz maps $f_i : \mathbb{R}^k \to X$ so that

$$\mathcal{H}^k \left(E \setminus \bigcup_{i=1}^{\infty} f_i \left(\mathbb{R}^k\right) \right) = 0.$$

A measure μ is k-rectifiable if there is a k-rectifiable set E and a Borel function f such that

$$d\mu = f \, d\mathcal{H}^k\big|_E.$$

- **Density**

For a measure μ on $\mathbb{R}^n$, the k-dimensional *upper density* at $x \in \mathbb{R}^n$ is defined as

$$\Theta^*(\mu,x) = \Theta^*_k(\mu,x) := \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k},$$

where ω_k is the volume of the k-dimensional unit ball. Similarly, the *lower density* is defined as

$$\Theta_*(\mu,x) = \Theta^k_*(\mu,x) := \liminf_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k}.$$

For a Borel set $E \subset \mathbb{R}^n$, the k-th upper and lower densities are defined by replacing μ with $\mathcal{H}^k|_E$.

By [9, Theorem 2.56], we know that if μ is a Radon measure on $\mathbb{R}^n$ and E is a Borel subset, then:

(a) If $\Theta^*(\mu, x) < t$ whenever $x \in E$, then

$$\mu(E) \leq 2^k t \mathcal{H}^k(E).$$

(b) If $E \subset U$ for some bounded open domain U and $\Theta^*(\mu, x) > t$ whenever $x \in E$, then

$$\mu(E) \geq t \mathcal{H}^k(E).$$

A basic consequence of this result is that if $\Theta_k^*(\mu, x)$ is positive and finite for $\mathcal{H}^k$-almost every $x \in \text{Spt}\,\mu$ then there is a Borel function f such that $d\mu = f\, d\mathcal{H}^k|_{\text{Spt}\,\mu}$. See [112, Proposition 2.16].

- **Tangent measures and rectifiability**

Let μ be a measure on $\mathbb{R}^n$ and $\alpha \geq 0$. Define an *α-tangent measure* ν_x of μ at $x \in \mathbb{R}^n$ to be a measure so that there is a sequence of positive numbers $r_i \downarrow 0$ such that

$$\frac{\mu(x + r_i(\cdot))}{\omega_\alpha r_i^\alpha} \overset{*}{\rightharpoonup} \nu_x$$

as $i \to \infty$, where we have denoted ω_α the Gamma function extension of volumes of unit balls. Denote by $\text{Tan}_\alpha(\mu, x)$ the collection of all such ν_x, and $\text{Tan}(\mu, x) := \bigcup_{\alpha \geq 0} \text{Tan}_\alpha(\mu, x)$.

Theorem 2.2.6 ([112, Theorem 4.8]). *A measure μ on $\mathbb{R}^n$ is k-rectifiable for some integer $1 \leq k \leq n$ if and only if for μ-almost all $x \in \mathbb{R}^n$, $\text{Tan}(\mu, x) = \text{Tan}_k(\mu, x) = \{\nu_x\}$ is a singleton, where ν_x is of the form $c_x \mathcal{H}^k|_\tau$ for some $c_x > 0$ and $\tau \subset \mathbb{R}^n$ is some k-dimensional subspace. A set $E \subset \mathbb{R}^n$ is k-rectifiable if and only if $\mathcal{H}^k|_E$ is k-rectifiable.*

Definition 2.2.7. For a Radon measure w, whenever for some integer $1 \leq k \leq n$ the space

$$\text{Tan}(w, x) = \text{Tan}_k(w, x) = \{\nu_x\}$$

is a singleton, where $\nu_x = c_x \mathcal{H}^k|_\tau$ for some $c_x > 0$ and $\tau \subset \mathbb{R}^n$ a k-dimensional subspace, we call τ the tangent space of w at x.

- **Varifold, area criticality and monotonicity formula**

Let us now introduce the notion of varifolds. We only present the definitions and facts in $\mathbb{R}^n$ as they can be naturally extended to the case of manifolds such as torus. We refer to [6, 61, 125] for more discussions on this topic.

Denote for integers $1 \le k \le n$ the Grassmannian manifold $G(k,n)$ consisting of k-dimensional subspaces of $\mathbb{R}^n$. Notice that $G(k,n)$ is a compact smooth manifold.

A k-varifold μ is a Radon measure on $\mathbb{R}^n \times G(k,n)$. Let π be the projection that sends $(x,\tau) \in \mathbb{R}^n \times G(k,n)$ to x. Define the area distribution $\|\mu\| := \pi_{\#}\mu$ (see (2.2.1) for definition).

Call a varifold μ to be *stationary* in an open domain $U \subset \mathbb{R}^n$ if for all smooth vector fields $\Phi \in C_0^\infty(U;\mathbb{R}^n)$

$$\int_{U\times G(k,n)} \mathrm{Tr}\left(P_\tau D\Phi(x)\right) d\mu(x,\tau) = 0, \tag{2.2.6}$$

where P_τ is the orthogonal projection matrix from $\mathbb{R}^n$ to the k-subspace $\tau \subset \mathbb{R}^n$. The following theorem states that stationary varifolds are critical points of the area functional.

Theorem 2.2.8 ([6]). *A k-varifold μ on $\mathbb{R}^n$ is a critical point of the functional $\|\mu\|(U) < \infty$ for some bounded open domain $U \subset \mathbb{R}^n$ if and only if μ is stationary in the sense of* (2.2.6) *in U.*

Suppose μ is a stationary k-varifold on an open domain $U \subset \mathbb{R}^n$, then the following identity holds

$$\frac{\|\mu\|(B_r(x))}{r^k} - \frac{\|\mu\|(B_s(x))}{s^k} = \int_{U\times G(k,n)} \frac{|(I-P_\tau)(y-x)|^2}{|y-x|^{k+2}} d\mu(y,\tau) \ge 0, \tag{2.2.7}$$

for any $x \in U$ and $0 < s < r < \mathrm{dist}(x,\partial U)$. In particular, the density limit

$$\Theta(\|\mu\|, x) := \lim_{r\to 0^+} \frac{\|\mu\|(B_r(x))}{\omega_k r^k}$$

exists for every $x \in U$.

- **Ważewski parametrization**

We now introduce the Ważewski parametrization of connected compact metric spaces (called continuum) that have finite length. This will allow us to do fine analysis on sets having finite length.

Theorem 2.2.9 ([4, Theorem 4.4]). *Let X be a non-singleton continuum such that $\mathcal{H}^1(X) < \infty$, then $\mathcal{H}^1(X) > 0$ and there is a Lipschitz mapping $\gamma : [0,1] \to X$ satisfying the following properties:*

(i) *γ is closed, surjective and has degree zero.*

(ii) *The multiplicity $m(\gamma, x) = 2$ for $\mathcal{H}^1$-almost all $x \in X$.*

(iii) *γ has constant speed equaling to $2\mathcal{H}^1(X)$, and in particular, for any Borel function $f : X \to [0,\infty]$ (and in particular any integrable Borel function)*

$$\fint_X f(x) d\mathcal{H}^1(x) = \int_0^1 f(\gamma(t)) dt. \tag{2.2.8}$$

In particular, a continuum X is 1-rectifiable whenever $\mathcal{H}^1(X) < \infty$.

2.2.4 Torus and its topology

In the following, we sketch some important known topological properties of torus for audiences who are not familiar with them. Most of the materials in this section are covered in the textbook [93].

We write the standard flat torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. If not particularly mentioned, we denote

$$\pi : \mathbb{R}^n \to \mathbb{T}^n$$

the standard projection that maps $x \in \mathbb{R}^n$ to the equivalence class of x modulo $\mathbb{Z}^n$. As $\mathbb{Z}^n$ is a discrete subgroup of $\mathbb{R}^n$, standard theory shows that π is also a locally isometric diffeomorphism.

- **Path and path composition**

A path is a continuous map $\gamma : [0,1] \to \mathbb{T}^n$. Given two paths γ_1 and γ_2 with $\gamma_1(1) = \gamma_2(0)$, define the path composition

$$\gamma_1\gamma_2(t) := \begin{cases} \gamma_1(2t) & 0 \le t < 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1. \end{cases} \tag{2.2.9}$$

It is not difficult to show that for two Lipschitz paths γ_1 and γ_2 with $\gamma_1(1) = \gamma_2(0)$ the path composition $\gamma_1\gamma_2$ is still Lipschitz and the total length satisfies

$$\ell(\gamma_1\gamma_2) = \ell(\gamma_1) + \ell(\gamma_2). \tag{2.2.10}$$

- **Homotopy classes on torus**

Call a path γ to be closed if $\gamma(0) = \gamma(1)$. Two continuous closed paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{T}^n$ are homotopic based at $x_0 = \gamma_1(0) = \gamma_2(0)$ if there exists a continuous map

$$H : [0,1]^2 \to \mathbb{T}^n$$

such that $H(\cdot, 0) = \gamma_1$ and $H(\cdot, 1) = \gamma_2$, and H also satisfies $H(0,t) = H(1,t) = x_0$ for all $t \in [0,1]$. By standard theory, the collection of homotopy classes in $\mathbb{T}^n$ based at any point x_0 endowed with the path composition forms a group, which we denote by $\pi_1(\mathbb{T}^n, x_0)$. There is a natural isomorphism from $\pi_1(\mathbb{T}^n, x_0)$ to $\mathbb{Z}^n$. We will further discuss this isomorphism in Section 2.2.4.4.

- **Covering spaces of torus**

A covering space (g, X) of a topological space X^* consists of a continuous surjection $g : X \to X^*$ so that for each $x \in X^*$ there is an open neighborhood U such that $g^{-1}(U)$ is a union of disjoint open sets in X, restricted on which g is a homeomorphism onto U. In this paper we consider for each subgroup $G \le \mathbb{Z}^n$ the quotient space

$$\mathbb{T}^n_G := \mathbb{R}^n / G := \{[x] \ ; \ x \sim y \text{ if and only if } x - y \in G\}.$$

By standard theory the spaces $\mathbb{T}^n_G$ are smooth manifolds and the projection $\pi : \mathbb{R}^n \to \mathbb{T}^n$ can be factored as

$$\pi = \pi_G \circ \pi^G,$$

where π, $\pi^G : \mathbb{R}^n \to \mathbb{T}^n_G$ and $\pi_G : \mathbb{T}^n_G \to \mathbb{T}^n$ are locally diffeomorphic and isometric projections.

- **Path lifting property**

Given a covering map $g : X \to X^*$ and a path $\gamma : [0,1] \to X^*$, a lift of the path is another path $\tilde{\gamma} : [0,1] \to X$ such that $\gamma = g \circ \tilde{\gamma}$. The path lifting property states that for each lift $\tilde{x}_0 \in X$ of the starting point $x_0 = g(0)$, there is a unique lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$.

In the case of the covering map $\pi : \mathbb{R}^n \to \mathbb{T}^n$, given a base point $x_0 \in \mathbb{T}^n$ and its lift $y_0 \in \pi^{-1}(x_0)$, each closed path γ in $\mathbb{T}^n$ based at $\gamma(0) = x_0$ can be lifted to a unique path $\tilde{\gamma}$ in $\mathbb{R}^n$ starting at $\tilde{\gamma}(0) = y_0$. The vector $\tilde{\gamma}(1) - y_0 \in \mathbb{Z}^n$ turns out to be independent of y_0 and the choice of γ in its homotopy class. The map from the homotopy class $[\gamma]$ to $\tilde{\gamma}(1) - y_0$ is known to be an isomorphism, denoted by i_{x_0}, from $\pi_1(\mathbb{T}^n, x_0)$ to $\mathbb{Z}^n$. If the base point $x_0 \in \mathbb{T}^n$ is unambiguous, we do not distinguish $\mathbb{Z}^n$ and the homotopy group based at x_0.

One also notice that because the covering map π is a local isometry, the length $\ell(\gamma)$ is equal to the length of its lift $\ell(\tilde{\gamma})$ whenever γ is Lipschitz.

- **Periodic extension of measures**

For any Radon measure w on $\mathbb{T}^n$, we can extend w uniquely to a periodic Radon measure $\tilde{w}$ on $\mathbb{R}^n$. Define for any $\phi \in C_0^\infty(\mathbb{R}^n)$ the following linear functional

$$L_w(\phi) := \int_{\mathbb{T}^n} \sum_{y \in \pi^{-1}(x)} \phi(y) dw(x).$$

By the Riesz representation theorem, this defines a unique periodic Radon measure $\tilde{w}$ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \phi \, d\tilde{w} = L_w(\phi).$$

In particular, for any $\alpha \geq 0$ and subset $E \subset \mathbb{T}^n$ such that $\mathcal{H}^\alpha(E) < \infty$, the periodic extension of $\mathcal{H}^\alpha\big|_E$ is exactly $\mathcal{H}^\alpha\big|_{\pi^{-1}(E)}$.

Similar periodic extension can be done for matrix-valued Radon measures. Specifically for a matrix-valued Radon measure of the form $d\theta = \sigma \, d\|\theta\|$ we periodically extend the Radon measure $\|\theta\|$ as above to be a periodic Radon measure w on $\mathbb{R}^n$ and then define the periodic extension θ^* of θ as

$$d\theta^*(y) := \sigma(\pi(y)) \, dw(y). \tag{2.2.11}$$

2.3 Conductive medium and its average behavior

In this section we present some elementary results on mixtures of conductive materials that are presented by (especially singularly supported) matrix-valued measures. We call such objects medium and define them precisely as follows.

Definition 2.3.1. Define a *conductive medium* (or simply *medium*) to be a positive semi-definite matrix-valued Radon measure on $\mathbb{T}^n$ (see Section 2.2.2.2). A medium θ is called *isotropic* if $\theta(E) = \lambda(E) I_{n\times n}$ for any Borel set $E \subset \Omega$ and some number $\lambda(E) \geq 0$. Here λ is in fact a Radon measure.

Define the mass distribution of a medium θ as its total variation with respect to the trace

$$\|\theta\|(E) := \mathrm{Tr}(\theta(E)). \tag{2.3.1}$$

According to the discussions in Section 2.2.2.2, we know that any medium θ can be uniquely decomposed as

$$d\theta = \sigma \, d\|\theta\|, \tag{2.3.2}$$

where $\|\theta\|$ is a Radon measure and σ is a positive semi-definite matrix field such that

$$\mathrm{Tr}(\sigma(x)) = 1 \text{ for } \|\theta\|\text{-almost every } x \in \mathbb{T}^n.$$

In particular, if θ is isotropic, we obtain that $\sigma \equiv \frac{1}{n} I_{n\times n}$. If not particularly mentioned, we always identify Radon measures with isotropic media.

Physically the matrix field σ represents the local anisotropy of the material and the mass distribution $\|\theta\|$ represents the local magnitude of conductance as a sum of all directions. Our main goal in this section is to present some basic facts on the average conductive behavior of a given medium.

Definition 2.3.2. For a medium θ, define the *effective conductance tensor* (or simply *effective tensor*) as the positive semi-definite matrix $Q(\theta) \in \mathbb{R}^{n\times n}$ that satisfies

$$p \cdot Q(\theta) p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\nabla\varphi + p) \cdot \sigma(\nabla\varphi + p) d\|\theta\| \tag{2.3.3}$$

for every $p \in \mathbb{R}^n$ and we also define the *mean conductance* as the average of eigenvalues

$$M(\theta) := \frac{\mathrm{Tr}(Q(\theta))}{n}. \tag{2.3.4}$$

See Lemma 2.3.5 for the proof that (2.3.3) is indeed a quadratic form. The quantity in (2.3.3) is known to be quadratic in periodic homogenization in [100, Chapter 1] and in ergodic versions in [16, Chapter 1] for the case when $\|\theta\|$ is absolutely continuous with respect to Lebesgue measure. The main difference is that θ in this paper can have singular support.

The main character of this paper is the following singular version that generalizes the classical Wiener bound [174].

Lemma 2.3.3 (Singular Wiener bound)**.** *The effective tensor* $Q(\theta)$ *satisfies*

$$0 \le Q(\theta) \le \theta(\mathbb{T}^n). \tag{2.3.5}$$

In particular, the mean conductance satisfies

$$0 \le M(\theta) \le \frac{1}{n}\mathrm{Tr}\left(\theta(\mathbb{T}^n)\right) = \frac{1}{n}\|\theta\|(\mathbb{T}^n). \tag{2.3.6}$$

Proof. The lower bound immediately follows from the definition of $Q(\theta)$. The upper bound is obtained by taking the special test function $\varphi = 0$ in the formula. The bound for the mean conductance $M(\theta)$ is obtained by taking the average eigenvalue of $Q(\theta)$. □

Recall the classical Wiener bound (2.1.3) when $d\theta(x) = A(x)d\mathcal{L}^n$ for some matrix field A, one has

$$\left(\int_{\mathbb{T}^n} A^{-1}(x)d\mathcal{L}^n\right)^{-1} \le Q(\theta) \le \int_{\mathbb{T}^n} A(x)d\mathcal{L}^n.$$

The upper bound in (2.3.5) is a natural extension of the classical Wiener upper bound, but the lower bound in (2.3.5), although looks more trivial, is sharp in general. For example, one can consider any finite sum of Dirac delta

$$\rho := \sum_{i=1}^{N} \delta_{x_i},$$

where $x_i \in \mathbb{T}^n$ for $i = 1, \dots, N$, and then find that

$$Q(\rho) = 0$$

no matter how we choose N and the position x_i's. In fact, we will show in Theorem 2.5.1 that $Q(\theta) = 0$ whenever Spt θ is totally disconnected.

To classify the media that have different average behaviors, we now introduce some new terminologies.

Definition 2.3.4. Call a medium θ *trivial* if $Q(\theta) = 0$; *nontrivial* if $Q(\theta) \neq 0$; *positive* (or *resilient*) if $Q(\theta) > 0$ is positive definite; *maximal* if $Q(\theta) = \theta(\mathbb{T}^n)$, i.e., θ achieves the upper bound in (2.3.5).

We remark here that a medium θ is trivial if and only if $M(\theta) = 0$. It is maximal if and only if $M(\theta)$ reaches the upper bound in (2.3.6). Also note that a maximal medium is not necessarily positive.

In the following we show some basic properties of the effective tensor Q. In Section 2.3.1 we show that the quantity in (2.3.3) is indeed a quadratic form. In Section 2.3.2, we present basic properties of $Q(\theta)$ with respect to the addition of media. In Section 2.3.3 we show that $Q(\theta)$ is upper semi-continuous with respect to the weak* topology of media. In Section 2.3.4 we introduce the notion of efficient submedium. In Section 2.3.5 we derive the Euler-Lagrange equation for maximal media.

2.3.1 The effective quadratic form

Lemma 2.3.5. *The quantity defined in* (2.3.3) *is indeed a quadratic form.*

Let us write

$$C_\theta(\varphi,p) := \int_{\mathbb{T}^n} (\nabla\varphi + p)\cdot\sigma(\nabla\varphi + p)d\,\|\theta\| \text{ and } C_\theta(p) := \inf_{\varphi\in C^\infty(\mathbb{T}^n)} C_\theta(\varphi,p).$$

Before proving Lemma 2.3.5 we first show a technical lemma.

Lemma 2.3.6. *A sequence of functions $\varphi_j \in C^\infty(\mathbb{T}^n)$ satisfies*

$$C_\theta(\varphi_j,p) \downarrow C_\theta(p) \text{ as } j\to\infty \tag{2.3.7}$$

if and only if for every $\varepsilon > 0$ there is a $j_0 > 0$ such that for all $j > j_0$ and $\eta \in C^\infty(\mathbb{T}^n)$

$$\left|\int_{\mathbb{T}^n} (\nabla\varphi_j + p)\cdot\sigma\nabla\eta d\,\|\theta\|\right| \le \varepsilon\sqrt{\int_{\mathbb{T}^n} \nabla\eta\cdot\sigma\nabla\eta d\,\|\theta\|}. \tag{2.3.8}$$

Proof. To show the only if part, we take φ_j to be a minimizing sequence and for some $j_0 > 0$ and all $j > j_0$ we have

$$C_\theta(\varphi_j,p) \le C_\theta(p) + \varepsilon.$$

By minimality of $C_\theta(p)$ we have for all $\eta \in C^\infty(\mathbb{T}^n)$ and $t \neq 0$

$$C_\theta(\varphi_j,p) \le C_\theta(\varphi_j - t\eta, p) + \varepsilon,$$

which implies that

$$2t\int_{\mathbb{T}^n} (\nabla\varphi_j + p)\cdot\sigma\nabla\eta d\,\|\theta\| - t^2\int_{\mathbb{T}^n} \nabla\eta\cdot\sigma\nabla\eta d\,\|\theta\| \le \varepsilon.$$

If $\int_{\mathbb{T}^n} \nabla\eta \cdot \sigma\nabla\eta d\,\|\theta\| = 0$ the inequality (2.3.8) is trivial. If $\int_{\mathbb{T}^n} \nabla\eta \cdot \sigma\nabla\eta d\,\|\theta\| > 0$ then by arbitrariness of $t \neq 0$

$$\frac{\left|\int_{\mathbb{T}^n} (\nabla\varphi_j + p) \cdot \sigma\nabla\eta d\,\|\theta\|\right|^2}{\int_{\mathbb{T}^n} \nabla\eta \cdot \sigma\nabla\eta d\,\|\theta\|} \leq \varepsilon,$$

regardless of the choice of η.

To show the if part, we observe that if there is a $\delta > 0$ such that

$$C_\theta(\varphi_j, p) \geq C_\theta(p) + \delta$$

for all large j, then for all j there will be a smooth test function η_j such that

$$C_\theta(\varphi_j, p) \geq C_\theta(\varphi_j + \eta_j, p) + \delta/2$$

also for all large j. In other words, we have

$$-2\int_{\mathbb{T}^n} (\nabla\varphi_j + p) \cdot \sigma\nabla\eta_j d\,\|\theta\| - \int_{\mathbb{T}^n} \nabla\eta_j \cdot \sigma\nabla\eta_j d\,\|\theta\| \geq \delta/2,$$

while on the other hand, we have for all real number t

$$2t\int_{\mathbb{T}^n} (\nabla\varphi_j + p) \cdot \sigma\nabla\eta_j d\,\|\theta\| - t^2\int_{\mathbb{T}^n} \nabla\eta_j \cdot \sigma\nabla\eta_j d\,\|\theta\| \leq \frac{\left|\int_{\mathbb{T}^n} (\nabla\varphi_j + p) \cdot \sigma\nabla\eta_j d\,\|\theta\|\right|^2}{\int_{\mathbb{T}^n} \nabla\eta_j \cdot \sigma\nabla\eta_j d\,\|\theta\|} \leq \varepsilon.$$

By taking $t = -1$, we have

$$-2\int_{\mathbb{T}^n} (\nabla\varphi_j + p) \cdot \sigma\nabla\eta_j d\,\|\theta\| - \int_{\mathbb{T}^n} \nabla\eta_j \cdot \sigma\nabla\eta_j d\,\|\theta\| \leq \varepsilon \ll \delta/2,$$

as $j \to \infty$ by the assumption (2.3.8), which is a contradiction. □

Proof of Lemma 2.3.5. We first observe that for all real constant λ

$$\begin{aligned} C_\theta(\lambda p) &= \inf_{\varphi\in C^\infty(\mathbb{T}^n)} C_\theta(\varphi, \lambda p) \\ &= \lambda^2 \inf_{\varphi\in C^\infty(\mathbb{T}^n)} C_\theta(\varphi/\lambda, p) \\ &= \lambda^2 C_\theta(p). \end{aligned}$$

It then suffice to show that

$$(p_1, p_2)_\theta := \frac{1}{2}\left(C_\theta(p_1 + p_2) - C_\theta(p_1) - C_\theta(p_2)\right)$$

is a bilinear form. By Lemma 2.3.6

$$C_\theta(p_1 + \lambda p_2 + \mu p_3) = \lim_{j\to\infty} C_\theta(\varphi_1^j + \lambda\varphi_2^j + \mu\varphi_3^j, p_1 + \lambda p_2 + \mu p_3),$$

where φ_i^j's are minimizing sequences of $C_\theta(\cdot, p_i)$'s for each $i = 1, 2, 3$. Now we know that

$$\begin{aligned}(p_1, \lambda p_2 + \mu p_3)_\theta + o_j(1) &= \frac{1}{2}\Big(C_\theta(\varphi_1^j + \lambda\varphi_2^j + \mu\varphi_3^j, p_1 + \lambda p_2 + \mu p_3) - C_\theta(\varphi_1^j, p_1) \\ &\quad - C_\theta(\lambda\varphi_2^j + \mu\varphi_3^j, \lambda p_2 + \mu p_3)\Big) \\ &= \int_{\mathbb{T}^n} (\nabla\varphi_1^j + p_1)\cdot\sigma(\nabla(\lambda\varphi_2^j + \mu\varphi_3^j) + \lambda p_2 + \mu p_3) d\,\|\theta\| \\ &= \lambda\int_{\mathbb{T}^n} (\nabla\varphi_1^j + p_1)\cdot\sigma(\nabla\varphi_2^j + p_2) d\,\|\theta\| \\ &\quad + \mu\int_{\mathbb{T}^n} (\nabla\varphi_1^j + p_1)\cdot\sigma(\nabla\varphi_3^j + p_3) d\,\|\theta\| \\ &= \lambda(p_1, p_2)_\theta + \mu(p_1, p_3)_\theta + o_j(1).\end{aligned}$$

Sending $j \to \infty$ on both sides, we obtain the bilinearity of $(p_1, p_2)_\theta$.

□

2.3.2 Submedium and super-additivity

In this subsection we discuss some basic behaviors of $Q(\theta)$ with respect to addition of media θ's. Let us start with the notion of submedia.

Definition 2.3.7. A *submedium* of θ is a medium θ' such that for all Borel set $A \subset \mathbb{T}^n$

$$\theta'(A) \le \theta(A).$$

We also denote $\theta' \le \theta$. We further write $\theta' < \theta$ if $\theta' \ne \theta$. For a Borel set E, the *restriction* of θ on E is defined as

$$\theta_E(A) := \theta(E \cap A), \text{ for every Borel set } A \subset \mathbb{T}^n.$$

Notice that if θ' is a submedium of θ on U, then the subtraction

$$\tilde{\theta} := \theta - \theta'$$

also defines a submedium. We call $\tilde{\theta}$ the *complement* of θ' with respect to θ. We are interested in how $Q(\theta) = Q(\theta' + \tilde{\theta})$ behaves as we view θ as a sum of θ' and its complement. It turns out that we don't in general have additivity, but instead an inequality (see Lemma 2.3.8 (3) below). This non-additivity comes from the intersection of supports of each summand. Indeed, we can consider the sum of θ_1 and θ_2 defined on $\mathbb{T}^1$:

$$d\theta_1 := 1_{\{0\leq s\leq 1/2\}}ds \text{ and } d\theta_2 := 1_{\{1/2\leq s\leq 1\}}ds,$$

where $s:[0,1]\to\mathbb{T}^1$ is a unit speed parametrization. Notice that both

$$Q(\theta_1) = Q(\theta_2) = 0,$$

but $Q(\theta_1+\theta_2) = 1 > 0$.

Lemma 2.3.8. *We show the following basic properties of $Q(\theta)$.*

(1) Let θ' be a submedium of θ, then

$$Q(\theta') \leq Q(\theta).$$

(2) Suppose two media θ and θ' satisfy that $\operatorname{Spt}\theta\cap\operatorname{Spt}\theta' = \varnothing$, then

$$Q(\theta+\theta') = Q(\theta) + Q(\theta).$$

(3) (Super-additivity) Given a sequence of media θ_i such that

$$\sum_{i=1}^{\infty} \|\theta_i\|(\mathbb{T}^n) < \infty,$$

then the following medium

$$\theta_\infty := \sum_{i=1}^{\infty}\theta_i$$

is well-defined and we have the following super-additivity

$$Q(\theta_\infty) \geq \sum_{i=1}^{\infty} Q(\theta_i).$$

In particular, if all θ_i are maximal, then the equality holds and θ_∞ is also maximal.

There is in fact no countable additivity even if the supports of the summand are disjoint from each other (see Example 2.5.5). Here we only present finite additivity under the disjointness assumption, but we will discuss in further details on such problems in Section 2.5.

Proof. To prove property (1) we decompose $d\theta' = \sigma' \, d\|\theta'\|$. For every $\varphi \in C^\infty(\mathbb{T}^n)$, we have

$$\int_{\mathbb{T}^n} (\nabla\varphi + p) \cdot \sigma'(x) \, (\nabla\varphi + p) \, d\|\theta'\|(x) \le \int_{\mathbb{T}^n} (\nabla\varphi + p) \cdot \sigma(x) \, (\nabla\varphi + p) \, d\|\theta\|(x).$$

The inequality $Q(\theta') \le Q(\theta)$ follows by taking infimum over all $\varphi \in C^\infty(\mathbb{T}^n)$.

For property (2), notice that the set Spt θ' has positive distance from Spt θ, and both are compact by definition. This implies that the restriction of any test function $\varphi \in C^\infty(\mathbb{T}^n)$ on Spt $\theta' \cup$Spt θ can be represented as the restriction of $\varphi_1 + \varphi_2$ with

$$\varphi_1, \varphi_2 \in C^\infty(\mathbb{T}^n) \text{ that satisfy Spt } \theta \subset \text{Spt } \varphi_1, \text{ Spt } \theta' \subset \text{Spt } \varphi_2 \text{ and Spt } \varphi_1 \cap \text{Spt } \varphi_2 = \varnothing.$$

This establishes the proof when one takes the infimum over the test function φ.

To show property (3), we notice that $\sum_{i=1}^\infty \theta_i$ is a well-defined positive semi-definite matrix-valued Radon measure, and

$$\left\| \sum_{i=1}^\infty \theta_i \right\| (\mathbb{T}^n) = \sum_{i=1}^\infty \|\theta_i\| (\mathbb{T}^n) < \infty.$$

We write

$$d\theta_\infty := \sigma_\infty \, d\|\theta_\infty\| \text{ and } d\theta_i := \sigma_i \, d\|\theta_i\| \text{ for each } i \ge 1.$$

To show the inequality we observe that for all $N > 0$ and $\varphi \in C^\infty(\mathbb{T}^n)$

$$\begin{aligned}
\int_{\mathbb{T}^n} (\nabla\varphi + p) \cdot \sigma_\infty \, (\nabla\varphi + p) \, d\|\theta_\infty\| &= \int_{\mathbb{T}^n} \mathrm{Tr}\left((\nabla\varphi + p) \otimes (\nabla\varphi + p) \, d\theta_\infty\right) \\
&= \sum_{i=1}^\infty \int_{\mathbb{T}^n} \mathrm{Tr}\left((\nabla\varphi + p) \otimes (\nabla\varphi + p) \, d\theta_i\right) \\
&\ge \sum_{i=1}^N \int_{\mathbb{T}^n} \mathrm{Tr}\left((\nabla\varphi + p) \otimes (\nabla\varphi + p) \, d\theta_i\right) \\
&= \sum_{i=1}^N \int_{\mathbb{T}^n} (\nabla\varphi + p) \cdot \sigma_i \, (\nabla\varphi + p) \, d\|\theta_i\| .
\end{aligned}$$

By taking the infimum over all smooth test functions φ we obtain

$$Q(\theta_\infty) \ge \sum_{i=1}^N Q(\theta_i).$$

Sending $N \to \infty$ establishes the inequality we desire.

Suppose all θ_i's are maximal, then by the singular Wiener upper bound (2.3.5)

$$\sum_{i=1}^{\infty} Q(\theta_i) = \sum_{i=1}^{\infty} \theta_i(\mathbb{T}^n) = \theta_\infty(\mathbb{T}^n) \geq Q(\theta_\infty),$$

which finishes the proof.

□

2.3.3 Weak* convergence, upper semi-continuity and faithful convergence

In this subsection we study the continuity property of $Q(\theta)$ with respect to the weak* topology of media. Let us introduce the notion of weak* convergence.

Definition 2.3.9. Say that a sequence of media θ_i weakly* converges to another medium θ_∞ if for any continuous function $\varphi \in C(\mathbb{T}^n)$ there is

$$\lim_{i\to\infty} \int_{\mathbb{T}^n} \varphi(x) d\theta_i(x) = \int_{\mathbb{T}^n} \varphi(x) d\theta_\infty(x).$$

We use the notation $\theta_i \overset{*}{\rightharpoonup} \theta_\infty$.

The effective tensor $Q(\theta)$ is generally discontinuous with respect to the weak* topology of θ as above. Such phenomena occur often because of homogenization and have been studied in the context of G, H and Γ-convergence theories [41, 64, 119, 137]. We are currently aware of at least two phenomena when the discontinuity occurs:

- (Change of topology) When there is a significant topological difference between the supports of θ_∞ and θ_i's. To see this phenomenon, one can consider the medium

  $$d\theta_\delta(s) := 1_{\{0\leq s\leq 1-\delta\}} ds, \text{ for } s \in \mathbb{T}^1,$$

  where s is a unit speed parametrization of $\mathbb{T}^1$ and $\delta > 0$. Notice that for all $\delta > 0$, $Q(\theta_\delta) = 0$ but $Q(\theta_0) = 1$. On the other hand, Spt θ_δ does not contain any closed paths belonging to a nonzero homotopy class, but Spt $\theta_0 = \mathbb{T}^1$ does contain one.

- (Homogenization) When θ_i weakly* converge to a constant θ_∞, but θ_∞ does not faithfully represent the limit of the effective tensor. A very simple and well-known example is the 1-D periodic homogenization on $\mathbb{T}^1$ when $d\theta_\varepsilon(s) = a(s/\varepsilon)ds$. As $\varepsilon \to 0^+$, the media $\theta_\varepsilon \overset{*}{\rightharpoonup} \langle a \rangle := \int_0^1 a(s)ds$, but the effective tensor $Q(\theta_\varepsilon) \to \langle a^{-1} \rangle^{-1}$.

The characterization of discontinuities are left to future works. Although the discontinuity is ubiquitous we can still show that $Q(\theta)$ is upper semi-continuous.

Lemma 2.3.10. *If θ_i weakly* converges to another medium θ_∞, then for any $p \in \mathbb{R}^n$ we have*

$$\limsup_{i\to\infty} \; p \cdot Q(\theta_i)p \le p \cdot Q(\theta_\infty)p,$$

and in particular

$$\limsup_{i\to\infty} M(\theta_i) \le M(\theta_\infty).$$

Proof. We write

$$d\theta_i = \sigma_i \, d\,\|\theta_i\| \;\text{ and }\; d\theta_\infty = \sigma_\infty \, d\,\|\theta_\infty\|\,.$$

We first show that for all $p \in \mathbb{R}^n$ the quantity

$$p \cdot Q(\theta)p$$

is upper semi-continuous with respect to the weak* topology. Indeed, let ψ be an arbitrary smooth test function, we have for every $i > 0$

$$p \cdot Q(\theta_i)p \le \int_{\mathbb{T}^n} (\nabla\psi + p) \cdot \sigma_i(x)(\nabla\psi + p) d\,\|\theta_i\|\,(x) \overset{i\to\infty}{\longrightarrow} \int_{\mathbb{T}^n} (\nabla\psi + p) \cdot \sigma_\infty(x)(\nabla\psi + p) d\,\|\theta_\infty\|\,,$$

where the convergence is by applying the weak* convergence in Definition 2.3.9 component-wise. This implies that for every ψ

$$\limsup_{i\to\infty} p \cdot Q(\theta_i)p \le \int_{\mathbb{T}^n} (\nabla\psi + p) \cdot \sigma_\infty(x)(\nabla\psi + p) d\,\|\theta_\infty\|\,.$$

This completes the proof of the upper semi-continuity by taking an infimum in ψ. The upper semi-continuity of the mean conductance $M(\theta)$ follows immediately from the fact that

$$M(\theta_i) = \frac{1}{n} \sum_{j=1}^{n} e_j \cdot Q(\theta_i)e_j,$$

where $\{e_j\}_{j=1}^n$ forms an orthonormal basis of $\mathbb{R}^n$. □

One of the advantages of using singular matrix-valued measures is that sometimes they can reduce the dimension of the problem and give sharper insights. For example, when modeling the leaf vein networks (also read Example 2.3.13 and Section 2.6), one can simply look at singularly supported measures instead of matrix fields that have high conductance in an ε-tubular neighborhood of the network. An important aspect of this approach is that one has to check whether the singularization procedure faithfully indicates the conductive properties of the original nonsingular setup. To that end we introduce the following stronger notion of convergence.

Definition 2.3.11. Say that a sequence of media θ_i *faithfully converge* to θ_∞ if $\theta_i \overset{*}{\rightharpoonup} \theta_\infty$ and

$$\lim_{i\to\infty} Q(\theta_i) = Q(\theta_\infty).$$

Despite the general discontinuity of Q, we show a principle of faithful continuity when the limit medium is a submedium of all the media in the sequence.

Lemma 2.3.12. *Suppose θ_i is a sequence of media weakly* converging to θ_∞, and for all $i \geq 1$ we have $\theta_\infty \leq \theta_i$, then θ_i converge to θ_∞ faithfully.*

Example 2.3.13. In [96], the author proved an example of faithful convergence that is outside the assumptions in Lemma 2.3.12. Let $\Gamma \subset \mathbb{T}^2$ be a finite union of C^2 curves, having no tangential cusps at joining nodes. Define $\Gamma_\delta := \bigcup_{x\in\Gamma} B_{\delta/2}(x)$ and

$$d\theta_\delta(x) := \left(1 + \frac{1_{\Gamma_\delta}}{\delta}\right) I_{2\times 2}\, d\mathcal{L}^2, \text{ where } d\mathcal{L}^2 \text{ is the Lebesgue measure on } \mathbb{T}^2,$$

then θ_δ converges faithfully to $I_{2\times2}\, d\mathcal{L}^2 + I_{2\times2}\, d\mathcal{H}^1\big|_\Gamma$ as $\delta \to 0^+$. Notice that $I_{2\times2}\, d\mathcal{L}^2 + I_{2\times2}\, d\mathcal{H}^1\big|_\Gamma$ is not a submedium of θ_δ for any $\delta > 0$.

Proof of Lemma 2.3.12. It suffices to show that

$$\lim_{i\to\infty} Q(\theta_i) = Q(\theta_\infty).$$

By Lemma 2.3.10, we know that

$$\limsup_{i\to\infty} Q(\theta_i) \leq Q(\theta_\infty),$$

and then it suffices to show the reverse inequality. Indeed, because $\theta_\infty \le \theta_i$ for all $i \ge 1$, we have by Lemma 2.3.8 (1), the following inequalities

$$Q(\theta_i) \ge Q(\theta_\infty) \text{ for all } i \ge 1.$$

This finishes the proof.

□

2.3.4 Efficient submedium

There is often redundancy in a medium from the viewpoint of its effective tensor, especially when the medium is singularly supported. To talk about such properties we define the efficient part of a medium.

Definition 2.3.14. Call a submedium θ_* to be *efficient* in θ if $Q(\theta_*) = Q(\theta)$. Call θ_* to be *minimally efficient* in θ if for any $\theta' \le \theta_*$ satisfying $Q(\theta') = Q(\theta_*) = Q(\theta)$, then $\theta' = \theta_*$. Call θ *saturated* if it is equal to all its efficient submedium.

We are aware of at least two types of proper efficient submedia.

- (Dimension reduction) The medium $I_{2\times 2}\, d\mathcal{L}^2 + I_{2\times 2}\, d\mathcal{H}^1\big|_\Gamma(x)$ in Example 2.3.13 has proper efficient submedium, where Γ is the union of finitely many C^2 curves with nondegenerate joining angles at nodes. Indeed, one can check that if at some $x_0 \in \Gamma$ there is a $r_0 > 0$ such that $\Gamma \cap B_{r_0}(x_0)$ is a C^2 curve, then

  $$I_{2\times 2}\, d\mathcal{L}^2 + P_{\tau_x}\, d\mathcal{H}^1\big|_{\Gamma\cap B_{r_0}(x_0)}(x) + I_{2\times 2}\, d\mathcal{H}^1\big|_{\Gamma\smallsetminus B_{r_0}(x_0)}(x)$$

  is a proper efficient submedium, with P_{τ_x} the orthogonal projection from $\mathbb{R}^2$ to the 1-D tangent space τ_x of Γ at $x \in \Gamma \cap B_{r_0}(x_0)$. This can be done by modifying the smooth test functions in (2.3.3), which eliminates the normal contributions without costs in the tangential direction. As a consequence $I_{2\times 2}\, d\mathcal{L}^2 + P_{\tau_x}\, d\mathcal{H}^1\big|_\Gamma(x)$ is a proper efficient submedium if Γ is, outside a finite set of singular points, the union of finitely many C^2 curves.

- (Trivial components) Suppose for a medium θ there is a component $E \subset \operatorname{Spt}\theta$ such that the restriction θ_E is trivial, then by Lemma 2.5.10, $\theta - \theta_E$ is a proper efficient submedium of θ.

Let us show the existence of minimally efficient parts for nontrivial media. We leave the characterization of minimally efficient submedia of a given medium (or equivalently the characterization of saturated media) to future works.

Lemma 2.3.15. *Nontrivial media always have nonzero minimal efficient submedia.*

Proof. Consider a nontrivial medium θ and the following set of submedia

$$L := \{\theta' \leq \theta \ ; \ Q(\theta') = Q(\theta)\}.$$

We aim to solve the minimization problem

$$\min_{\theta' \in L} \mathrm{Tr}\left(\theta'(\mathbb{T}^n)\right). \tag{2.3.9}$$

Suppose θ_i is a minimizing sequence, then by weak* compactness, we can find a converging subsequence, not relabeled, such that

$$\theta_i \overset{*}{\rightharpoonup} \theta_\infty.$$

By the upper semi-continuity Lemma 2.3.10, we know that

$$Q(\theta_\infty) \geq \limsup_{i\to\infty} Q(\theta_i) = Q(\theta).$$

Notice that $\theta_\infty \leq \theta$, and then we have $\theta_\infty \in L$. We also know that $\theta_\infty \neq 0$ because θ is nontrivial.

Notice that θ_∞ is minimally efficient because if there is another $\tilde{\theta} < \theta_\infty$ and $\tilde{\theta} \in L$, then the fact that θ_∞ being a minimizer of (2.3.9) is violated.

□

2.3.5 The Euler-Lagrange equation for maximal media

In this subsection we present the Euler-Lagrange equation for maximal media.

Lemma 2.3.16. *A $d\theta = \sigma d\|\theta\|$ is maximal if and only if for every $\Phi \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$*

$$\int_{\mathbb{T}^n} \mathrm{Tr}\left(\sigma(x)\nabla\Phi(x)\right) d\|\theta\|(x) = 0. \tag{2.3.10}$$

Proof. We first recall that θ is maximal if and only if $Q(\theta) = \theta(\mathbb{T}^n)$. As both $Q(\theta)$ and $\theta(\mathbb{T}^n)$ are positive semi-definite, it is not difficult to find that θ is maximal if and only if its mean conductance (average eigenvalue of $Q(\theta)$) reaches the upper Wiener bound (2.3.6)

$$M(\theta) = \frac{1}{n}\mathrm{Tr}\left(Q(\theta)\right) = \frac{1}{n}\left\|\theta\right\|(\mathbb{T}^n). \tag{2.3.11}$$

Notice that if we write $\Phi = \sum_{i=1}^n \phi^i e_i$ with e_i's being an orthonormal basis for $\mathbb{R}^n$

$$\begin{aligned} M(\theta) &= \frac{1}{n}\mathrm{Tr}\left(Q(\theta)\right) \\ &= \frac{1}{n}\sum_{i=1}^n e_i \cdot Q(\theta)e_i \\ &= \frac{1}{n}\sum_{i=1}^n \inf_{\phi^i \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\nabla\phi^i(x) + e_i)\cdot\sigma(x)(\nabla\phi^i(x)+e_i)d\left\|\theta\right\|(x) \\ &= \frac{1}{n}\inf_{\Phi\in C^\infty(\mathbb{T}^n;\mathbb{R}^n)} \int_{\mathbb{T}^n} \mathrm{Tr}\Big((\nabla\Phi + I)^T\sigma(x)\left(\nabla\Phi+I\right)\Big)d\left\|\theta\right\|(x). \end{aligned} \tag{2.3.12}$$

Therefore by applying (2.3.11) the medium θ is maximal if and only if for all $\Phi \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$

$$\int_{\mathbb{T}^n} \mathrm{Tr}\Big((\nabla\Phi + I)^T\sigma(x)\left(\nabla\Phi+I\right)\Big)d\left\|\theta\right\|(x) \geq \left\|\theta\right\|(\mathbb{T}^n).$$

Simplifying the inequality we get

$$\int_{\mathbb{T}^n} \mathrm{Tr}\left(\nabla\Phi^T\sigma(x)\nabla\Phi\right)d\left\|\theta\right\|(x) + 2\int_{\mathbb{T}^n}\mathrm{Tr}\left(\sigma(x)\nabla\Phi\right)d\left\|\theta\right\|(x) \geq 0.$$

Replacing Φ by $h\Phi$ with $|h| > 0$ we see that

$$|h|\int_{\mathbb{T}^n} \mathrm{Tr}\left(\nabla\Phi^T\sigma(x)\nabla\Phi\right)d\left\|\theta\right\|(x) + 2\frac{h}{|h|}\int_{\mathbb{T}^n}\mathrm{Tr}\left(\sigma(x)\nabla\Phi\right)d\left\|\theta\right\|(x) \geq 0,$$

which shows (2.3.10) after sending $h \to 0$ from both sides. The reverse follows immediately when we plug (2.3.10) into the last formula in (2.3.12).

□

2.4 Maximal medium, stationary varifold and dimension bound

In this section we characterize maximal media by applying some results from geometric measure theory, especially the topic on varifolds and PDE for measures. Specifically we discuss Theorem B and (2.1.11), which states the following formal identity

$$\text{Conductance Maximality} \quad = \quad \text{Area Criticality}. \tag{2.4.1}$$

On the left of the identity we mean the media that achieve the upper Wiener bound (2.3.5), and on the right we mean stationary varifolds, which are known to be critical points of the generalized area functional (see Section 2.2.3.7 for more details). Based on this identity we prove a pointwise dimension bound for maximal medium (see Theorem 2.4.5).

At first glance it is far from being clear why the equality (2.4.1) makes sense. We explain this by introducing a mapping, first introduced in [13, Remark 3.2], from the space of varifolds to the space of media. From this transformation we view medium, a matrix-valued measure, as a varifold having *variable* and *fractional* dimension (see Lemma 2.4.7 and Definition 2.4.3). Specifically we define the mapping $\mathcal{T}$ as follows: First disintegrate a k-varifold μ, by Theorem 2.2.5

$$d\mu(x,\tau) = d\rho_x(\tau)d\left\|\mu\right\|(x),$$

where ρ_x is a family of probability measures on the Grassmannian manifold $G(k,n)$ and $\|\mu\| = \pi_{\#}\mu$ is the pushforward of μ with respect to the projection $\pi(x,\tau) = x$. Second, we define the matrix field

$$\sigma_\mu(x) := \frac{1}{k}\int_{G(k,n)} P_\tau d\rho_x(\tau), \tag{2.4.2}$$

where P_τ is the orthogonal projection matrix to the k-dimensional subspace $\tau \subset \mathbb{R}^n$. We then define $\mathcal{T}(\mu)$ as the following medium

$$d\mathcal{T}(\mu) := \sigma_\mu d\left\|\mu\right\|. \tag{2.4.3}$$

Notice that $\mathrm{Tr}\left(\sigma_\mu\right) = 1$ and $\left\|\mathcal{T}(\mu)\right\| = \left\|\mu\right\|$.

As we have seen in Lemma 2.3.16 (see also the discussions in [90, Section 6]), the maximization of conductance is essentially solving the distributional PDE

$$\nabla\cdot\theta = 0 \tag{2.4.4}$$

under various conditions (two-phase, anisotropic, etc.) on the medium θ. If $\theta = \mathcal{T}(\mu)$ for some varifold μ, then plug this into (2.4.4) we obtain

$$0 = \nabla\cdot\left(\sigma_\mu(x)d\left\|\mu\right\|(x)\right) = \frac{1}{k}\nabla\cdot\left(\int_{G(k,n)} P_\tau d\rho_x(\tau)d\left\|\mu\right\|(x)\right), \tag{2.4.5}$$

which is equivalently saying that μ is a stationary varifold (see Section 2.2.3.7). This proves the second part of the following rigorous version of (2.4.1).

Theorem 2.4.1. *The mapping $\mathcal{T}$ as defined in* (2.4.3) *is surjective and continuous with respect to the weak* topology of media and varifolds. Moreover, the following statements are equivalent for a fixed medium θ:*

(a) The medium θ is maximal.

(b) The medium θ satisfies $\nabla \cdot \theta = 0$ in the distributional sense (see Lemma 2.3.16).

(c) All varifold realizations $\mu \in \mathcal{T}^{-1}(\theta)$ are stationary.

(d) There exists a stationary varifold $\mu \in \mathcal{T}^{-1}(\theta)$.

Proof. The proof of Theorem 2.4.1 is complete by Lemma 2.4.7 below, where we provide a more precise discussion on the surjectivity and continuity of the mapping $\mathcal{T}$.

□

Our main contribution in this section is a pointwise dimension bound for maximal media. As we have pointed out in the introduction, for a medium in $\mathbb{T}^3$ of the form

$$d\theta := \sigma \, d\mathcal{H}^2\big|_{\mathbb{T}^2 \times \{0\}},$$

where σ is a constant positive semi-definite matrix, the rank of σ satisfies the following bound when θ is maximal

$$\operatorname{rank}(\sigma) \leq \dim_{\mathcal{H}}\left(\mathbb{T}^2 \times \{0\}\right) = 2.$$

More generally a *global version* of such dimension bound was established in [18] for $\mathcal{A}$-free measures, including maximal medium. At a heuristic level, the theory, in terms of maximal medium, states that the rank of the matrix field $\sigma = \frac{d\theta}{d\|\theta\|}$ does not exceed the dimension of $\|\theta\|$ (see [18, Corollary 1.4 and Proposition 3.1] for more precise discussions). In this case, we call the rank of the matrix field σ is bounded by the dimension of $\|\theta\|$.

To establish a *pointwise* dimension bound, the compromise is that one should use a new slightly weaker notion of "rank". We define the new "rank" as follows.

Definition 2.4.2 (Realizable dimension)**.** Given a positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$, we define its *realizable dimension* as

$$\dim_{\mathrm{r}}(A) := \frac{\mathrm{Tr}\,(A)}{\lambda_{\max}(A)},$$

where $\lambda_{\max}(A)$ is the maximal eigenvalue of A.

Note that $\dim_{\mathrm{r}}(A) \leq \mathrm{rank}(A)$ with equality if and only if A is an orthogonal projection matrix. The interest of this new notion is that it sharply characterizes whether a medium θ is in the range of k-varifolds under the transformation $\mathcal{T}$ (see Section 2.4.1).

To state the dimension bound result more precisely, we define two notions of dimensions. The first is the semi-continuous version of realizable dimensions of the anisotropy matrix fields σ in media.

Definition 2.4.3. For a medium $d\theta = \sigma\, d\,\|\theta\|$, we define its *lower realizable dimension* at $x \in \mathrm{Spt}\ \theta$ as

$$\underline{\dim}_{\mathrm{r}}(\theta)(x) := \sup_{\delta>0} \operatorname{ess\,inf}\{\dim_{\mathrm{r}}(\sigma(y))\ ;\ |y-x| \leq \delta,\ y \in \mathrm{Spt}\ \theta\}, \tag{2.4.6}$$

and symmetrically the *upper realizable dimension*

$$\overline{\dim}_{\mathrm{r}}(\theta)(x) := \inf_{\delta>0} \operatorname{ess\,sup}\{\dim_{\mathrm{r}}(\sigma(y))\ ;\ |y-x| \leq \delta,\ y \in \mathrm{Spt}\ \theta\}. \tag{2.4.7}$$

Here we take "ess sup" and "ess inf" with respect to the Radon measure $\|\theta\|$. Define the *realizable dimension* of θ as

$$\dim_{\mathrm{r}}(\theta)(x) = \underline{\dim}_{\mathrm{r}}(\theta)(x) = \overline{\dim}_{\mathrm{r}}(\theta)(x) \tag{2.4.8}$$

if equality holds. Note that $\underline{\dim}_{\mathrm{r}}(\theta)(x) \neq \dim_{\mathrm{r}}(\sigma(x))$ and $\overline{\dim}_{\mathrm{r}}(\theta)(x) \neq \dim_{\mathrm{r}}(\sigma(x))$ in general.

Besides the realizable dimension, we also consider the local dimensions from the viewpoints of fractal geometry. The following definition of local dimension can be found in [169, 178] and the book [32].

Definition 2.4.4. We define the *lower local dimension* of a medium θ as

$$\underline{\dim}_{\mathrm{loc}}(\theta)(x) := \liminf_{r\to 0^+} \frac{\log\|\theta\|\,(B_r(x))}{\log r},$$

and similarly the *upper local dimension*

$$\overline{\dim}_{\mathrm{loc}}(\theta)(x) := \limsup_{r\to 0^+} \frac{\log \|\theta\| (B_r(x))}{\log r}.$$

The *local dimension* of θ at x is defined as

$$\dim_{\mathrm{loc}}(\theta)(x) := \lim_{r\to 0^+} \frac{\log \|\theta\| (B_r(x))}{\log r},$$

if the limit exists.

The main theorem of this section describes the relation between the above two dimensions in general.

Theorem 2.4.5. *Suppose a medium θ is maximal, then for all $x \in \operatorname{Spt} \theta$ we have*

$$\underline{\dim}_{\mathrm{r}}(\theta)(x) \le \underline{\dim}_{\mathrm{loc}}(\theta)(x). \tag{2.4.9}$$

In particular, the lower local dimension $\underline{\dim}_{\mathrm{loc}}(\theta)(x) \ge 1$ for all $x \in \operatorname{Spt} \theta$.

The proof of this theorem does not depend on the theory as used in [18], but simply a *fractional* version of the classical monotonicity formula for varifolds (see Lemma 2.4.11). We do not know if (2.4.9) still holds when the realizable dimension is replaced by the standard rank.

Let us outline the structure of this section. In Section 2.4.1 we characterize the media that can be realized by k-varifold for some $1 \le k \le n$. In Section 2.4.2, we prove Theorem 2.4.5 by introducing a fractional version of monotonicity formula. We also present some examples to show the sharpness of the bound (2.4.9). In Section 2.4.3 we show some applications of existing theorems on stationary varifolds to maximal media. In Section 2.4.4 we discuss Problem 2.1.3 in rigorous mathematical terms.

2.4.1 Varifold realizations of media

In this subsection we show some basic properties of the transformation $\mathcal{T}$ as defined in (2.4.3). Let us first denote $\mathcal{H}$ the space of all varifolds on $\mathbb{T}^n$, and $\mathcal{F}$ the space of all media. We denote for each integer $1 \le k \le n$ the space $\mathcal{H}_k$ of all k-varifolds and therefore we have $\mathcal{H} = \bigcup_{k=1}^{n} \mathcal{H}_k$.

Definition 2.4.6 (Realization map)**.** Call the mapping $\mathcal{T} : \mathcal{H} \to \mathcal{F}$ as defined in (2.4.3) the *realization map*. For each $1 \le k \le n$, call the restriction $\mathcal{T}_k := \mathcal{T}\big|_{\mathcal{H}_k}$ the k-th realization map.

Lemma 2.4.7. *For each $1 \le k \le n$, the k-th realization map $\mathcal{T}_k$ is continuous with respect to the weak* topology on $\mathcal{H}_k$ and $\mathcal{F}$. A medium $\theta \in \mathcal{F}$ is in the range of $\mathcal{T}_k$ if and only if the matrix field $\sigma = \frac{d\theta}{d\|\theta\|}$ satisfies*

$$\dim_{\mathrm{r}}(\sigma(x)) \ge k \text{ for } \|\theta\| \text{-almost all } x \in \mathbb{T}^n.$$

In particular, the first realization map $\mathcal{T}_1 : \mathcal{H}_1 \to \mathcal{F}$ is surjective.

To prove the lemma we observe that a medium $d\theta = \sigma \, d\,\|\theta\|$ is in the range of $\mathcal{T}_k$ if and only if the matrix field $\sigma(x)$ can be written in the form (2.4.2) for some Borel choice in x of probability measures ρ_x on the Grassmannian manifold $G(k,n)$. In the following lemma we prove a sharp criterion for the existence of such a representation for fixed x.

Lemma 2.4.8. *A positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$ that satisfies $\mathrm{Tr}\,(A) = 1$ takes the form*

$$A = \frac{1}{k}\int_{G(k,n)} P_\tau d\rho(\tau) \tag{2.4.10}$$

where P_τ is the orthogonal projection to the k-dimensional subspace τ and ρ is a probability measure on $G(k,n)$ if and only if the eigenvalues $\{\lambda_i\}_{i=1}^n$ of A satisfy

$$0 \le \lambda_i \le 1/k, \text{ for all } i = 1,\dots,n.$$

In particular, A can always be written in the form (2.4.10) *when $k = 1$.*

Proof. It is not difficult to see the only if part because

$$\mathrm{Tr}\left(\int_{G(k,n)} \frac{1}{k} P_\tau d\rho(\tau)\right) = \int_{G(k,n)} \frac{1}{k}\mathrm{Tr}\,(P_\tau) d\rho(\tau) = 1,$$

and for any unit vector p, we have

$$\int_{G(k,n)} \frac{1}{k}|P_\tau(p)|^2 d\rho(\tau) \le 1/k.$$

To show the reverse, we first rotate the coordinate and reduce to the case when $A = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ is a diagonal matrix. Observe that the set of A that are of the integral form (2.4.10) is convex. Therefore, by applying the Krein–Milman theorem, it suffices to show that the extreme points of the set

$$F := \left\{0 \le \lambda_i \le 1/k, \sum_{i=1}^{n} \lambda_i = 1\right\}$$

are all of the integral form (2.4.10). Let us first determine the extreme points of F. Notice that the extreme points of F are also extreme points of the square $\{0 \le \lambda_i \le 1/k\}$, which are of the form $\lambda_i = 0$ or $1/k$. Because of the constraint $\sum_{i=1}^{n} \lambda_i = 1$, we thus find that the extreme points of F are permutations of

$$\lambda_1 = 1/k, \dots, \lambda_k = 1/k, \lambda_{k+1} = 0, \dots, \lambda_n = 0.$$

On the other hand, each k-subset J of $\{1, \dots, n\}$ such that $\lambda_i = 1/k$ for $i \in J$ correspond to a k-dimensional space $V_J \subset \mathbb{R}^n$ spanned by the selection $\{e_i\}_{i \in J}$ of the orthonormal basis $\{e_i\}_{i=1}^{n}$. Therefore, each extreme point $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ takes the form

$$A = \int_{G(k,n)} \frac{1}{k} P_\tau d\delta_{V_J}(\tau),$$

for some k-subset J of $\{1, \dots, n\}$.

□

Denote $\mathcal{P}_k$ the space of probability measures on $G(k,n)$. Note that $\mathcal{P}_k$ is a Polish space under the weak topology, which coincides with the dual space $C_0^*(G(k,n))$ as $G(k,n)$ is compact [88, Section 2]. By Lemma 2.2.2, $\mathcal{P}_k$ is also compact with respect to the weak* topology.

Proof of Lemma 2.4.7. To show the realizability assertion, we first denote the full $\|\theta\|$-measure set

$$F := \{x \in \mathrm{Spt}\ \theta \ ; \ \dim_{\mathrm{r}}(\sigma(x)) \ge k\}$$

and consider

$$K := \left\{(x, \rho) \in F \times \mathcal{P}_k \ ; \ \sigma(x) = \frac{1}{k} \int_{G(k,n)} P_\tau d\rho(\tau)\right\} \subset F \times \mathcal{P}_k.$$

Because the relation $\sigma(x) = \frac{1}{k} \int_{G(k,n)} P_\tau d\rho(\tau)$ is Borel in x and continuous in ρ, the set K is Borel. By Lemma 2.4.8, for each $x \in F$ the slice

$$K_x := \left\{\rho \ ; \ \sigma(x) = \frac{1}{k} \int_{G(k,n)} P_\tau d\rho(\tau)\right\}$$

is compact and nonempty in $\mathcal{P}_k$. By [105, Theorem 28.8] and the Kuratowski-Ryll-Nardzewski measurable selection theorem [109], there is a Borel selector $R : F \to \mathcal{P}_k$ such that $R(x) \in K_x$ for all $x \in F$. This establishes the existence of a k-varifold in $\mathcal{T}^{-1}(\theta)$.

To show the continuity of the restriction $\mathcal{T}_k : \mathcal{H}_k \to \mathcal{F}$, we first observe that for every continuous test function φ

$$\psi(x,\tau) := \frac{\varphi(x)}{k} P_\tau$$

is continuous on $\mathbb{T}^n \times G(k,n)$. Suppose μ_l is a weakly* convergent sequence of k-varifolds with limit μ_∞, then if we write $d\mathcal{T}(\mu_l) = \sigma_l d\left\|\mu_l\right\|$ and $d\mathcal{T}(\mu_\infty) = \sigma_\infty d\left\|\mu_\infty\right\|$, we have as $l \to \infty$

$$\begin{aligned}\int_{\mathbb{T}^n} \varphi(x)\sigma_l(x) d\left\|\mu_l\right\|(x) &= \int_{\mathbb{T}^n \times G(k,n)} \psi(x,\tau) d\mu_l(x,\tau) \\ &\to \int_{\mathbb{T}^n \times G(k,n)} \psi(x,\tau) d\mu_\infty(x,\tau) \\ &= \int_{\mathbb{T}^n} \varphi(x)\sigma_\infty(x) d\left\|\mu_\infty\right\|(x).\end{aligned}$$

As $\varphi \in C(\mathbb{T}^n)$ is arbitrary, we know that $\mathcal{T}(\mu_l)$ weakly* converges to $\mathcal{T}(\mu_\infty)$ in the sense of Definition 2.3.9.

□

2.4.2 Dimensions and fractional monotonicity formula

In this subsection we prove Theorem 2.4.5, which is a fine estimate of the dimensions of a maximal medium.

The proof requires a *fractional* monotonicity formula (see Lemma 2.4.11) that is similar to that of a stationary varifold as we have seen in (2.2.7). Such monotonicity formulae was first designed to study the fine structures of stationary varifolds [6, 61, 165]. In the context of medium the monotonicity formula gives pointwise dimension bound (2.4.9).

Before presenting the monotonicity formula and the proof of Theorem 2.4.5, let us first discuss the sharpness of the inequality (2.4.9) in the following two examples.

Example 2.4.9. The inequality (2.4.9) is in general not an equality. Let θ be a medium on $\mathbb{T}^2$ defined as

$$d\theta(x) := ((1-\lambda)e_1 \otimes e_1 + \lambda e_2 \otimes e_2)\, d\mathcal{L}^2(x),$$

where $0 \le \lambda \le 1$, $\mathcal{L}^2$ is the Lebesgue measure on $\mathbb{T}^2$, e_1 and e_2 form an orthonormal basis for $\mathbb{R}^2$. For each $0 \le \lambda \le 1$ the medium is a maximal medium because for any $\Phi \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ we have by integration by parts

$$\begin{aligned}\int_{\mathbb{T}^2} \mathrm{Tr}\,(((1-\lambda)e_1 \otimes e_1 + \lambda e_2 \otimes e_2)\,\nabla\Phi) d\mathcal{L}^2(x) &= \int_{\mathbb{T}^1}\int_{\mathbb{T}^1} (1-\lambda)\partial_1\Phi_1(x_1,x_2)dx_1dx_2 \\ &\quad + \int_{\mathbb{T}^1}\int_{\mathbb{T}^1} \lambda\partial_2\Phi_2(x_1,x_2)dx_1dx_2 \\ &= 0.\end{aligned}$$

The local dimension of this medium coincides with the Lebesgue measure and is always 2. However, its realizable dimension is

$$\dim_{\mathrm{r}}(\theta) \equiv \dim_{\mathrm{r}}((1-\lambda)e_1 \otimes e_1 + \lambda e_2 \otimes e_2) = \frac{1}{\max\{\lambda, 1-\lambda\}},$$

which can be any real number in the interval $[1,2]$.

Example 2.4.10. The upper realizable dimension can be strictly greater than the local dimension. This example also shows that the upper realizable dimension does not capture the lower dimensional structures in a medium. To see this, let $\theta = \theta_1 + \theta_2$ be the sum of two maximal media on $\mathbb{T}^2$ that are of the form

$$d\theta_1(x) := \frac{1}{2} I_{2\times 2}\, d\mathcal{L}^2(x),$$

and

$$d\theta_2(x) = e_1 \otimes e_1 d\mathcal{H}^1\big|_{\{x_2=0\}}(x).$$

By Lemma 2.3.8, we know that θ is maximal and the local dimension of θ on $\{x_2 = 0\}$ is exactly 1. On the other hand, we know that the matrix field $\sigma = \frac{d\theta}{d\|\theta\|}$ satisfies

$$\sigma(x) = \begin{cases} \frac{1}{2}I, & x_2 \neq 0, \\ e_1 \otimes e_1, & x_2 = 0, \end{cases}$$

which has upper realizable dimension 2 everywhere on $\mathbb{T}^2$. Notice that $\sigma(x)$ has exactly lower realizable dimension 1 on $\{x_2 = 0\}$, while 2 elsewhere.

Let us now present the fractional version of the monotonicity formula.

Lemma 2.4.11. *Given a medium $d\theta = \sigma\, d\|\theta\|$. If θ is maximal and $x_0 \in \operatorname{Spt}\theta$, then the following quantity*

$$\frac{\|\theta\|(B_r(x_0))}{r^\alpha} \tag{2.4.11}$$

is monotone nondecreasing in $0 < r < 1/2$ for $\alpha = 1$. If $\kappa = \underline{\dim}_{\mathrm{r}}(\theta)(x_0) > 1$, then for all $\alpha \in [1,\kappa)$ there exists $r_\alpha > 0$ such that for all $0 < r < r_\alpha$, the quotient (2.4.11) is monotone nondecreasing. If $\underline{\dim}_{\mathrm{r}}(\theta)(y) \geq m$ for some constant $m > 1$ and all $y \in U \cap \operatorname{Spt}\theta$ in an open neighborhood $U \ni x_0$ then (2.4.11) is monotone nondecreasing for $\alpha = m$. In particular, this gives the standard monotoncity formula when m is an integer.

Proof. We may without loss assume that $x_0 = 0$ and apply a special $\Phi \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$ to the E-L equation (2.3.10). To this end we consider for small $\delta > 0$ an auxiliary function $\eta = \eta_{r,\delta} \in C_0^\infty([0,\infty))$ that satisfies $\eta(s) = 1$ for $0 \leq s \leq r-\delta$, $\eta(s) = 0$ for $s \geq r$ and η is strictly decreasing on $r-\delta < s < r$. By plugging $\Phi(x) = x\eta(|x|)$ into (2.3.10) and because $\operatorname{Tr}(\sigma) = 1$ we obtain

$$\begin{aligned}\int_{B_r} \eta d\|\theta\|(x) &= -\int_{B_r} (\sigma(x)x)\cdot\nabla(\eta(|x|))\, d\|\theta\|(x)\\ &= -\int_{B_r} \frac{x\cdot\sigma(x)x}{|x|}\eta'(|x|) d\|\theta\|(x).\end{aligned}$$

By definition of realizable dimension, we know that

$$\begin{aligned}\int_{B_r} \eta d\|\theta\|(x) &= -\int_{B_r} \frac{x\cdot\sigma(x)x}{|x|}\eta'(|x|) d\|\theta\|(x)\\ &\leq r\int_{B_r} \frac{\operatorname{Tr}(\sigma(x))}{\dim_{\mathrm{r}}(\sigma(x))}|\eta'|(|x|) d\|\theta\|(x).\end{aligned} \tag{2.4.12}$$

Because we always have $\underline{\dim}_{\mathrm{r}}(\theta)(x) \geq 1$

$$\int_{B_r} \eta d\|\theta\|(x) \leq r\int_{B_r} |\eta'|(|x|) d\|\theta\|(x). \tag{2.4.13}$$

If we write

$$J_\delta(r) = \int_{B_r} \eta d\|\theta\|(x),$$

the inequality (2.4.13) implies that

$$J_\delta(r) \leq r J_\delta'(r),$$

which shows that

$$\frac{J_\delta(r)}{r}$$

is monotone nondecreasing in $r > 0$. Sending $\delta \to 0$, we know that J_δ/r converges monotonically to $\frac{\|\theta\|(B_r)}{r}$. This shows that $\frac{\|\theta\|(B_r)}{r}$ is monotone nondecreasing in $0 < r < 1/2$.

Let us now discuss the case when $\kappa = \underline{\dim}_{\mathrm{r}}(\theta)(0) > 1$. By the definition of $\underline{\dim}_{\mathrm{r}}(\theta)(x)$ (see Definition 2.4.2) and the inequality (2.4.12), we know that for every $\alpha \in [1, \kappa)$ there is an $r_\alpha > 0$ such that for all $0 < r < r_\alpha$ we have

$$\int_{B_r} \eta d \|\theta\| (x) \leq \frac{r}{\alpha} \int_{B_r} \mathrm{Tr}\,(\sigma(x)) |\eta'|(|x|) d \|\theta\| (x).$$

Following the same argument as in the case for $\alpha = 1$ in (2.4.13), we obtain

$$\frac{\|\theta\| (B_r)}{r^\alpha}$$

is monotone nondecreasing in $0 < r < r_\alpha$.

In the case $\underline{\dim}_{\mathrm{r}}(\theta)(y) \geq m$ for some constant $m > 1$ and all y in an open neighborhood $U \ni x_0$, we obtain that

$$\dim_{\mathrm{r}}(\sigma(y)) \geq m$$

for $\|\theta\|$-a.e. $y \in U$. Plug this into (2.4.12) and follow the same argument we obtain the monotonicity of (2.4.11) with $\alpha = m$. □

Proof of Theorem 2.4.5. We without loss focus on the case that $x_0 = 0 \in \operatorname{Spt} \theta$ and argue by contradiction that

$$\underline{\dim}_{\mathrm{r}}(\theta)(0) > \underline{\dim}_{\mathrm{loc}}(\theta)(0). \tag{2.4.14}$$

We choose a number $\alpha \in (\underline{\dim}_{\mathrm{loc}}(\theta)(0), \underline{\dim}_{\mathrm{r}}(\theta)(0))$. Because $\alpha < \underline{\dim}_{\mathrm{r}}(\theta)(0)$, by the monotonicity formula in Lemma 2.4.11, we know that there is an $r_\alpha > 0$ such that

$$\frac{\|\theta\| (B_r)}{r^\alpha}$$

is monotone nondecreasing for $0 < r < r_\alpha$. Because $\|\theta\|$ is a Radon measure on $\mathbb{T}^n$ and hence a finite measure, we know that

$$\frac{\|\theta\| (B_r)}{r^\alpha} \leq C \tag{2.4.15}$$

for some $C > 0$ and all $0 < r < r_\alpha$. However, on the other hand, because

$$\liminf_{r\to 0^+} \frac{\log \|\theta\|(B_r)}{\log r} = \underline{\dim}_{\text{loc}}(\theta)(0) < \alpha$$

there exists a sequence of $r_j \to 0^+$ such that

$$\limsup_{j\to\infty} \frac{\log \|\theta\|(B_{r_j})}{\log r_j} < \alpha,$$

which implies that for a small $\delta > 0$ and all large j

$$r_j^{\alpha-\delta} \le \|\theta\|(B_{r_j}).$$

Therefore we have

$$\frac{1}{r_j^\delta} \le \frac{\|\theta\|(B_{r_j})}{r_j^\alpha}.$$

This contradicts the bound (2.4.15). Therefore, there should be no such an α contained in the interval $(\underline{\dim}_{\text{loc}}(\theta)(0), \underline{\dim}_{\text{r}}(\theta)(0))$. This contradicts (2.4.14) and hence proves the theorem. □

2.4.3 Applications of stationary varifolds

In this subsection we make clear the applications of the known theory of stationary varifolds in the characterization of maximal media. Most of the materials in this subsection are known. Specifically we apply the celebrated rectifiability theorem on stationary varifolds given by Allard [6]. There are many recent results in this topic, which we refer to [6, 125] for more discussions and references.

The following theorem is a result of Allard's rectifiability theorem [7, Theorem 14].

Theorem 2.4.12. *Suppose a medium θ is maximal and satisfies the following lower density inequality for some integer $1 \le k \le n-1$*

$$\liminf_{r\to 0^+} \frac{\|\theta\|(B_r(x))}{r^k} > 0 \tag{2.4.16}$$

for $\|\theta\|$-almost all $x \in \mathbb{T}^n$. Then $\|\theta\|$ is k-rectifiable, and there is a unique realization μ_θ that is a rectifiable stationary k-varifold.

For dimension $k = 1, n$ there are stronger results. First we have a result by Allard-Almgren.

Theorem 2.4.13 (Allard-Almgren [8]). *Suppose μ is a stationary 1-varifold on $U \subset\subset \mathbb{R}^n$ that satisfies the lower density bound for $\|\mu\|$-almost every x*

$$\liminf_{r\to 0^+} \frac{\|\mu\|(B_r(x))}{r} \geq \delta > 0$$

for some $\delta > 0$, then μ is 1-rectifiable with

$$d\mu(x,\tau) = \xi(x) d\delta_{T_x\mu} d\|\mu\|(x),$$

where the density $\xi(x) = \lim_{r\to 0^+} \frac{\|\mu\|(B_r(x))}{2r}$ is $\|\mu\|$-almost everywhere well-defined, $T_x\mu \subset T_xM$ is the 1-dimensional tangent space of the 1-rectifiable support Spt $\|\mu\|$ *at x and μ further satisfies*

1. *the support* Spt $\|\mu\|$ *is, up to an $\mathcal{H}^1$-null closed set S, a countable union of straight line segments, which are open relative to* Spt $\|\mu\|$*;*
2. *on each geodesic line segment there is a constant $c \geq \delta$ such that $\xi \equiv c$;*
3. *at every $x \in S$, there exists a unique stationary tangent cone consisting of finitely many half lines with densities (see Figure 2.3);*
4. *if the density ξ is discretely valued, then for every compact subset $K \subset\subset U$ the number of line segments that have nontrivial intersection with K is finite.*

The above theorem implies the following result.

Corollary 2.4.14. *Let θ be a maximal media that satisfies the 1-dimensional lower density bound*

$$\liminf_{r\to 0^+} \frac{\|\theta\|(B_r(x))}{r} \geq \delta > 0 \tag{2.4.17}$$

for $\|\theta\|$-almost every $x \in \mathbb{T}^n$. Then θ admits a unique 1-rectifiable stationary varifold realization as described in Corollary 2.1.2.

For isotropic media, or equivalently media that have realizable dimension $k = n$, we have the following characterization.

Theorem 2.4.15. *Let θ be an isotropic maximal medium, then $d\|\theta\| = c\, d\mathcal{L}^n$ for some constant $c > 0$.*

This result for the case when $\|\theta\|$ is absolutely continuous with respect to the Lebesgue measure has been established earlier, see [100, Section 1.6].

Proof. It is not difficult to see that

$$d\mu(x,\tau) = \frac{1}{n} d\delta_{\mathbb{R}^n}(\tau) d\|\theta\|(x)$$

is the unique n-realization of θ. By the monotonicity formula (see Section 2.2.3.7), we know that

$$\frac{\|\theta\|(B_r(x))}{r^n}$$

is monotone nondecreasing in $r > 0$ for every $x \in \mathbb{T}^n$. By the Lebesgue-Besicovitch differentiation theorem (see [112, Theorem 2.10]), $\|\theta\|$ is absolutely continuous with respect to the Lebesgue measure, that is, for some $\xi \geq 0$ and $\xi \in L^1(\mathbb{T}^n)$

$$d\|\theta\|(x) = \xi(x) d\mathcal{L}^n.$$

Now, by Theorem 2.4.1 we have for every $\Phi \in C_0^\infty(U)$

$$0 = \int_U \nabla \cdot \Phi(x) d\|\theta\|(x) = \int_U \nabla \cdot \Phi(x)\xi(x) d\mathcal{L}^n,$$

which shows that $\xi(x) \equiv c$ for some constant $c > 0$ and all $x \in U$. □

2.4.4 Stationary networks and the question of the maximal valency of leaf vein patterns

As we have established the connection between the stationary varifolds and leaf vein patterns in Theorem 2.4.1, it is natural to ask what geometric properties of the leaf vein patterns can we derive from this relation. In this subsection, we provide an example question on stationary 1-varifold on $\mathbb{T}^2$ with density exactly 1 almost everywhere. In this example we formulate mathematically the question on the maximal valencies, that is, the maximal number of edges joining at one node, of leaf vein patterns.

By Theorem 2.4.13 and Corollary 2.4.14, such objects are networks composed of finitely many nodes and edges $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N} \subset \mathbb{T}^2$ and $\mathcal{E}$ is a collection of straight line segments in $\mathbb{T}^2$ with endpoints in $\mathcal{N}$ satisfying that for every node $x \in \mathcal{N}$ there is an integer $k \geq 2$ such that the edges joining at x can be written as $e_1, \dots, e_k \in \mathcal{E}$, and the edges always satisfy, according to (2.1.13), the balance condition

$$\sum_{i=1}^{k} T_i = 0, \tag{2.4.18}$$

where for each $1 \le i \le k$, the vector T_i is the unit tangent vector of the line segment e_i starting from the endpoint x.

Definition 2.4.16. We define a *periodic planar stationary network* (simply call *stationary network* in the following context) as $\Gamma = (\mathcal{N}, \mathcal{E})$ as described above that satisfies (2.4.18) for all $x \in \mathcal{N}$. We do not distinguish Γ and the closed set consisting of all points in $\mathcal{N}$ and edges in $\mathcal{E}$. For a stationary network Γ, define the *valency* $\mathrm{V}_\Gamma(x)$ at $x \in \mathbb{T}^2$ as

$$\mathrm{V}_\Gamma(x) \coloneqq \lim_{r \to 0^+} \#\partial B_r(x) \cap \Gamma.$$

To talk about the maximal valency problem, we need another concept, which is the irreducibility of a stationary network.

Definition 2.4.17. A stationary network is called *irreducible* if it can not be written as the union of two distinct stationary networks.

Our question on the maximal valency states as follows.

Problem 2.4.18 (Maximal valency)**.** Is there a universal constant $C > 0$ such that for any irreducible stationary network Γ, the maximal valency

$$V(\Gamma) \coloneqq \max_{x \in \Gamma} V_\Gamma(x) \le C?$$

If the bound C exists, is $C = 4, 5$ or 6 ?

The reason for using the notion "irreducible" is Lemma 2.4.19 below, which states that a reducible stationary network can be perturbed infinitesimally without losing stationarity. This indicates that the maximal valency of reducible stationary networks is unstable in the sense that it changes after small and simple perturbations of the network while the stationarity is preserved (see Figure 2.5).

Lemma 2.4.19. *Given two stationary networks Γ_1 and Γ_2, for any $\varepsilon > 0$ there is a vector $|p| = 1$ such that*

$$\Gamma_1 \cup \Gamma_2 + \varepsilon p$$

is a stationary network.

(a) A reducible stationary network with maximal valency 6.

(b) After small perturbations the maximal valency becomes 4.

Figure 2.5: Instability of maximal valencies of reducible stationary networks.

Proof. Note that $\Gamma_1 \cup \Gamma_2$ is a stationary network whenever the intersection of the edges is at most finite. This can always be achieved by a small perturbation. □

One might also ask whether there are only finitely many such irreducible objects. In the following we prove that there are infinitely many irreducible stationary networks. Note that the following construction does not enumerate all the irreducible stationary networks.

Lemma 2.4.20. *There are infinitely many irreducible $\mathbb{Z}^2$-periodic stationary networks. All of them have maximal valency 4.*

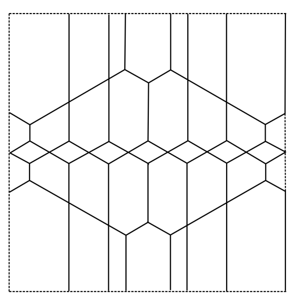

Figure 2.6: An irreducible stationary network that has $k = 7$ parallelograms in the middle. This example was first made in [96].

Proof. We prove this by constructing infinitely many such patterns. Let $k > 0$ be an arbitrary odd number and we put k equilateral parallelograms with inner angles $\pi/3, 2\pi/3$ (diamonds) inside a unit square $(0,1]^2$. In Figure 2.6, we put $k = 7$ such parallelograms along the line $y = 0.5$. There is no obstacle to set k to be arbitrarily large as long as k is odd.

□

2.5 The support of nontrivial media

In this section we present three theorems about the support of a medium when its effective tensor is nonzero. These results present the interplay between the geometry of the support Spt θ and the nonzero effective tensor $Q(\theta) \neq 0$ under different assumptions. They are also important preliminary results for the discussion of the 1-D attainability of the lower Wiener bound in Section 2.6. In Section 2.5.1 we also present several examples that can show the sharpness of these theorems.

Our first result concerns the dimension of a nontrivial medium and shows that the support of a nontrivial medium is not totally disconnected, and hence has at least Hausdorff dimension 1. This result is sharp in the sense that the dimension of the support of a nontrivial medium can be any real number in $[1, n]$, see Example 2.5.6 for more details.

Theorem 2.5.1. *Let θ be a nontrivial medium, then the support* Spt θ *is not totally disconnected, that is, there is a nonsingleton component in* Spt θ*. This implies that the 1D Hausdorff measure*

$$\mathcal{H}^1(\text{Spt } \theta) > 0. \tag{2.5.1}$$

In particular, the Hausdorff dimension $\dim_{\mathcal{H}}(\text{Spt } \theta) \geq 1$*.*

Our second result concerns the decomposition of θ into the countable sum of its restrictions to the components of Spt θ. In general there is no countable additivity due to Example 2.5.5. We also show the existence of a maximal medium having uncountably many connected components in Example 2.5.7.

Theorem 2.5.2. *Let θ be a nontrivial medium and suppose* Spt θ *have countably many components E_i with*

$$E_i \subset \text{Spt } \theta, \ \text{Spt } \theta = \bigcup_{i=1}^{\infty} E_i \text{ and } E_i \cap E_j = \varnothing \text{ for } i \neq j.$$

Define the restrictions $\theta_i := \theta\big|_{E_i}$ then

$$Q(\theta) = \sum_{i=1}^{\infty} Q(\theta_i).$$

In particular, if θ is saturated then all θ_i's are nontrivial.

Our third result concerns the countable decomposition of θ solely when Spt θ has finite $\mathcal{H}^1$ measure. The key improvement of this theorem is that it does not require the medium to have only countably many components.

Theorem 2.5.3. *Let θ be a nontrivial medium such that $\mathcal{H}^1(\operatorname{Spt}\theta) < \infty$. Then there exist countably many 1-rectifiable mutually disjoint connected components $E_i \subset \operatorname{Spt}\theta$ such that the submedia $\theta_i := \theta\big|_{E_i}$ satisfy*

$$Q(\theta) = \sum_{i=1}^{\infty} Q(\theta_i). \tag{2.5.2}$$

We postpone the proofs of Theorem 2.5.1, Theorem 2.5.2 and Theorem 2.5.3 to Section 2.5.2 and 2.5.3. In Section 2.5.1 we present some examples.

2.5.1 Examples on trivial and nontrivial media

In this subsection we present some examples to show the sharpness of the above theorems. The nontrivial media are weaker notions than the maximal ones, which means that we should expect less regularity properties of such media than, for example, Theorem 2.4.12 and Theorem 2.4.13. We list several examples (or non-examples) to show that why Theorem 2.5.1, Theorem 2.5.2 and Theorem 2.5.3 are sharp. There are also some examples that are related to the results obtained in the previous sections.

Example 2.5.4. In this example we show that the reverse of Theorem 2.5.1 is generally false, that is, a medium can still be trivial even if the support has positive $\mathcal{H}^1$ measures. We first parametrize $\mathbb{T}^1$ with unit speed coordinate $x \in [-1/2, 1/2)$. Define the medium θ as

$$d\theta(x) = |x|^{\alpha} d\mathcal{L}^1(x) = |x|^{\alpha} dx,$$

where $\alpha > 1$. Notice that Spt $\theta = \mathbb{T}^1$ but it is not difficult to check that θ is a trivial medium.

Example 2.5.5. In Lemma 2.3.8 (2) we proved additivity for two media θ_1 and θ_2 satisfying Spt $\theta_1 \cap$ Spt $\theta_2 = \varnothing$. In this example we show that there is generally no countable additivity for a sequence of media θ_i even if we assume Spt $\theta_i \cap$Spt $\theta_j = \varnothing$ for $i \neq j$. Indeed, on $\mathbb{T}^1$ we define the media

$$\theta_j := \mathcal{H}^1\big|_{\overline{D_j}},$$

where D_j is the union of open subintervals of $(0,1)$, with each interval having length $1/3^j$, obtained by taking the deleted intervals at the j-th step in the construction of the standard Cantor set. More precisely, for $j \geq 1$ the sets D_j satisfy

$$D_1 = (1/3, 2/3) \text{ and } D_{j+1} = \bigcup_{k=0}^{3^j-1} \left(\frac{3k+1}{3^{j+1}}, \frac{3k+2}{3^{j+1}} \right) \setminus D_j.$$

See Figure 2.7 for the corresponding D_j's when $j = 1, 2, 3$. Note that Spt $\theta_j = \overline{D}_j$ and it is not difficult to show that

$$\operatorname{dist}(\text{Spt } \theta_i, \text{Spt } \theta_j) = \operatorname{dist}(\overline{D}_i, \overline{D}_j) = 1/3^{\max\{i,j\}} > 0$$

for all $i \neq j$. Now we can find that

$$Q(\theta_j) = 0 \text{ for all } j \geq 1,$$

while $\sum_{j=1}^{\infty} \theta_j$ is exactly the Lebesgue measure of $\mathbb{T}^1$, which is a maximal medium.

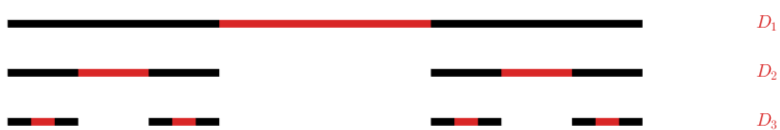

Figure 2.7: The first three steps in constructing the standard Cantor set. The red intervals correspond to the deleted intervals D_j in step $j = 1, 2$ and 3.

Example 2.5.6. In this example we show that it is difficult to go beyond Theorem 2.5.1 and prove a similar pointwise theory like Theorem 2.4.5 and the fractional montonicity formula like Lemma 2.4.11 for maximal media. In fact we show that a medium can have “arbitrarily” rough support even if they are close to the upper Wiener bound (2.3.5). We start with the medium θ on $\mathbb{T}^n$ as defined below

$$d\theta := e_1 \otimes e_1 \, d\mathcal{H}^1\big|_{\{x'=0\}},$$

where we parametrize $\mathbb{T}^n$ by $x = (x_1, x') \in [0,1) \times [0,1)^{n-1}$ and $e_1 = \nabla x_1$ is the unit vector on x_1-axis. Notice that θ is maximal. Now let ξ be an arbitrary medium, then by Lemma 2.3.8 (1), for every $\varepsilon > 0$, the medium $\theta_\varepsilon := \theta + \varepsilon\xi$ is nontrivial. As $\varepsilon > 0$ is arbitrary, the medium θ_ε can be made arbitrarily close to the upper Wiener bound (2.3.5). However, the dimension of Spt θ_ε can be any number between 1 and n as ξ is chosen arbitrarily.

Example 2.5.7. In this example we present a maximal medium θ such that Spt θ has uncountably many components. Let $\beta = \frac{\log 2}{\log 3}$, and consider on $(x_1, x_2) \in \mathbb{T}^2$ the following Radon measure

$$dw(x_1,x_2) := d\mathcal{H}^\beta\big|_C(x_1)dx_2,$$

where $C \subset \mathbb{T}^1$ is the standard Cantor set. By standard result (see [79, Theorem 1.14]), we know that $\mathcal{H}^\beta\big|_C$ is a probability measure with support equal to C. This shows that w is a probability measure supported on $C \times \mathbb{T}^1$.

We define the medium θ to be

$$\|\theta\| = w \text{ and } \sigma := \frac{d\theta}{d\,\|\theta\|} = e_2 \otimes e_2,$$

where $e_1 = \nabla x_1$ and $e_2 = \nabla x_2$ form an orthonormal basis for $\mathbb{R}^2$. By using Fubini's theorem, it is not difficult to check that θ is maximal on $\mathbb{T}^2$, but Spt θ is composed of uncountably many components.

2.5.2 Proof of Theorem 2.5.1 and Theorem 2.5.2

The proofs of both Theorem 2.5.1 and Theorem 2.5.2 hinge on the following fact from topology. This lemma adapts from the arguments in [15, Proposition 3.1.7].

Lemma 2.5.8. *Let X be a closed subset of $\mathbb{T}^n$, then for every component $E \subset X$ and its open neighborhood $V \subset \mathbb{T}^n$ there is another open subset K of $\mathbb{T}^n$ such that $E \subset K \subset V$, and $K \cap X$ is closed in $\mathbb{T}^n$.*

The argument for proving this lemma also works for compact components in a closed set of $\mathbb{R}^n$.

Proof. We consider the set $W = V \cap X$, and if W is also closed then we are done. Otherwise $(\partial V) \cap X$ is non-empty. Notice that $E \subset W$ is a component of X and therefore it is also a component for $\overline{W}$. By [75, Lemma 1.4.4], E is also a quasi-component of $\overline{W}$. Recall that a quasi-component in $\overline{W}$ is the intersection of all subsets of $\overline{W}$ that contain a fixed $x \in X$ and are both open and closed in the subset topology. Therefore for every $y \in (\partial V) \cap X$, because $y \notin E$ there is an open subset $H_y \subset \mathbb{T}^n$ such that $E \subset H_y$, $y \notin H_y$ and $H_y \cap \overline{W}$ is also closed. This means that $\overline{W} \setminus H_y$ form an open cover for $(\partial V) \cap X$ in the subset topology of $\overline{W}$. As $(\partial V) \cap X$ is compact, we may choose finitely many $y_1, \dots, y_m \in (\partial V) \cap X$ such that $\overline{W} \setminus H_{y_i}$, $i = 1, \dots, m$ cover $(\partial V) \cap X$. Let $H = \cap_{i=1}^m H_{y_i}$, then H by the previous construction is still open and contains E, and $H \cap \overline{W}$ is closed. Moreover, we know that $H \cap \overline{W} \cap (\partial V) \cap X = \varnothing$, which means that $H \cap \overline{W} \subset\subset V$, and hence we may finish the proof by just choosing $K = V \cap H$.

□

Proof of Theorem 2.5.1. We argue by contradiction and assume that the compact set

$$X := \text{Spt } \theta \subset \mathbb{T}^n$$

is totally disconnected. We claim that in this case the mean conductance

$$M(\theta) = 0.$$

To prove the claim, we apply Lemma 2.5.8 to the totally disconnected set X. Notice that all components of X are singletons, and therefore for every small $\delta > 0$ and every $x \in X$ there is an open subset $K_\delta(x) \subset \mathbb{T}^n$ such that $x \in K_\delta(x)$, $\text{diam}(K_\delta) < \delta$ and $K_\delta \cap X$ is closed. Now by compactness of X we can choose finitely many x_i for $i = 1, \dots, m$ such that the open sets $K_\delta(x_i)$'s cover X and each $I_i := K_\delta(x_i) \cap X$ is closed in $\mathbb{T}^n$. In other words, I_i's are both closed and open in X in the subset topology. After finitely many times of intersections and complementations of I_i's, we obtain for X a new finite cover J_k's that are disjoint from each other and also both open and closed in X in the subset topology. Moreover the diameters $\text{diam}(J_k) < \delta$.

For each k, let $\psi_k \in C^\infty(\mathbb{T}^n)$ be a smooth function such that $0 \le \psi_k \le 1$, $\psi_k \equiv 1$ in a neighborhood of J_k and $\psi_k = 0$ outside a slightly larger neighborhood. By the previous constructions on J_k's we can further allow

$$\text{diam}(\text{Spt } \psi_k) < \delta \text{ and } \psi_k \psi_l = 0 \text{ for all } k \neq l.$$

Because $\mathbb{T}^n$ is a flat torus for each k we may choose a smooth isometric chart ξ_k from a δ geodesic ball containing J_k to the ball $B_\delta(0)$ in $\mathbb{R}^n$. Now we define

$$\Phi_0(x) = -\sum_k \xi_k \psi_k \in C^\infty(\mathbb{T}^n; \mathbb{R}^n),$$

and see that, if we write $d\theta = \sigma d\|\theta\|$ and apply (2.3.12)

$$M(\theta) \le \frac{1}{n}\int_{\mathbb{T}^n} \mathrm{Tr}\left((\nabla\Phi_0 + I)^T \sigma(\nabla\Phi_0 + I)\right) d\|\theta\|(x) = 0.$$

This contradicts the assumption that θ is nontrivial and implies that the support Spt θ must contain at least one non-singleton connected component E. We finish the proof of this theorem by applying the following lemma.

Lemma 2.5.9. *Let E be a connected subset of a continuum X. Then $\mathcal{H}^1(E) \ge$ diam(E).*

The proof of this lemma can be found in [4, Lemma 2.11].

□

To prove Theorem 2.5.2 we require the following lemma that improves Lemma 2.3.8 (2).

Lemma 2.5.10. *Let θ be a medium and $E \subset$ Spt θ be a component. If we write θ_E to be the restriction of θ on E then*

$$Q(\theta) = Q(\theta_E) + Q(\theta - \theta_E).$$

Proof. The difficulty here is that the support of θ_E may intersect the support of its complement $\theta - \theta_E$. To overcome this, we apply Lemma 2.5.8 to the component E. We then obtain for each $\delta > 0$ an open subset $K_\delta \subset \mathbb{T}^n$ such that $E \subset K_\delta$, $\mathrm{dist}(E, \mathbb{T}^n \setminus K_\delta) < \delta$ and $K_\delta \cap$ Spt θ is closed in $\mathbb{T}^n$. We denote $J_\delta := K_\delta \cap$ Spt θ, and define the restrictions

$$\theta_\delta := \theta\big|_{J_\delta}.$$

Notice that $\theta_\delta \overset{*}{\rightharpoonup} \theta_E$ and $\theta_E \le \theta_\delta$. By applying Lemma 2.3.12, we know that θ_δ faithfully converges to θ_E as $\delta \to 0^+$, that is, we have

$$\lim_{\delta\to 0^+} Q(\theta_\delta) = Q(\theta_E).$$

On the other hand, by the construction of θ_δ, the support Spt θ_δ is disjoint from the support of its complement Spt $(\theta - \theta_\delta)$. By Lemma 2.3.8 (2), we have for every $\delta > 0$

$$Q(\theta) = Q(\theta_\delta) + Q(\theta - \theta_\delta). \tag{2.5.3}$$

Because $\theta - \theta_\delta$ weakly* converges to $\theta - \theta_E$, by sending $\delta \to 0$ in (2.5.3) and applying Lemma 2.3.10

$$Q(\theta) = Q(\theta_E) + \limsup_{\delta \to 0^+} Q(\theta - \theta_\delta) \leq Q(\theta_E) + Q(\theta - \theta_E).$$

We then finish the proof by applying Lemma 2.3.8 (3).

□

Proof of Theorem 2.5.2. Notice that if E and F are disjoint components of Spt θ, then F is also a component of Spt $(\theta - \theta_E)$, where θ_E is the restriction of θ on E.

Let E_i be the countable collection of components of Spt θ and $\theta_i := \theta\big|_{E_i}$ be the restrictions of θ on E_i. By iterating Lemma 2.5.10, we obtain the identity

$$Q(\theta) = \sum_{i=1}^{N} Q(\theta_i) + Q\left(\theta - \sum_{i=1}^{N} \theta_i\right),$$

for all $N \geq 1$. In particular, we have for all $N \geq 1$ the following identity for the mean conductances

$$M(\theta) = \sum_{i=1}^{N} M(\theta_i) + M\left(\theta - \sum_{i=1}^{N} \theta_i\right).$$

Notice that by Lemma 2.3.8 (3) we always have $Q(\theta) \geq \sum_{i=1}^{\infty} Q(\theta_i)$, and if the inequality is strict, then there is a constant $c > 0$ such that for all $N \geq 1$

$$M\left(\theta - \sum_{i=1}^{N} \theta_i\right) = M(\theta) - \sum_{i=1}^{N} M(\theta_i) \geq c > 0.$$

This is impossible because $\left(\theta - \sum_{i=1}^{N} \theta_i\right) \overset{*}{\rightharpoonup} 0$ as $N \to \infty$, and by Lemma 2.3.10 we have

$$0 \geq \limsup_{N \to \infty} M\left(\theta - \sum_{i=1}^{N} \theta_i\right) \geq c > 0,$$

which is a contradiction. □

2.5.3 Proof of Theorem 2.5.3

In this proof we will have to select carefully a countable family of connected components based on the only assumptions that $Q(\theta) \neq 0$ and $\mathcal{H}^1(\mathrm{Spt}\ \theta) < \infty$.

We begin with the following quantitative version of the inequality (2.5.1).

Lemma 2.5.11. *Suppose a medium θ is nontrivial, then*

$$\mathcal{H}^1(\mathrm{Spt}\ \theta) \geq 1/2.$$

Proof. We argue by contradiction and assume that $\mathcal{H}^1(\mathrm{Spt}\ \theta) < 1/2$. We claim that in this case, the set Spt θ can be decomposed as a finite collection of subsets that have positive distance from each other and have diameters smaller than 1/2.

First of all, by definition (see Section 2.2.3.3), we know that for every $\delta > 0$

$$\inf \left\{ \sum_{j=1}^{\infty} \mathrm{diam}(U_j)\ ;\ \mathrm{diam}(U_j) \leq \delta, \mathrm{Spt}\ \theta \subset \bigcup_{j=1}^{\infty} U_j \right\} < 1/2.$$

This shows that, by using the compactness of Spt θ, we can find $N \in \mathbb{N}_+$ open balls B_j such that

$$\sum_{j=1}^{N} \mathrm{diam}(B_j) < 1/2 \text{ and } \mathrm{Spt}\ \theta \subset \bigcup_{j=1}^{N} B_j =: K.$$

We finish the proof of the claim by showing that the diameter of each component of K is less than $1/2$. To prove this we denote $K^* \subset K$ a connected component of K. As K is a finite union of balls, the component K^* is also a finite union of balls $\{B_{j_l}\} \subset \{B_j\}_{j=1}^N$. By the triangle inequality

$$\mathrm{diam}(K^*) \leq \sum_l \mathrm{diam}(B_{j_l}) \leq \sum_{j=1}^{N} \mathrm{diam}(B_j) < 1/2.$$

This proves the claim.

Now, because each connected component K^* of K has diameter at most $1/2$, K^* as a subset of $\mathbb{T}^n$ must be isometrically diffeomorphic to a connected open domain in $\mathbb{R}^n$. Indeed, the preimage $\pi^{-1}(B_r)$ of any open ball $B_r \subset \mathbb{T}^n$ of radius $0 < r \leq 1/2$ under the standard projection $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is a countable union of disjoint balls in $\mathbb{R}^n$ of the same radius, and π restricted on any of these balls defines an isometric diffeomorphism onto B_r. Applying this observation to a fixed ball of radius $1/2$ that covers K^* (there is always such a ball because $\mathrm{diam}(K^*) < 1/2$), we can find a smooth isometric diffeomorphism $\phi : K^* \to \mathbb{R}^n$. The proof is then done by a similar construction of the smooth test vector field Φ_0 as in the proof of Theorem 2.5.1.

□

Proof of Theorem 2.5.3. We construct E_i's inductively and denote $\theta_i = \theta\big|_{E_i}$. Because θ is nontrivial, by Theorem 2.5.1 we know that there is always a connected component of Spt θ having positive $\mathcal{H}^1$ measure. Therefore, we can find E_1 to satisfy

$$\mathcal{H}^1(E_1) \geq \frac{1}{2} \sup\{\mathcal{H}^1(F) \ ; \ F \text{ is a connected component of Spt } \theta\} > 0.$$

By Lemma 2.5.8, for every $\varepsilon > 0$ we can find an open set J_ε such that $J_\varepsilon \subset J_\delta$ whenever $\delta > \varepsilon > 0$ and

$$E_1 \subset J_\varepsilon, \ \operatorname{dist}(E_1, \partial J_\varepsilon) \leq \varepsilon, \ \mathcal{H}^1(J_\varepsilon \cap \text{Spt } \theta \setminus E_1) \leq \varepsilon \tag{2.5.4}$$

and $J_\varepsilon \cap \text{Spt } \theta$ is closed. We claim that when $\varepsilon > 0$ is sufficiently small, we have

$$Q(\theta\big|_{J_\varepsilon}) = Q(\theta_1).$$

Indeed, on one hand we have by Lemma 2.3.12 the identity $\lim_{\varepsilon \to 0^+} Q(\theta\big|_{J_\varepsilon}) = Q(\theta_1)$. On the other hand, we have by Lemma 2.5.10

$$Q(\theta\big|_{J_\delta}) = Q(\theta\big|_{J_\varepsilon}) + Q(\theta\big|_{J_\delta \setminus J_\varepsilon})$$

for $0 < \varepsilon < \delta \ll 1$. When both $\delta, \varepsilon \to 0^+$ with $\delta > \varepsilon$, by (2.5.4), we know that $\mathcal{H}^1(J_\delta \cap \text{Spt } \theta \setminus J_\varepsilon) \to 0$. This implies that according to Lemma 2.5.11

$$Q(\theta\big|_{J_\delta}) = Q(\theta\big|_{J_\varepsilon})$$

for all $\delta > \varepsilon > 0$ whenever both are small. Therefore $Q(\theta\big|_{J_\varepsilon}) = Q(\theta_1)$ for sufficiently small $\varepsilon > 0$.

We define K_1 to be such an open set J_ε with sufficiently small $\varepsilon > 0$ that satisfies

$$E_1 \subset K_1, \ Q(\theta\big|_{K_1}) = Q(\theta_1)$$

and Spt $\theta \cap K_1$ is closed. If $\theta - \theta\big|_{K_1}$ is trivial, then by Lemma 2.3.8 we are done. Otherwise by Theorem 2.5.1 there is a connected component E_2 of Spt θ that is disjoint from K_1 and

$$\mathcal{H}^1(E_2) \geq \frac{1}{2} \sup\{\mathcal{H}^1(F) \ ; \ F \text{ is a connected component of Spt } \theta \setminus K_1\} > 0.$$

Note that by the construction of K_1, all the connected components of Spt $\theta \setminus K_1$ are also components of Spt θ. Suppose we have obtained $E_1, \dots, E_i$ for some $i \geq 2$. For E_i we can find by a similar argument as before, an open set K_i such that

$$E_i \subset K_i,\ Q(\theta\big|_{K_i}) = Q(\theta_i),\ K_i \cap \text{Spt } \theta \text{ is closed}$$

and $K_j \cap K_{j'} = \varnothing$ for $1 \leq j < j' \leq i$. If $\theta - \sum_{j=1}^{i} \theta\big|_{K_j}$ is trivial, then we are done. Otherwise by applying Theorem 2.5.1 to $\theta - \sum_{j=1}^{i} \theta\big|_{K_j}$, we can find a connected component E_{i+1} of Spt θ disjoint from all K_j for $j \leq i$ and

$$\mathcal{H}^1(E_{i+1}) \geq \frac{1}{2} \sup\{\mathcal{H}^1(F)\ ;\ F \text{ is a connected component of Spt } \theta \setminus \bigcup_{j=1}^{i} K_j\} > 0. \quad (2.5.5)$$

We finish the proof by showing that the mean conductance satisfies

$$\lim_{i\to\infty} M\left(\theta - \sum_{j=1}^{i} \theta\big|_{K_j}\right) = 0.$$

Suppose for some $c > 0$ and all large i we have

$$M\left(\theta - \sum_{j=1}^{i} \theta\big|_{K_j}\right) \geq c > 0.$$

Sending $i \to \infty$ and denote

$$\tilde{\theta} = \theta - \sum_{j=1}^{\infty} \theta\big|_{K_j}.$$

By applying the singular Wiener bound (2.3.5) and the upper semi-continuity in Lemma 2.3.10, we know that

$$\frac{1}{n} \|\tilde{\theta}\| (\mathbb{T}^n) \geq M(\tilde{\theta})(\mathbb{T}^n) \geq c > 0.$$

As $\tilde{\theta}$ is a nontrivial medium, by Theorem 2.5.1 again, there is always a component $K \subset$ Spt $\tilde{\theta}$ such that $\mathcal{H}^1(K) > 0$. On one hand, we know that

$$\text{Spt } \tilde{\theta} \subset \text{Spt } \theta \setminus \bigcup_{j=1}^{\infty} K_j,$$

and therefore $K \cap E_j = \varnothing$ for all $j \geq 1$. On the other hand, for all $i \geq 1$ there is a connected component $\tilde{K}_i$ of Spt $\theta \setminus \bigcup_{j=1}^{i} K_j$ that contains K and satisfies by (2.5.5)

$$\mathcal{H}^1(E_{i+1}) \geq \frac{1}{2}\mathcal{H}^1(\tilde{K}_i) \geq \frac{1}{2}\mathcal{H}^1(K) > 0.$$

This contradicts the fact that $\sum_{i=1}^{\infty} \mathcal{H}^1(E_i) \leq \mathcal{H}^1(\text{Spt } \theta) < \infty$.

□

2.6 Loopiness, reticulation and the lower Wiener bound

In this section we characterize the lower attainability property of the singular lower Wiener bound (2.3.5) for *network-like* media. To be more specific, we focus on media of the form $d\theta = I_{n\times n}\,dw$, where the Radon measure w satisfies

$$0 < \limsup_{r\to 0^+} \frac{w(B_r(x))}{r} < \infty \tag{2.6.1}$$

for w-almost every $x \in \mathbb{T}^n$ and the following coercivity condition

$$\limsup_{r\to 0^+} \frac{w(B_r(x))}{r} > c > 0 \tag{2.6.2}$$

for $\mathcal{H}^1$-almost every $x \in \operatorname{Spt} w = \operatorname{Spt} \theta$. Note that the subtle point here is that the coercivity condition (2.6.2) is satisfied for $\mathcal{H}^1$-almost every $x \in \operatorname{Spt} w$ instead of for w-almost every x. By the discussions in Section 2.2.3.5, there is an equivalent definition for such medium

$$d\theta(x) = a(x) I_{n\times n}\, d\mathcal{H}^1\big|_{\Gamma}(x), \tag{2.6.3}$$

where $\Gamma = \operatorname{Spt} w = \operatorname{Spt} \theta$ is a closed subset of $\mathbb{T}^n$ and $\mathcal{H}^1(\Gamma) < \infty$, and the coercivity condition on a takes the form

$$a \in L^1\left(\mathcal{H}^1\big|_{\Gamma}\right) \text{ and } a(x) \geq \Lambda^{-1} \text{ for } \mathcal{H}^1\big|_{\Gamma}\text{-almost every } x \in \mathbb{T}^n, \tag{2.6.4}$$

for some constant $\Lambda > 0$. We refer to Example 2.5.4 for the necessity of the coercivity condition.

Such medium arise as one considers in classical theory the matrix field $\bar{\sigma}$ of the form with small $\delta > 0$

$$\bar{\sigma}(x) := \begin{cases} \frac{a(x)}{\delta} I_{n\times n} & \operatorname{dist}(x,\Gamma) < (\delta/\omega_{n-1})^{1/(n-1)} \\ \delta I_{n\times n} & \text{elsewhere.} \end{cases}$$

That is, when the high conductive material concentrates on the 1D set Γ, one should study the medium of the form (2.6.3) as an effective model. We refer to [96] for a justification of the faithful convergence of $\bar{\sigma}(x)\, d\mathcal{L}^2$ to $d\theta$ as in (2.6.3) as $\delta \to 0$ in the special case that the dimension $n = 2$ and Γ is a finite union of C^2 curves.

Most of the results in this section are motivated by the modeling of leaf venation patterns. Specifically we view $\mathbb{T}^n$ as a small piece of leaves, $\Gamma = \operatorname{Spt}\theta$ the geometric structure of the leaf veins on $\mathbb{T}^n$ and $a(x)$ the local conductance of veins at $x \in \Gamma$. The goal is to show, as pointed out in the introduction, that a periodic planar network is resilient to fluctuations if and only if it is reticulate. This formal theorem will be stated in a rigorous way in Theorem 2.6.5 and Theorem 2.6.6.

It turns out naturally that the answer is hidden in the study of the effective tensor $Q(\theta)$, and in particular, the positive definiteness of $Q(\theta)$. By standard homogenization theory, the effective conductance of the medium θ in direction $p \in \mathbb{R}^n$ should be

$$p \cdot Q(\theta)p.$$

Here the vector p can be viewed as the effective exterior pressure gradient applied on the medium θ. Suppose the medium experiences a random fluctuation in p, then one would expect that $p \cdot Q(\theta)p$ to be as large as possible in all direction p. However, due to the vast variety of leaf veins, we do not expect a quantitative way to analyze, but at least we can allow $p \cdot Q(\theta)p$ to be positive in all p. This is equivalently saying that $Q(\theta)$ has to be positive definite, or in other words, the medium θ has to be positive.

Although the above simple explanation is clear, there is still a gap between this argument and the original setting in [58, 104], where the minimization of the total dissipation under random fluctuations was considered. In Appendix 2.7.1 we try to fix this gap by deriving the positivity of the effective tensor $Q(\theta)$ from a continuum version of the dissipation minimization problem in [58, 104].

To characterize positive medium θ of the form (2.6.3), we need to introduce some topological terminologies. We begin with the set H_E that collects all the homotopy classes in the subset $E \subset \mathbb{T}^n$.

Definition 2.6.1. For any subset $E \subset \mathbb{T}^n$, we denote $\pi_1(E, x_0) \leq \pi_1(\mathbb{T}^n, x_0)$ the subgroup of the homotopy classes containing closed paths in E based at $x_0 \in E$. We denote the collection of homotopy classes in E as the set

$$H_E := \bigcup_{x_0 \in E} i_{x_0}\left(\pi_1(E, x_0)\right) \subset \mathbb{Z}^n, \tag{2.6.5}$$

where $i_{x_0} : \pi_1(\mathbb{T}^n, x_0) \to \mathbb{Z}^n$ is the isomorphism as introduced in Section 2.2.4.4. Call $\mathbb{Z}H_E := \mathrm{Span}_{\mathbb{Z}}(H_E)$, $\mathbb{R}H_E := \mathrm{Span}_{\mathbb{R}}(H_E)$ and $H_E^{\perp} := (\mathbb{R}H_E)^{\perp}$ the real orthogonal complement of $\mathbb{R}H_E$ in $\mathbb{R}^n$.

Definition 2.6.2. Let $E \subset \mathbb{T}^n$. Denote $H_E \subset \mathbb{Z}^n$ the collection of homotopy classes in E as defined in Definition 2.6.1. Call E to be trivial if $H_E = 0$. Call E to be *loopy* if the real span $\mathbb{R}H_E = \mathbb{R}^n$. Call E to be *reticulate* if it has a loopy connected component. Call E to be *quasi-laminate* if $H_E = \mathrm{Span}_{\mathbb{Z}}(p)$ for some $p \in \mathbb{Z}^n$.

Remark 2.6.3. The terms "loopy" and "reticulate" generally share the same meaning for leaf venation patterns in biological literature [154, 156]. In dimension $n = 2$ they are indeed the same mathematically (see Lemma 2.6.15). In higher dimensions they are different notions because the loops can be detached in the extra dimensions. A quick example in $\mathbb{T}^3$ is the following set

$$\mathbb{T}^1 \times \{0\} \times \{1/2\} \cup \{0\} \times \{1/2\} \times \mathbb{T}^1 \cup \{1/2\} \times \mathbb{T}^1 \times \{0\}. \tag{2.6.6}$$

This detaching argument does not work in dimension $n = 2$, because in $\mathbb{T}^2$ two closed paths that belong to nonparallel homotopy classes will eventually intersect with each other and form a reticulate set, see Lemma 2.6.15 and Lemma 2.6.16. Intuitively this is related to the simple fact that on $\mathbb{R}^2$ any two nonparallel straight lines will always intersect with each other, while in higher dimensions $n \geq 3$, two nonparallel lines generically do not intersect with each other, and if they do they must be on the same plane. In this higher-dimensional scenario, one can construct a loopy set like (2.6.6), in which there are three linearly independent closed paths that do not intersect with each other, i.e. the union is not connected and therefore not reticulate.

Remark 2.6.4. It is natural to ask whether E being loopy can be equivalently defined as H_E spanning $\mathbb{Z}^n$ instead of $\mathbb{R}^n$. It turns out that in dimension $n = 2$, the two ways of defining loopy are equivalent. This can be seen by Lemma 2.6.15 (d). However, in higher dimensions $n \geq 3$, H_E spanning $\mathbb{Z}^n$ is a stronger property than loopiness. Here let us present an explicit example of a loopy set $F \subset \mathbb{T}^3$, of which the homotopy class collection H_F does not span $\mathbb{Z}^3$: we parametrize $\mathbb{T}^3$ by

$(x, y, z) \in [0,1)^3$, and define

$$\begin{aligned} F &:= \{2y = x \text{ mod } \mathbb{Z} \text{ and } z = 0\} \cup \{x = 0,\ z = 1/2\} \cup \{x = y = 1/2\} \\ &=: F_1 \cup F_2 \cup F_3. \end{aligned} \tag{2.6.7}$$

Note that $H_{F_1} = \text{Span}_{\mathbb{Z}}\{(2,1,0)\}$, $H_{F_2} = \text{Span}_{\mathbb{Z}}\{(0,1,0)\}$ and $H_{F_3} = \text{Span}_{\mathbb{Z}}\{(0,0,1)\}$ (these sets can be easily derived by using the path lifting property and the simple structure of F_i's). Because F_1, F_2 and F_3 are mutually disjoint, we have $H_F = H_{F_1} \cup H_{F_2} \cup H_{F_3}$ and

$$\mathbb{Z}H_F = (2\mathbb{Z}) \times \mathbb{Z}^2 < \mathbb{Z}^3, \text{ while } \mathbb{R}H_F = \mathbb{R}^3.$$

The intuition here is very similar to what we have just discussed in Remark 2.6.3. In dimension $n = 2$, even if we do not initially observe the existence of $(1,0)$ and $(0,1)$ in H_E, the closed paths having nonparallel homotopy classes in E will have to intersect with each other, resulting in the occurrence of $(1,0)$ and $(0,1)$ in H_E that span the whole $\mathbb{Z}^2$. However, in higher dimensions, the closed paths can be detached and contained in different components (such as the components F_1, F_2 and F_3 in F as in (2.6.7)).

Our main theorem of this section is to show the direct relation between the kernel of $Q(\theta)$ and the collection of homotopy classes $H_\Gamma \subset \mathbb{Z}^n$ in the support $\Gamma = \text{Spt}\ \theta$.

Theorem 2.6.5. *Let θ be a network-like medium as defined in* (2.6.3) *with the weight a satisfying the coercivity condition* (2.6.4)*, then we have the following identity*

$$H_\Gamma^\perp = \ker Q(\theta). \tag{2.6.8}$$

In particular, $Q(\theta)$ is positive definite if and only if Γ is loopy.

In dimension $n = 2$, we can make some further improvements.

Theorem 2.6.6. *Suppose $n = 2$ and assume the same as Theorem 2.6.5. We characterize the topological properties of Γ for different forms of $Q(\theta)$:*

- *$Q(\theta) = 0$ if and only if Γ is trivial.*

- $Q(\theta) = q \otimes q$ *for some* $q \in \mathbb{R}^2 \setminus \{0\}$ *if and only if* $q = rz$ *for some* $r \in \mathbb{R}$, $z \in \mathbb{Z}^2$ *and* Γ *is quasi-laminate in direction* q.

- $Q(\theta)$ *is positive definite if and only if* Γ *is reticulate.*

Theorem 2.6.6 is a corollary of Theorem 2.6.5 combined with the fact that loopiness is equivalent to reticulation in dimension $n = 2$ in Lemma 2.6.15.

The proof of Theorem 2.6.5 consists of three steps. First, in Section 2.6.1 we use Theorem 2.5.3 to decompose θ as a countable sum of its restrictions to its connected components. This step reduces the problem to the case when $\Gamma = \text{Spt}\,\theta$ is a 1-rectifiable non-singleton compact connected set with finite $\mathcal{H}^1$-measure.

Second, in Section 2.6.2 we prove the inclusion $H_\Gamma^\perp \subset \ker Q(\theta)$. The key is to construct in a neighborhood of Γ the linear function $p \cdot x$ for $p \in H_\Gamma^\perp$. This is accomplished by analyzing the unique lift in the covering space $\mathbb{R}^n/\mathbb{Z}H_\Gamma$ of a surjective Ważewski parametrization γ of Γ.

In the last step, we first show a change of variable result in Section 2.6.3. This result helps us to estimate $Q(\theta)$ and then we can finish the proof of Theorem 2.6.5 by showing the following lower bound

$$p \cdot Q(\theta)p \geq \frac{|p|^2}{\Lambda c_{n,\Gamma}},$$

for all $p \in H_\Gamma$, where $\Lambda > 0$ is the coercivity constant in (2.6.4) and $c_{n,\Gamma} > 0$ depends only on n and the geometry of Γ. As we have pointed out in the introduction, this is accomplished by a modification procedure that improves the estimation of lengths and multiplicities of Lipschitz closed paths in Γ without changing the homotopy classes. See Section 2.6.4 for more details.

2.6.1 Decomposition of medium

Let us show that, to prove Theorem 2.6.5, the assumption (2.6.3) on θ can be reduced to media of the form

$$d\theta(x) = a(x) I_{n\times n}\, d\mathcal{H}^1\big|_\Gamma(x), \tag{2.6.9}$$

where a is the positive function as before in (2.6.4), but Γ is a 1-rectifiable closed connected set in $\mathbb{T}^n$ such that $\mathcal{H}^1(\Gamma)$ is positive and finite.

To be more precise, we prove the following lemma.

Lemma 2.6.7. *Suppose Theorem 2.6.5 is correct for $\theta = \theta_w$ of the form* (2.6.9) *with a satisfying* (2.6.4) *and the support $\Gamma \coloneqq \text{Spt}\,\theta = \text{Spt}\,w$ being a closed connected 1-rectifiable set with $0 < \mathcal{H}^1(\Gamma) < \infty$. Then Theorem 2.6.5 is correct.*

Proof. Let us start with a nontrivial medium θ. By Theorem 2.5.1 and Theorem 2.5.3, we know that there are $\theta_j \le \theta$ such that

$$Q(\theta) = \sum_{j=1}^{\infty} Q(\theta_j), \tag{2.6.10}$$

where θ_j are the restrictions of θ on $\Gamma_j \coloneqq \text{Spt}\,\theta_j \subset \Gamma$, which are mutually disjoint closed connected sets that satisfy $0 < \mathcal{H}^1(\Gamma_j) < \infty$. By Theorem 2.2.9, each Γ_j is also 1-rectifiable.

Suppose $q \in H_\Gamma^\perp$, then because $\Gamma_j \subset \Gamma$, we know that for all j, the vector $q \in H_{\Gamma_j}^\perp$. By the assumption, we know that $q \in \ker Q(\theta_j)$ for all j. This implies that by (2.6.10)

$$Q(\theta)q = \sum_{j=1}^{\infty} Q(\theta_j)q = 0.$$

That is, $q \in \ker Q(\theta)$.

To prove the reverse inclusion, we first assume $q \in \ker Q(\theta)$, and then it suffices to show that for any connected component $E \subset \text{Spt}\,\theta$, we have $q \in H_E^\perp$. Indeed, because $\mathcal{H}^1(\Gamma) < \infty$, by [4, Proposition 3.4], each connected component $E \subset \Gamma$ is also path-connected and hence for every $x \in E$

$$\pi_1(\Gamma, x) = \pi_1(E, x).$$

Therefore, by definition (2.6.5) we have

$$H_\Gamma = \bigcup_{x\in\Gamma} i_x(\pi_1(\Gamma, x)) = \bigcup_{E}\bigcup_{x\in E} i_x(\pi_1(\Gamma, x)) = \bigcup_{E}\bigcup_{x\in E} i_x(\pi_1(E, x)) = \bigcup_{E} H_E, \tag{2.6.11}$$

where the union is taken with respect to all the connected components $E \subset \Gamma$. This implies that

$$q \in \bigcap_E H_E^\perp = H_\Gamma^\perp.$$

To prove the claim we may without loss assume that E is a nonsingleton set, as otherwise $H_E^\perp$ is trivially $\mathbb{R}^n$. Because θ takes the form (2.6.3), it is not difficult to derive that

$$d\theta\big|_E = a(x) I_{n\times n} d\mathcal{H}^1\big|_E.$$

By the discussions in the prior paragraph, E is both compact and path-connected, and hence by Lemma 2.5.9, we also obtain that for any $x \in E$ the following inequality

$$\liminf_{r\to 0^+} \frac{\theta(E\cap B_r(x))}{r} \geq \liminf_{r\to 0^+} \frac{\mathcal{H}^1(E\cap B_r(x))}{\Lambda r} I_{n\times n} \geq \frac{I_{n\times n}}{\Lambda} > 0$$

for $\Lambda > 0$ the coercivity constant. This shows that

$$\text{Spt } \theta\big|_E = E.$$

Therefore the support of $\theta\big|_E$ is a continuum. Also because $0 < \mathcal{H}^1(E) < \infty$, by Theorem 2.2.9, the medium $\theta\big|_E$ satisfies the assumption in Lemma 2.6.7 with Γ replaced by E, and therefore $q \in H_E^{\perp}$.

Let us finish the proof by discussing the case that θ is trivial. By Theorem 2.5.1, we do not worry about the case that the support $\Gamma = \text{Spt } \theta$ is also totally disconnected. In this case we automatically have the identity (2.6.8). Suppose otherwise the support Γ has a nonsingleton connected component E. By Lemma 2.3.8 (1) we have $Q(\theta) = Q(\theta\big|_E) = 0$. As $0 < \mathcal{H}^1(E) \leq \mathcal{H}^1(\Gamma) < \infty$, we have by Theorem 2.2.9 the set E is also 1-rectifiable. Therefore by the assumption, we have $H_E^{\perp} = \ker Q(E) = \mathbb{R}^n$. The proof is then finished by applying the identity (2.6.11). □

2.6.2 One side inclusion

In this subsection we show one side inclusion $H_\Gamma^{\perp} \subset \ker Q(\theta)$. This is accomplished by applying the Ważewski parametrization theorem in Theorem 2.2.9.

Lemma 2.6.8. *Assume the same as Theorem 2.6.5, then we have*

$$H_\Gamma^{\perp} \subset \ker Q(\theta).$$

The proof of this inclusion does not depend on the coercivity constant Λ as in (2.6.4).

Proof. By Lemma 2.6.7, it suffices to prove for medium θ of the form (2.6.9) and $\Gamma = \text{Spt } \theta$ a closed 1-rectifiable connected set in $\mathbb{T}^n$ such that $0 < \mathcal{H}^1(\Gamma) < \infty$. As Γ satisfies the conditions in Theorem 2.2.9, there is a surjective constant speed closed Lipschitz path $\gamma : [0,1] \to \Gamma$.

Let $p \in H_\Gamma^\perp$ be a nonzero vector, then by definition (2.3.3)

$$p \cdot Q(\theta)p := \inf_{\varphi \in C^\infty(\mathbb{T}^n)} \int_{\mathbb{T}^n} |\nabla\varphi(x) + p|^2 a(x) d\mathcal{H}^1\big|_\Gamma(x)$$

To show that $Q(\theta)p = 0$ it suffices to show that in an open neighborhood U of Γ here is a smooth function $\varphi_p \in C_0^\infty(U)$ such that $\nabla\varphi_p = -p$ on Γ.

To that end, we first write $H = \mathbb{Z}H_\Gamma$ and recall the following covering maps (see Section 2.2.4.3)

$$\pi = \pi_H \circ \pi^H,\ \pi^H : \mathbb{R}^n \to \mathbb{T}_H^n = \mathbb{R}^n/H \text{ and } \pi_H : \mathbb{T}_H^n \to \mathbb{T}^n.$$

As $p \perp H$, we know that for any $x, y \in \mathbb{R}^n$ such that $x - y \in H$, $p \cdot x = p \cdot y$. This shows that there is a unique function h_p on $\mathbb{T}_H^n$ such that

$$p \cdot x = h_p(\pi^H(x)) \text{ for } x \in \mathbb{R}^n. \tag{2.6.12}$$

We claim that the function $h_p \in C^\infty(\mathbb{T}_H^n)$. First notice that because $H \leq \mathbb{Z}^n$ is a discrete subgroup of $\mathbb{R}^n$, the projection π^H is a locally isometric diffeomorphism onto $\mathbb{T}_H^n$, and for sufficiently small radius $r > 0$ the preimage $(\pi^H)^{-1}(B_r)$ of each geodesic ball $B_r \subset \mathbb{T}_H^n$ is a countable union of disjoint balls in $\mathbb{R}^n$ having the same radius r. Restricting π^H on a fixed component in the preimage $(\pi^H)^{-1}(B_r)$ defines an isometric diffeomorphism from the component to B_r, which by (2.6.12) shows the differentiability of h_p in B_r. As the position of B_r can be chosen arbitrarily, we have proved that $h_p \in C^\infty(\mathbb{T}_H^n)$.

Notice that it is impossible to find a function h on $\mathbb{T}^n$ such that $p \cdot x = h(\pi(x))$ for all $x \in \mathbb{R}^n$. Otherwise one would obtain

$$0 \neq p \cdot p = h(\pi(p)) = h(\pi(0)) = p \cdot 0 = 0,$$

which is impossible. This is one of the main reasons why we invoke the intermediate space $\mathbb{T}_H^n = \mathbb{R}^n/H$.

Let $\hat{\gamma} : [0,1] \to \mathbb{T}_H^n$ denotes the unique path lifting of γ (starting at some fixed point lift) and write $\hat{\Gamma}$ to be the image of $\hat{\gamma}$. We claim that in a small neighborhood $\hat{U}$ of $\hat{\Gamma}$ the covering map $\pi_H : \mathbb{T}_H^n \to \mathbb{T}^n$ is an isometric diffeomorphism onto a neighborhood U of Γ. This claim finishes the proof because we can define

$$\varphi_p := -\eta \cdot \left[h_p \circ \left(\pi_H\big|_{\hat{U}}\right)^{-1}\right] \in C^\infty(\mathbb{T}^n),$$

where $\eta \in C^{\infty}(\mathbb{T}^n)$ satisfies $0 \le \eta \le 1$, $\eta \equiv 1$ near Γ and $\eta \equiv 0$ outside a compact subset of U.

To prove the claim, we observe that π_H is a locally diffeomorphic isometry, and therefore it suffices to show $\pi_H|_{\hat{U}}$ is injective. Let us first show that $\pi_H|_{\hat{\Gamma}}$ is injective. Suppose there are $0 \le t_1 < t_2 \le 1$ such that

$$\hat{\gamma}(t_1) \ne \hat{\gamma}(t_2) \text{ but } \pi_H\left(\hat{\gamma}(t_1)\right) = \pi_H\left(\hat{\gamma}(t_2)\right).$$

Let q_1 and q_2 be vectors in $\mathbb{R}^n$ such that we can write $\hat{\gamma}(t_1)$ and $\hat{\gamma}(t_2)$ respectively as

$$\hat{\gamma}(t_1) = q_1 + H \text{ and } \hat{\gamma}(t_2) = q_2 + H.$$

Let $z = q_2 - q_1 \in \mathbb{R}^n$, then by the assumption that $\hat{\gamma}(t_1) \ne \hat{\gamma}(t_2)$, we know that $z \ne 0$ and $z + H \cap H = \varnothing$. Because $\pi_H\left(\hat{\gamma}(t_1)\right) = \pi_H\left(\hat{\gamma}(t_2)\right)$, we obtain that the vector $z \in \mathbb{Z}^n$ and $\pi_H \circ \hat{\gamma} : [t_1, t_2] \to \Gamma$ is a closed path that belongs to the homotopy class

$$z + h \in \mathbb{Z}^n \setminus H$$

for some $h \in H$. This contradicts the assumption that $H = \mathbb{Z}H_\Gamma \supset H_\Gamma$ includes all the homotopy classes in Γ.

To prove that $\pi_H|_{\hat{U}}$ is injective for some small neighborhood $\hat{U}$, we argue by contradiction. We assume that there is a constant $c > 0$ and for all small $\delta > 0$ there are x_δ, y_δ in $\mathbb{T}^n_H$ having distance to $\hat{\Gamma}$ bounded by δ such that

$$\operatorname{dist}(x_\delta, y_\delta) \ge c > 0$$

independent of $\delta > 0$ but $\pi_H(x_\delta) = \pi_H(y_\delta)$. Notice that $\hat{\Gamma}$ is a compact subset in $\mathbb{T}^n_H$ because π_H is a local isometry and hence $\hat{\gamma}'(t) = \gamma'(t)$ is bounded and therefore $\hat{\Gamma}$ is bounded and closed in $\mathbb{T}^n_H$. The proof is done by a compactness argument as we send $\delta \to 0$, which will lead to a contradiction to our previous claim that $\pi_H|_{\hat{\Gamma}}$ is injective.

□

2.6.3 Change of variables

Another technical lemma is the change of variables using Lipschitz paths. Let us first recall a standard result for Lipschitz paths. We also refer to Section 2.2.3.2

for more discussions on the preliminary concepts that are required in this subsection.

Lemma 2.6.9 ([4, Remark 3.6]). *Suppose* Γ *is the image of a Lipschitz path* $\gamma : [0,1] \to \Gamma \subset \mathbb{T}^n$, *then for any nonnegative Borel function* f *(or integrable ones) on* Γ *we have*

$$\int_0^1 f(\gamma(t))|\gamma'|dt = \int_\Gamma f(x)m(\gamma,x)d\mathcal{H}^1,$$

where $m(\gamma,x) := \#\gamma^{-1}(x)$ *is the multiplicity of* γ *at* x.

We have the following technical results.

Lemma 2.6.10. *Suppose* $\gamma : [0,1] \to \Gamma$ *is a constant speed Lipschitz path with*

$$m_\gamma := \sup_{x\in\Gamma} \#\gamma^{-1}(x) < \infty \text{ and path length } \ell(\gamma) = |\gamma'|,$$

then we have

$$p \cdot Q(\theta)p \geq \frac{1}{m_\gamma \ell(\gamma)} \inf_{\varphi\in C^\infty(\mathbb{T}^n)} \int_0^1 \left|\frac{d}{dt}\left[\varphi(\gamma(t))\right] + p\cdot\gamma'(t)\right|^2 a(\gamma(t))dt. \tag{2.6.13}$$

Proof. Let $K := \gamma([0,1])$ be the image. Note that for every $\varphi \in C^\infty(\mathbb{T}^n)$ we have by Lemma 2.6.9

$$\begin{aligned}\int_\Gamma |\nabla\varphi + p|^2 a\, d\mathcal{H}^1 &\geq \int_K |\nabla\varphi + p|^2 a\, d\mathcal{H}^1 \\ &\geq \frac{\ell(\gamma)}{m_\gamma} \int_0^1 |P_{\tau_{\gamma(t)}}(\nabla\varphi(\gamma(t)) + p)|^2 a(\gamma(t))dt \\ &= \frac{1}{\ell(\gamma)m_\gamma} \int_0^1 \left|\frac{d}{dt}\left[\varphi(\gamma(t))\right] + p\cdot\gamma'(t)\right|^2 a(\gamma(t))dt,\end{aligned}$$

where $\tau_{\gamma(t)}$ is the 1-D tangent space of γ at t. The proof is done by taking the infimum over φ. □

2.6.4 Proof of Theorem 2.6.5

We require the following technical lemmas for proving the reverse inclusion $H_\Gamma^\perp \supset \ker Q(\theta)$. We refer to Section 2.2.4 for the preliminaries of some concepts in this subsection.

Lemma 2.6.11. *Let* $E \subset \mathbb{T}^n$ *be a subset such that* $\mathcal{H}^1(E) < \infty$. *For any nontrivial closed path* $\gamma^* : [0,1] \to E$, *there is another constant speed Lipschitz closed path*

$\gamma : [0,1] \to E$ such that γ belongs to the same homotopy class as γ^*, the path length $\ell(\gamma) < \infty$ and $\ell(\gamma) \le \ell(\gamma^*)$, and the lift $\tilde{\gamma}$ of γ on $\mathbb{R}^n$ starting at any $\tilde{y} \in \pi^{-1}(\gamma(0))$ is injective.

Proof. Denote $x_0 = \gamma(0) = \gamma^*(0)$ and take some $\tilde{y} \in \pi^{-1}(x_0)$. Let $\tilde{\gamma}^*$ be the unique lift of γ^* on $\mathbb{R}^n$ starting at $\tilde{\gamma}^*(0) = \tilde{y}$, and denote the homotopy class as $q = \tilde{\gamma}^*(1) - \tilde{y} \in \mathbb{Z}^n \setminus \{0\}$. Because $\tilde{\gamma}^*$ is continuous, its image is compact and connected in $\mathbb{R}^n$. On the other hand, the periodic extension (see Section 2.2.4.5) $\mathcal{H}^1|_{\pi^{-1}(E)}$ of $\mathcal{H}^1|_E$ is locally finite on $\mathbb{R}^n$. This implies that

$$\mathcal{H}^1(\tilde{\gamma}^*([0,1])) = \mathcal{H}^1|_{\pi^{-1}(E)}(\tilde{\gamma}^*([0,1])) < \infty.$$

Now we know that $\tilde{\gamma}^*([0,1])$ is a connected compact set in $\mathbb{R}^n$ that has finite $\mathcal{H}^1$ measure. By applying [4, Proposition 3.4], there is an injective Lipschitz path $\tilde{\gamma} : [0,1] \to \tilde{\gamma}^*([0,1])$ such that $\tilde{\gamma}(0) = \tilde{y}$ and $\tilde{\gamma}(1) = q + \tilde{y}$. The proof is done by setting $\gamma = \pi \circ \tilde{\gamma}$. Notice that by [4, Proposition 3.4] and Lemma 2.6.9

$$\ell(\gamma) \le \mathcal{H}^1(\tilde{\gamma}^*([0,1])) \le \ell(\tilde{\gamma}^*) = \ell(\gamma^*).$$

On the other hand we also have $\ell(\gamma) \le \mathcal{H}^1(\tilde{\gamma}^*([0,1])) < \infty$. □

Lemma 2.6.12. *Let $E \subset \mathbb{T}^n$ be a connected closed subset such that $\mathcal{H}^1(E) < \infty$. Then for any point $x_0 \in E$ we have*

$$H_E = \mathbb{Z}H_E = i_{x_0}(\pi_1(E, x_0)),$$

where $i_{x_0} : \pi_1(\mathbb{T}^n, x_0) \to \mathbb{Z}^n$ is the isomorphism defined in Section 2.2.4.4.

Proof. Let $q = i_{y_0}([\xi]_{y_0})$ be a vector in H_E, where $y_0 \in E$ and $[\xi]_{y_0}$ is a homotopy class of a closed path ξ in E based at y_0, and i_{y_0} is the isomorphism as defined in Section 2.2.4.4. It then suffices to show that $q \in i_{x_0}(\pi_1(E, x_0))$. The case $x_0 = y_0$ is trivial. In the case $x_0 \ne y_0$, as E is connected compact and has finite $\mathcal{H}^1$ measure, by [4, Proposition 3.4] there is an injective constant speed Lipschitz path $\eta : [0,1] \to E$ such that $\eta(0) = x_0$ and $\eta(1) = y_0$. Consider the path composition

$$\hat{\xi} := \eta^{-1}\xi\eta,$$

which defines a closed path in E based at x_0. Note that because η is injective, it is not difficult to compute the homotopy class $i_{x_0}([\hat{\xi}]_{x_0}) = i_{y_0}([\xi]_{y_0}) = q$. This shows that $q \in i_{x_0}(\pi_1(E, x_0))$. □

Lemma 2.6.13. *Let $E \subset \mathbb{T}^n$ be a connected closed subset such that $\mathcal{H}^1(E) < \infty$. Then there is a constant $c = c_{n,E} > 0$ such that for each $q \in H_E \setminus \{0\}$ we can find a Lipschitz closed path γ_q in Γ that belongs to q, having injective lift in $\mathbb{R}^n$ and*

$$1 \leq |q| \leq \ell(\gamma_q) \leq c|q|.$$

Notice that the key difficulty of this lemma is that the length $\ell(\gamma)$ of a closed path γ in E may not be bounded by $\mathcal{H}^1(E)$ because of multiplicity.

Proof. By Lemma 2.6.12, we can view H_E as the isomorphic image of $\pi_1(E, x_0)$ in $\mathbb{Z}^n$ for some $x_0 \in E$. From now on in this proof we do not distinguish $\pi_1(E, x_0)$ and H_E. Let $u_1, \dots, u_k$ be a minimal collection of the generators of the group $H_E \leq \mathbb{Z}^n$. By Lemma 2.6.11 the following quantity is well-defined

$$\lambda_E := \max_{1 \leq j \leq k} \inf \{\ell(\gamma_j) \ ; \ \gamma_j \text{ is a closed path belonging to } u_j\} < \infty.$$

For each $1 \leq j \leq k$ we denote γ_j a closed path in E based at x_0 that belongs to u_j and

$$\ell(\gamma_j) \leq 2 \inf \left\{\ell(\gamma_j^*) \ ; \ \gamma_j^* \text{ is a closed path belonging to } u_j\right\}.$$

Now for each $q \in H_E$, there are $c_i \in \mathbb{Z}$ such that

$$q = \sum_{i=1}^{k} c_i u_i.$$

Correspondingly we define

$$\gamma_q^* := \prod_{i=1}^{k} \gamma_i^{c_i}$$

as the path composition based at x_0. Notice that γ_q^* is a closed path in Γ that belongs to q. By Lemma 2.6.11 we can find a Lipschitz path $\gamma_q : [0,1] \to \gamma_q^*([0,1])$ that belongs to q and has a unique injective lift $\tilde{\gamma}_q$ on $\mathbb{R}^n$ starting at some $\tilde{\gamma}_q(0) = \tilde{y} \in \pi^{-1}(\gamma_q(0))$. Because the lift $\tilde{\gamma}_q$ connects $\tilde{y}$ and $\tilde{y} + q$, and $\gamma_q' = \tilde{\gamma}_q'$, we have

$$\ell(\gamma_q) = \ell(\tilde{\gamma}_q) \geq |q|.$$

On the other hand, by the construction of γ_q

$$\ell(\gamma_q) \leq \ell(\gamma_q^*) = \sum_{i=1}^{k} |c_i| \ell(\gamma_i) \leq 2\lambda_E \sum_{i=1}^{k} |c_i|. \tag{2.6.14}$$

It then suffices to show that $\sum_{i=1}^{k}|c_i|$ is uniformly bounded by $|q|$. Indeed, the set of vectors $u_i \in \mathbb{Z}^n$ is $\mathbb{Z}$-linearly independent and therefore they are also $\mathbb{R}$-linearly independent, see Lemma 2.7.4. This implies that $\sum_{i=1}^{k}|c_i|$ defines another norm on $\mathbb{R}H_E$. As we live in a finite dimensional space there is always a constant $c = c_{n,E} > 0$ such that

$$c^{-1}|q| \leq \sum_{i=1}^{k}|c_i| \leq c|q|.$$

By combining (2.6.14) we finish the proof.

□

Proof of Theorem 2.6.5. By Lemma 2.6.7 and Lemma 2.6.8, we know that it suffices to prove the following inclusion

$$\ker Q(\theta) \subset H_\Gamma^\perp,$$

where θ takes the form (2.6.9) and $\Gamma = \operatorname{Spt}\theta$ is a closed connected subset of $\mathbb{T}^n$ such that $0 < \mathcal{H}^1(\Gamma) < \infty$.

To prove this inclusion we claim that there is a constant $c > 0$ such that

$$\inf_{q \in H_\Gamma \setminus \{0\}} \frac{q \cdot Q(\theta)q}{|q|^2} \geq c > 0.$$

This claim proves the inclusion $\ker Q(\theta) \subset H_\Gamma^\perp$ because if $p \in \ker Q(\theta)$ then we can write $p = p_1 + p_2$ with $p_1 \in \mathbb{R}H_\Gamma$ and $p_2 \perp H_\Gamma$. By Lemma 2.6.8, we know that

$$0 = Q(\theta)p = Q(\theta)p_1.$$

This implies that $p_1 \in \ker Q(\theta)$. If $p_1 \neq 0$, then by Lemma 2.6.12, $H_\Gamma = \mathbb{Z}H_\Gamma$ is the $\mathbb{Z}$-span of a finite collection of vectors, and therefore the set $\{q/|q|\ ;\ q \in H_\Gamma \setminus \{0\}\}$ is dense in $\partial B_1(0) \cap \mathbb{R}H_\Gamma$ by Lemma 2.7.3, which implies that

$$\frac{p_1 \cdot Q(\theta)p_1}{|p_1|^2} \geq \inf_{q \in H_\Gamma \setminus \{0\}} \frac{q \cdot Q(\theta)q}{|q|^2} \geq c > 0,$$

which is a contradiction to the fact that $p_1 \in \ker Q(\theta)$.

To prove the claim we observe that for $q \in H_\Gamma \setminus \{0\}$, by Lemma 2.6.10 and Lemma 2.6.13 we can find a constant speed closed Lipschitz path γ_q in Γ that belongs to

q and

$$\begin{aligned} q\cdot Q(\theta)q &\geq \inf_{\varphi\in C^\infty(\mathbb{T}^n)} \frac{1}{m_{\gamma_q}\ell(\gamma_q)}\int_0^1 \left|\frac{d}{dt}\left[\varphi(\gamma_q(t))\right] + q\cdot\gamma_q'(t)\right|^2 a(\gamma_q(t))dt \\ &\geq \inf_{\varphi\in C^\infty(\mathbb{T}^n)} \frac{1}{\Lambda m_{\gamma_q}\ell(\gamma_q)}\left(\int_0^1 \frac{d}{dt}\left[\varphi(\gamma_q(t))\right] + q\cdot\gamma_q'(t)dt\right)^2 \\ &= \inf_{\varphi\in C^\infty(\mathbb{T}^n)} \frac{1}{\Lambda m_{\gamma_q}\ell(\gamma_q)}\left(q\cdot\int_0^1 \gamma_q'(t)dt\right)^2 \\ &= \frac{|q|^4}{\Lambda m_{\gamma_q}\ell(\gamma_q)}, \end{aligned}$$

where m_{γ_q} is the maximal multiplicity of γ_q. Notice that the lift $\tilde{\gamma}_q$ of γ_q starting at some $\tilde{\gamma}_q(0) = \tilde{y} \in \pi^{-1}(\gamma_q(0))$ is an injective map, and $\gamma_q = \pi\circ\tilde{\gamma}_q$. This shows that

$$m_{\gamma_q} \leq \max_{x\in[0,1)^n} \#\left\{\tilde{\gamma}_q([0,1])\cap(\mathbb{Z}^n + x)\right\} \leq \lceil\ell(\tilde{\gamma}_q)\rceil = \lceil\ell(\gamma_q)\rceil \leq \ell(\gamma_q) + 1 \leq 2\ell(\gamma_q). \quad (2.6.15)$$

Combining Lemma 2.6.13, we can finish the proof by observing

$$q\cdot Q(\theta)q \geq \frac{|q|^4}{\Lambda m_{\gamma_p}\ell(\gamma_q)} \geq \frac{|q|^2}{\Lambda c_{n,\Gamma}} \quad (2.6.16)$$

for some $c_{n,\Gamma} > 0$ depending only on the dimension n and $\Gamma = \operatorname{Spt}\theta$.

□

Remark 2.6.14. One might wonder why there is a fourth power in $|q|$ in the intermediate step of (2.6.16). The main reason is that when we estimate the integral in Lemma 2.6.10 we allow the closed paths γ_q to have multiplicities, especially when $q = \lambda\tilde{q}$ for some $\lambda\in\mathbb{Z}_+$ and $\tilde{q}\in H_\Gamma$ and $\gamma_q = \gamma_{\tilde{q}}^\lambda$ is the path composition of $\gamma_{\tilde{q}}$ with itself for λ times. In this case, $\ell(\gamma_q)$ and m_{γ_q} are not constants, but linear functions of $|q|$. Generally, the efforts in Lemma 2.6.13 and (2.6.15) are just made to bound the effects of the length and multiplicity of γ_q when we use the formula in Lemma 2.6.10. It turns out that when Γ is connected, closed and $\mathcal{H}^1(\Gamma) < \infty$, one can find closed paths γ_q in Γ so that the growth of both lengths $\ell(\gamma_q)$ and the multiplicities m_{γ_q} are indeed linear in $|q|$, which then leads to the lower bound (2.6.16).

2.6.5 Loopiness and reticulation in dimension $n = 2$

In this subsection we prove Theorem 2.6.6. Let us begin with the following characterization of loopy sets in dimension $n = 2$.

Lemma 2.6.15. *Let $E \subset \mathbb{T}^2$ be a closed subset such that $\mathcal{H}^1(E) < \infty$ and denote $\tilde{E} = \pi^{-1}(E)$ be the periodic extension. Denote $e_1 = (1,0)$ and $e_2 = (0,1)$ as the standard orthonormal basis for $\mathbb{R}^2$. The following statements are equivalent:*

(a) *E is loopy.*

(b) *E is reticulate.*

(c) *There is a reticulate compact connected component $E^* \subset E$ such that $E \setminus E^*$ is trivial.*

(d) *There is a point $x \in \tilde{E}$ such that for each $i = 1, 2$ there is a Lipschitz path $\gamma_i : [0,1] \to \tilde{E}$ with $\gamma_i(0) = x$ and $\gamma_i(1) = x + e_i$.*

To prove this lemma we require the following technical lemma.

Lemma 2.6.16. *Let γ_1 and γ_2 be two closed paths in $\mathbb{T}^2$ and their homotopy classes are not parallel to each other, then the images of γ_1 and γ_2 must intersect at some point.*

By the same proof of this lemma we can show the following corollary.

Corollary 2.6.17. *Suppose $L_i(t) := y_i + tq_i \in \mathbb{R}^2$ are two straight lines with q_1, q_2 linearly independent, then for any continuous functions $\Gamma_i : \mathbb{R} \to \mathbb{R}^2$ such that*

$$\sup_{t \in \mathbb{R},\, i=1,2} \operatorname{dist}(\Gamma_i(t), L_i(t)) < \infty$$

we have the images of Γ_1 and Γ_2 must have nontrivial intersection.

Proof of Lemma 2.6.16. We denote the nonzero vectors (n_1, m_1) and (n_2, m_2) in $\mathbb{Z}^2$ to be the homotopy classes of γ_1 and γ_2 respectively. By the nonparallel assumption we know that $|n_1 m_2 - n_2 m_1| > 0$. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be the lift of γ_1 and γ_2 respectively on $\mathbb{R}^2$ such that the starting points are $y_i \in \pi^{-1}(\gamma_i(0))$ and

$$\widetilde{\gamma}_i(1) = y_i + (n_i, m_i),$$

for all $i = 1, 2$.

Now we construct $\Gamma_i : \mathbb{R} \to \mathbb{R}^2$ by defining

$$\Gamma_i(t) = \widetilde{\gamma}_i(t - \lfloor t \rfloor) + \lfloor t \rfloor (n_i, m_i)$$

for $i = 1, 2$ and all $t \in \mathbb{R}$. Notice that Γ_i's are continuous on $\mathbb{R}$ because of the definition of $\widetilde{\gamma}_i$'s.

Notice that γ_1, γ_2 have intersection points if and only if the images of Γ_i's have intersection points. It then suffices to consider the intersection of Γ_1 and Γ_2. We know that because Γ_i's are continuous and $\Gamma_i(t+l) - \Gamma_i(t) = l(n_i, m_i)$ for all $t \in \mathbb{R}$, $l \in \mathbb{Z}$ and $i = 1, 2$, there is a constant $C > 0$ such that

$$\operatorname{dist}(\Gamma_i(t), (y_i + t(n_i, m_i))) \leq C$$

for all $i = 1, 2$ and $t \in \mathbb{R}$. Let $L_i(t) = y_i + t(n_i, m_i)$ be the linearized paths. Define the matrix $A = \begin{pmatrix} n_1 & -n_2 \\ m_1 & -m_2 \end{pmatrix}$ with $\det A = n_2 m_1 - n_1 m_2 \neq 0$. Consider the continuous map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$\Phi(t, s) = A^{-1} \left(y_2 - y_1 + (\Gamma_2(s) - L_2(s)) - (\Gamma_1(t) - L_1(t)) \right).$$

The boundedness $\operatorname{dist}(\Gamma_i(t), L_i(t)) \leq C$ implies Φ maps some closed ball $B_R \subset \mathbb{R}^2$ to itself. By Brouwer's fixed-point theorem, there exists $(t_0, s_0) \in B_R$ with $\Phi(t_0, s_0) = (t_0, s_0)$, yielding:

$$\Gamma_1(t_0) = \Gamma_2(s_0).$$

Projecting via π gives $\gamma_1(t_0) = \gamma_2(s_0)$ in $\mathbb{T}^2$, completing the proof.

□

Proof of Lemma 2.6.15. By Lemma 2.6.16 we know that (d) $\implies$ (c) $\implies$ (b) $\implies$ (a). By the same reason and also Lemma 2.6.12 we deduce that (a) $\implies$ (c), and therefore (a), (b) and (c) are equivalent. It then suffice to show that (d) can be implied by other three conditions. We begin with (c) and assume there are two closed paths γ_1 and γ_2 based at the same point $x_0 \in E^*$. It suffices to show that E^* contains homotopy classes of the form $(1, 0)$ and $(0, 1)$.

Denote (a, b) and (c, d) as the nonzero homotopy classes of γ_1 and γ_2 respectively, then by the assumption (d) we can assume that $g := ad - bc \neq 0$. Notice that by taking integer combinations

$$(-d) \cdot (a, b) + b \cdot (c, d) = (bc - ad, 0), \quad (-c) \cdot (a, b) + a \cdot (c, d) = (0, ad - bc),$$

we obtain classes of the form $q_1 = (-g, 0)$ and $q_2 = (0, g)$. To show the existence of classes of the form $(1, 0)$ (the class $(0, 1)$ can be deduced symmetrically) we denote

γ_i^* as some closed paths based at the common x_0 that belong to q_i for $i = 1, 2$. Denote $\tilde{\gamma}_i^*$ as the lift of γ_i^* based at some common $y_0 \in \pi^{-1}(x_0)$ and define

$$\Gamma_i(t) = \tilde{\gamma}_i^*(t - \lfloor t \rfloor) + \lfloor t \rfloor q_i.$$

The proof is done by observing that Γ_1 must intersect $\Gamma_2 + e_1$ according to Corollary 2.6.17.

□

Proof of Theorem 2.6.6. The proof is done by combining Theorem 2.6.5 and Lemma 2.6.15. □

2.7 Appendix

2.7.1 A formal derivation of resilience from the expected total dissipation

In this section we derive the notion of resilience introduced in Theorem A, that is, the positivity of the effective tensor $Q(\theta)$, from the minimization of expected total dissipation in the hydraulic system of a leaf under random fluctuations [58, 104].

Let us model a small piece of leaf by a smooth bounded planar domain $\mathscr{D} \subset \mathbb{R}^2$ along with a positive definite matrix field $\sigma = L + Q : \mathscr{D} \to \mathbb{R}^{2\times 2}$ that represents the local conductance of the leaf. Here L represents the conductance of the lower order veins, which are supported near a tree-like network. The field $Q \geq Q_0$ is a constant positive definite matrix, regarded as the effective tensor of $\theta + Q_0\, d\mathcal{L}^2$ where θ is a network-like medium as in Theorem A and Q_0 is a small positive definite matrix that represents the background medium. That is, we are at a scale where the higher order veins are considered well-mixed with the background materials. We are concerned about the positive definiteness of $Q - Q_0 \approx Q(\theta)$ as the leaf grows.

By Darcy's law, the velocity field j satisfies the following relation with the pressure function ϕ:

$$j = -\sigma \nabla \phi. \tag{2.7.1}$$

Suppose the source distribution is m_i and the sink distribution is m_o, where m_i and m_o are nonnegative finite measures on $\partial\mathscr{D}$ and $\mathscr{D}$ respectively. The mass conservation indicates that $m_o(\mathscr{D}) = m_i(\partial\mathscr{D})$. Combining the Darcy's law (2.7.1) with the Gauss law, one obtains the Neumann problem

$$\begin{cases} \nabla \cdot (\sigma \nabla \phi)(x) = m_o(x) & x \in \mathscr{D} \\ \sigma \nabla \phi \cdot \vec{n}(x) = m_i(x) & x \in \partial\mathscr{D}, \end{cases} \tag{2.7.2}$$

where $\vec{n}(x)$ is the unit outer normal of $\partial\mathscr{D}$ at x. Under random fluctuations in m_i and m_o, the total dissipation in this scenario takes the form

$$P(\sigma) = \mathbb{E}\left[\int_{\mathscr{D}} \nabla\phi \cdot \sigma \nabla \phi\right], \tag{2.7.3}$$

where the pressure function ϕ satisfies (2.7.2) and the expectation is taken with respect to m_i and m_o. As the Gauss law explicitly controls the velocity field j, the total dissipation $P(\sigma)$ represents the effective resistance of the construction σ under a random choice of the source-sink distributions m_i and m_o. Therefore, for a leaf to maintain its function, it is natural to minimize the effective resistance, or equivalently the total dissipation $P(\sigma)$.

We are especially interested in the effects of the higher order veins, which leads us to compute the gradient of $P(\sigma) = P(L+Q)$ as a function of Q. Indeed, let B be any symmetric matrix, we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} P(\sigma + tB) &= -\mathbb{E}\left[\int_{\mathscr{D}} \nabla\phi \cdot B \nabla\phi\right] \\ &= -\mathrm{Tr}(BK), \end{aligned} \tag{2.7.4}$$

where $K := \mathbb{E}\left[\int_{\mathscr{D}} \nabla\phi \otimes \nabla\phi\right]$ and ϕ satisfies the equation (2.7.2). This implies that the gradient flow of Q takes the form

$$\frac{d}{dt}Q = K = \mathbb{E}\left[\int_{\mathscr{D}} \nabla\phi \otimes \nabla\phi\right].$$

Note that if m_i and m_o are random, then the tensor K is generically positive definite, which means that in this scenario the effective tensor $Q_t - Q_0$ of the higher order veins regardless of the background has to be positive definite.

2.7.2 Periodic homogenization of media

For a positive semi-definite matrix-valued Radon measure $d\xi = \beta\, d\|\xi\|$ on a bounded open domain $U \subset\subset \mathbb{R}^n$, one can consider the following mesoscopic effective tensor

$$p \cdot Q(\xi; U)p := \inf_{\phi \in C_0^\infty(U)} \fint_U (\nabla\phi + p) \cdot \beta(\nabla\phi + p)\, d\|\xi\| = \inf_{u \in p\cdot x + C_0^\infty(U)} \fint_U \nabla u \cdot \beta \nabla u\, d\|\xi\|,$$

where $⨍_U := \frac{1}{|U|}\int_U$ with $|U|$ the Lebesgue measure of U.

In this section we show the following periodic homogenization result.

Lemma 2.7.1. *Let θ^* be the periodic extension (see Section 2.2.4.5) of a medium θ on $\mathbb{T}^n$, then*

$$\lim_{R\to\infty} Q(\theta^*; (-R,R)^n) = Q(\theta).$$

Such homogenization results are known for the case when θ is absolutely continuous with respect to the Lebesgue measure, see [100, Chapter 1] and an ergodic version in [16, Chapter 1]. A singular version of homogenization can be found in [38]. Here we present a proof for media including general anisotropy. The proof does not use either the BBS tangent space theory as in [38] or any Sobolev estimates on the corrector equation.

Let us start with an auxiliary lemma. First we denote

$$N_R(x) := \#(-R,R)^n \cap \pi^{-1}(x),$$

where $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the standard projection. Note that

$$N_{R,-}(x) := \inf_{x\in\mathbb{T}^n} N_R(x) \text{ and } N_{R,+}(x) := \sup_{x\in\mathbb{T}^n} N_R(x)$$

satisfy

$$\lim_{R\to\infty} \frac{N_{R,\pm}}{(2R)^n} = 1.$$

Lemma 2.7.2. *For every $R > 0$ one has*

$$Q(\theta) \le \liminf_{R\to\infty} Q(\theta^*; (-R,R)^n).$$

Proof. For every $\phi \in C_0^\infty((-R,R)^n))$ we define

$$v_\phi(x) := \frac{1}{(2R)^n} \sum_{y\in\pi^{-1}(x)} \phi(y).$$

Notice that

$$\nabla v_\phi(x) = \frac{1}{(2R)^n} \sum_{y\in\pi^{-1}(x)} \nabla\phi(y).$$

In particular by Cauchy-Schwartz inequality we have

$$\nabla v_\phi(x) \cdot \sigma(x) \nabla v_\phi(x) \le \frac{N_R(x)}{(2R)^{2n}} \sum_{y\in\pi^{-1}(x)} \nabla\phi(y) \cdot \sigma(x) \nabla\phi(y). \tag{2.7.5}$$

By the definition of periodic extension in Section 2.2.4.5, the periodic extension of the medium $d\theta(x) = \sigma(x)\, d\,\|\theta\|\,(x)$ takes the form

$$d\theta^*(y) = \sigma(\pi(y))\, dw(y),$$

where the Radon measure w is the periodic extension of $\|\theta\|$. This implies that

$$\begin{aligned}
\fint_{(-R,R)^n} |\nabla\phi + p|^2_{\sigma(\pi(y))} dw(y) &= \int_{\mathbb{T}^n} \frac{1}{(2R)^n} \sum_{y\in\pi^{-1}(x)} |\nabla\phi(y)|^2_{\sigma(x)} d\,\|\theta\|\,(x) + 2\int_{\mathbb{T}^n} \nabla v_\phi \cdot \sigma p\, d\,\|\theta\| \\
&\quad + \fint_{(-R,R)^n} p\cdot\sigma(\pi(y))p\, dw(y) \\
&\geq \int_{\mathbb{T}^n} \frac{1}{c_R(x)} |\nabla v_\phi|^2_\sigma d\,\|\theta\|\,(x) + 2\int_{\mathbb{T}^n} \nabla v_\phi(x)\cdot\sigma(x)p\, d\,\|\theta\|\,(x) \\
&\quad + \int_{\mathbb{T}^n} c_R(x)|p|^2_\sigma\, d\,\|\theta\|\,(x) + o_R(1)\int_{\mathbb{T}^n} |p|^2_\sigma\, d\,\|\theta\|\,(x) \\
&= \int_{\mathbb{T}^n} \frac{1}{c_R(x)} \left|\nabla v_\phi + c_R(x)p\right|^2_\sigma d\,\|\theta\| + o_R(1),
\end{aligned}$$

where $|q|^2_\sigma := q\cdot\sigma q$, $c_R(x) := \frac{N_R(x)}{(2R)^n}$ and we have used (2.7.5). By using triangle inequality

$$\fint_{(-R,R)^n} |\nabla\phi + p|^2_{\sigma(\pi(y))} dw(y) \geq \int_{\mathbb{T}^n} \frac{1}{c_R(x)} \left(\left|\nabla v_\phi + p\right|_\sigma - |1 - c_R(x)||p|_\sigma\right)^2 d\,\|\theta\| + o_R(1).$$

This shows that for any $\varepsilon > 0$

$$\fint_{(-R,R)^n} |\nabla\phi + p|^2_{\sigma(\pi(y))} dw(y) \geq \int_{\mathbb{T}^n} \frac{1-\varepsilon}{c_R(x)} \left|\nabla v_\phi + p\right|^2_\sigma + \left(1 - \frac{1}{\varepsilon}\right) \frac{|1 - c_R(x)|^2}{c_R(x)} |p|^2_\sigma d\,\|\theta\| + o_R(1).$$

Therefore, for any $\varepsilon > 0$ and $R > 0$

$$p\cdot Q(\theta^*; (-R,R)^n)p \geq \frac{(2R)^n(1-\varepsilon)}{N_{R,+}} p\cdot Q(\theta)p + \left(1 - \frac{1}{\varepsilon}\right)\int_{\mathbb{T}^n} \frac{|1 - c_R(x)|^2}{c_R(x)} |p|^2_\sigma d\,\|\theta\| + o_R(1)$$

Sending $R \to \infty$ we have $\lim_{R\to\infty} c_R(x) = 1$ uniformly in $x \in \mathbb{T}^n$ and hence for all $\varepsilon > 0$

$$\liminf_{R\to\infty}\, p\cdot Q(\theta^*; (-R,R)^n)p \geq (1-\varepsilon)p\cdot Q(\theta)p.$$

This finishes the proof by sending $\varepsilon \to 0$. □

Proof of Lemma 2.7.1. By Lemma 2.7.2, it suffices to show that

$$\limsup_{R\to\infty} Q(\theta^*; (-R,R)^n) \leq Q(\theta).$$

To this end, we denote $\eta_R \in C_0^\infty((-R,R)^n)$ a cut-off function satisfying

- $0 \le \eta_R \le 1$ and $|\nabla \eta_R| \le 2$
- $\eta_R \equiv 1$ on $[-\lfloor R \rfloor + 1, \lfloor R \rfloor - 1]^n$ and $\eta_R \equiv 0$ near the boundary $\partial(-R,R)^n$.

By using this function we observe that any $\psi \in C^\infty(\mathbb{T}^n)$ can be extended to some smooth function $f_\psi^R \in C_0^\infty((-R,R)^n)$ via

$$f_\psi^R(y) := \eta_R(y)\psi(\pi(y)).$$

Notice that

$$\begin{aligned} ⨍_{(-R,R)^n} |\nabla f_\psi^R + p|^2_{\sigma(\pi(y))} dw(y) = \frac{(2\lfloor R \rfloor - 2)^n}{(2R)^n} \int_{\mathbb{T}^n} |\nabla \psi + p|^2_\sigma d\,\|\theta\| \\ + \frac{1}{(2R)^n} \int_{(-R,R)^n \setminus [-\lfloor R \rfloor + 1, \lfloor R \rfloor - 1]^n} |\nabla f_\psi^R + p|^2_{\sigma(\pi(y))} dw(y). \end{aligned}$$

Observe that for any fixed ψ, the second term is $o_R(1)$, and therefore

$$\frac{(2\lfloor R \rfloor - 2)^n}{(2R)^n} \int_{\mathbb{T}^n} |\nabla \psi + p|^2_\sigma d\,\|\theta\| + o_R(1) \ge p \cdot Q(\theta^*; (-R,R)^n) p$$

Sending $R \to \infty$ we obtain

$$\int_{\mathbb{T}^n} |\nabla \psi + p|^2_\sigma d\,\|\theta\| \ge \limsup_{R \to \infty} p \cdot Q(\theta^*; (-R,R)^n) p,$$

which finishes the proof by taking the infimum over all $\psi \in C^\infty(\mathbb{T}^n)$.

□

2.7.3 Some technical facts

In this section we present some technical facts that we use in the proofs but are not in the scope of the main topics of this article.

Lemma 2.7.3. *Suppose H is a discrete additive subgroup of $\mathbb{R}^n$. Then the set of points*

$$\left\{ \frac{q}{|q|} \,;\, q \in H \setminus \{0\} \right\}$$

is dense in the unit sphere of $\mathrm{Span}_\mathbb{R} H$.

Proof. Let $G = \mathrm{Span}_\mathbb{R} H$. Denote by $S_G = \{x \in G : |x| = 1\}$ the unit sphere in G. Since H is discrete, it is a lattice in G. Thus there exists $r = \dim_\mathbb{R} G$ and linearly independent vectors $v_1, \ldots, v_r \in G$ such that $H = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r$. The set $\{v_1, \ldots, v_r\}$ is an $\mathbb{R}$-basis for G.

Define a linear isomorphism $T : G \to \mathbb{R}^r$ by $T(v_i) = e_i$ (the standard basis vectors). Then $T(H) = \mathbb{Z}^r$. The Euclidean inner product on $\mathbb{R}^n$ induces an inner product on $\mathbb{R}^r$ via

$$\langle x, y\rangle_T = \langle T^{-1}x, T^{-1}y\rangle \text{ and } \|x\|_T = \sqrt{\langle x, x\rangle_T}.$$

Then T is an isometry: $|x| = \|T(x)\|_T$ for all $x \in G$.

For $q \in H \setminus \{0\}$, set $z = T(q) \in \mathbb{Z}^r \setminus \{0\}$. Then

$$T\left(\frac{q}{|q|}\right) = \frac{z}{\|z\|_T}.$$

Hence it suffices to prove that $\{z/\|z\|_T : z \in \mathbb{Z}^r \setminus \{0\}\}$ is dense in the T-unit sphere

$$S_T = T(S_G) = \{x \in \mathbb{R}^r : \|x\|_T = 1\}.$$

Let $u \in S_T$ and $\varepsilon > 0$. By Dirichlet's simultaneous approximation theorem (see for example [159, Theorem 1B]), for any integer $N > 0$ there exist $q_N \in \mathbb{Z}$ with $1 \le q_N \le N$ and $p_N \in \mathbb{Z}^r$ such that

$$\|q_N u - p_N\|_2 \le \frac{\sqrt{r}}{N^{1/r}}.$$

Then because all norms on $\mathbb{R}^r$ are equivalent

$$\|q_N u - p_N\|_T \le \frac{C\sqrt{r}}{N^{1/r}},$$

for some constant $C > 0$. Choose N large enough so that $C\sqrt{r}/N^{1/r} < \varepsilon$. For such N, we have $p_N \ne 0$ (otherwise $\|q_N u\|_T = q_N \ge 1$ contradicts the inequality for large N).

Now because $p_N/q_N \to u$ in $\|\cdot\|_T$ and $\|p_N/q_N\|_T \to \|u\|_T = 1$, we obtain

$$\left\|\frac{p_N}{\|p_N\|_T} - u\right\|_T = \left\|\frac{p_N/q_N}{\|p_N/q_N\|_T} - \frac{u}{\|u\|_T}\right\|_T \to 0$$

as $N \to \infty$. Thus u is approximated by elements of $\{z/\|z\|_T : z \in \mathbb{Z}^r \setminus \{0\}\}$. By the isometry T, the original set $\{q/|q| : q \in H \setminus \{0\}\}$ is dense in S_G.

□

Lemma 2.7.4. *Suppose $V = \{v_1, \ldots, v_k\}$ is a sequence of $\mathbb{Z}$-linearly independent vectors in $\mathbb{Z}^n$, then V is also $\mathbb{R}$-linearly independent.*

Proof. We argue by contradiction and assume that there are real numbers $c_1, \dots, c_k$, not all zeros, such that

$$c_1 v_1 + \dots + c_k v_k = 0.$$

By writing $c = (c_j)$ and $M = (M_{ij})$, where $M_{ij} = (v_j)_i \in \mathbb{Z}$ the i-th component of the vector v_j for $j = 1, \dots, k$ and $i = 1, \dots, n$, we obtain the linear equation $Mc = 0$.

As the coefficients of M are integers and $0 \neq c \in \ker M$, by Gaussian elimination one can always find a solution $\tilde{c} \in \mathbb{Q}^k \setminus \{0\}$ so that $M\tilde{c} = 0$. We can define

$$c^* = \tilde{c} \cdot d \in \mathbb{Z}^k$$

by choosing $d \in \mathbb{Z}$ that divides all the denominators of the components of $\tilde{c}$. This implies that $Mc^* = 0$ for some $c^* \in \mathbb{Z}^k \setminus \{0\}$, and hence $v_1, \dots, v_k$ are $\mathbb{Z}$-linearly dependent, contradicting the assumption.

□

Chapter 3

Homogenization of a vertical oscillating Neumann condition

This chapter is a joint work with William Feldman.

3.1 Introduction

In this paper we study the homogenization limit $\varepsilon \to 0$ of the following heat equation with the Neumann data oscillating in the "vertical" u-variable

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon & \text{in } B_1 \cap \{x_1 > 0\} \times (0,\infty) \\ \partial_1 u^\varepsilon = f\left(\frac{u^\varepsilon}{\varepsilon}\right) & \text{on } B_1 \cap \{x_1 = 0\} \times (0,\infty). \end{cases} \tag{3.1.1}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently regular 1-periodic function, B_1 is the unit ball in $\mathbb{R}^d$ with $d \geq 2$, and $\partial_1 = \partial_{x_1}$. We work in the upper-half unit ball as a model domain and denote

$$B_1^+ := B_1 \cap \{x_1 > 0\} \quad \text{and} \quad B_1' := B_1 \cap \{x_1 = 0\}.$$

We also study the homogenization of the steady state problem

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } B_1^+ \\ \partial_1 u^\varepsilon = f\left(\frac{u^\varepsilon}{\varepsilon}\right) & \text{on } B_1'. \end{cases} \tag{3.1.2}$$

Such problems arise when the graph of u is considered as an interface in contact with a heterogeneous boundary, see Figure 3.1. Our primary motivation comes from the modeling of capillary droplet motion on anisotropic rough surfaces [3, 5, 48, 50, 77, 89, 91, 144, 172, 173, 177]. Our model arises from a linearization of the mean curvature flow with prescribed contact angle condition [5, 89, 91, 172, 173] on a patterned surface. The appeal of (3.1.1) is that it is simply posed, yet still captures the interplay between multi-dimensionality, homogenization, rate independent pinning, and rate dependent gradient flow. We discuss the importance of these themes further in Section 3.1.2.

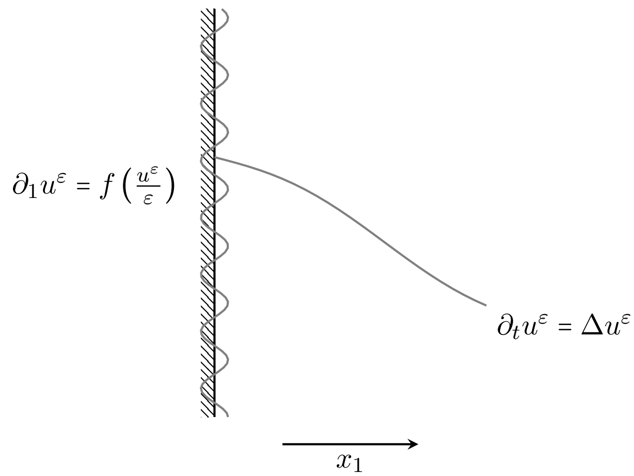

Figure 3.1: The graph of u^ε over B_1^+ is a moving interface interacting with an inhomogeneous medium via a Neumann condition at the boundary B_1'.

The full description of the homogenization limits of the above two problems requires detailed language from the theory of viscosity solutions. For the purposes of exposition we begin by describing our result at a formal level. We will present the PDE following the formalism common in the study of rate independent energetic systems (a la [130] and see Section 3.1.2 below). The homogenization limit of (3.1.1) is a heat equation with a singular pinned Neumann condition

$$\begin{cases} \partial_t u = \Delta u & \text{in } B_1^+ \times (0,\infty) \\ \partial_1 u \in \partial \mathcal{R}(\partial_t u; \nabla' u) & \text{on } B_1' \times (0,\infty). \end{cases} \tag{3.1.3}$$

Here $\nabla' u$ is the tangential gradient of u on B_1', and $\partial\mathcal{R}(\tau;p)$ is the subdifferential in τ of the following function

$$\mathcal{R}(\tau;p) := \begin{cases} L^*(p)\tau & \text{if } \tau \geq 0 \\ L_*(p)\tau & \text{if } \tau < 0. \end{cases} \tag{3.1.4}$$

Here L_* and L^* are certain homogenized coefficients depending on the tangential slope $p \in \mathbb{R}^{d-1}$.

We will show the following simple formula for the homogenized coefficients

$$L_*(p) = (\min f)\, 1_{\{p=0\}} + \langle f \rangle 1_{\{p \neq 0\}} \quad \text{and} \quad L^*(p) = (\max f)\, 1_{\{p=0\}} + \langle f \rangle 1_{\{p\neq 0\}}. \tag{3.1.5}$$

Here $\langle f\rangle := \int_0^1 f(u)du$ is the average value of f.

The rate independent evolution law at the Neumann boundary in (3.1.3) can be described heuristically as:

- When the tangential gradient $\nabla' u \neq 0$ the inner normal derivative $\partial_1 u = \langle f\rangle$ is pinned exactly at the average value. When the tangential gradient $\nabla' u = 0$ then the inner normal derivative
$$\partial_1 u \in [\min f, \max f]$$
is pinned in a nontrivial interval.

- If $\partial_t u > 0$, then $\partial_1 u \geq \langle f\rangle$ and if further $\nabla' u = 0$ then $\partial_1 u = \max f$ (in a certain weak sense).

- If $\partial_t u < 0$, then $\partial_1 u \leq \langle f\rangle$ and if further $\nabla' u = 0$ then $\partial_1 u = \min f$ (in a certain weak sense).

Due to the discontinuity of the homogenized coefficients L_* and L^*, our description here is not completely accurate, especially the final two bullet points which we are only able to interpret in the viscosity solutions / comparison principle sense and have no classical sense. See Definition 3.5.1, especially conditions (b) and (c), for a precise description in the language of viscosity solution theory.

Our central main result is the homogenization limit from (3.1.1) to (3.1.3). We will study solutions with a fixed Dirichlet data on the upper part of the parabolic boundary of $B_1^+ \times (0,\infty)$:

$$\partial_{\mathrm{p}}^+(B_1 \times (0,\infty)) := \big[(\partial B_1 \cap \{x_1 \geq 0\}) \times (0,\infty)\big] \cup \big[\overline{B_1^+} \times \{t=0\}\big]. \qquad (3.1.6)$$

For the introduction we write the result briefly, see Theorem 3.6.1 below for a more detailed statement including precise assumptions on the Dirichlet boundary data and heterogeneity f.

Theorem 3.1.1. *Suppose f is 1-periodic and regular on $\mathbb{R}$. Then a sequence u^ε of solutions to* (3.1.1)*, with a fixed Dirichlet condition on the parabolic boundary $\partial_{\mathrm{p}}^+(B_1^+ \times [0,\infty))$, converge locally uniformly on $(x,t) \in \overline{B_1^+} \times [0,\infty)$ to the unique continuous viscosity solution u to* (3.1.3) *with the same data on the parabolic boundary.*

A central element of the proof of this theorem is the comparison principle / uniqueness for (3.1.3). The important role played by comparison principle is well known in the homogenization of nonlinear elliptic and parabolic problems. In particular, the proof of Theorem 3.1.1 uses the half relaxed limit approach of Barles and Perthame [25, 26] which relies on comparison principle for semicontinuous sub and supersolutions.

Theorem 3.1.2. *Suppose u is a subsolution and v is a supersolution to* (3.1.3)*, as in Definition 3.5.1. If $u \le v$ on the parabolic boundary $\partial_{\mathrm{p}}^{+}(B_1^+ \times [0,\infty))$ (as in (3.1.6)) then $u \le v$ on the whole space-time domain $\overline{B_1^+} \times [0,\infty)$.*

There are two major difficulties in proving the comparison principle Theorem 3.1.2. The first is the failure of uniform obliqueness, and as a result the existence of free degenerate regions in (3.1.3)

$$\{\partial_1 u \neq \langle f \rangle\} \subset \{\nabla' u = 0\}.$$

As we shall see later that these sets are not empty and in the proof we need to discuss whether the location of the touching points are in such degenerate regions. Outside the degenerate set the solutions satisfy standard Neumann boundary conditions, but on the degenerate set the solutions satisfy a Dirichlet boundary condition with the boundary data depending only on the time variable. This makes the proof quite delicate as we don't know either the shape of the degenerate region or the values on such regions. The lack of uniform obliqueness also fails the construction of doubling test functions by Barles in [27] and therefore we use the inf / sup-convolution type of arguments (see Appendix 3.9.2 and also our previous work [82]). It is still interesting to explore if there is a doubling variable argument for the proof of comparison principle of (3.1.3).

Another difficulty in proving the comparison principle is that the boundary condition in (3.1.3) is essentially a differential inclusion. As we shall see in the following discussions, there is no uniqueness for the steady state equation. The crucial ingredients that ensure the uniqueness in the parabolic flow are the dynamic slope conditions (b) and (c) in Definition 3.5.1. As we mentioned before, these conditions, especially condition (c), can only be made precise by using viscosity solution notions.

The steady state equation for (3.1.3), which is also the homogenization limit of the elliptic problems (3.1.2), takes the form

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ \partial_1 u \in [L_*(\nabla' u), L^*(\nabla' u)] & \text{on } B_1'. \end{cases} \tag{3.1.7}$$

Thanks to the explicit formula (3.1.5) for L_* and L^* the above equation has the following equivalent form

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ (\partial_1 u - \langle f \rangle)|\nabla' u| = 0 \text{ and } \partial_1 u \in [\min f, \max f] & \text{on } B_1'. \end{cases} \tag{3.1.8}$$

There is, in general, non-uniqueness of solutions to Dirichlet boundary value problems for (3.1.7) and (3.1.8). This nonuniqueness also occurs before homogenization in (3.1.2), which is not so obvious, but it is in fact one consequence of the homogenization result that we describe next.

Despite the general non-uniqueness we can still determine the unique homogenization limits of special solutions to (3.1.2). On one hand by Perron's method, we can construct maximal subsolutions

$$u^\varepsilon_{\max}(x) = \max\{u(x)\ ;\ u \text{ is a subsolution to (3.1.2)}\}$$

and symmetrically minimal supersolutions

$$u^\varepsilon_{\min}(x) = \min\{u(x)\ ;\ u \text{ is a supersolution to (3.1.2)}\}.$$

We call both $u^\varepsilon_{\max}$ and $u^\varepsilon_{\min}$ the extremal solutions.

On the other hand, we can consider the corresponding energy of (3.1.1) and (3.1.2)

$$E_\varepsilon(u, B_1^+) := \int_{B_1^+} \frac{1}{2}|\nabla u|^2 + \int_{B_1'} \int_0^{u(x')} f(r/\varepsilon)\, dr\, dx'. \tag{3.1.9}$$

We define global energy minimizers $u^\varepsilon_{\text{glb}}$ satisfying

$$E_\varepsilon(u^\varepsilon_{\text{glb}}, B_1^+) \le E_\varepsilon(v, B_1^+),$$

for any appropriate test function v such that $v = u^\varepsilon_{\text{glb}} = g$ on $\partial B_1 \cap \{x_1 \ge 0\}$.

Again, it is not obvious whether these notions provide actually distinct solutions. In the case of standard Neumann data $\partial_1 u = f(x)$, they are all the same. We will see, via our homogenization result below, that these three solutions of (3.1.2) are often distinct, see also Proposition 3.1.4 later.

Typical arguments from Γ-convergence theory [41] show that the energies E_ε in (3.1.9) Γ-converge to the following energy

$$E_*(u, B_1^+) := \int_{B_1^+} \frac{1}{2}|\nabla u|^2 + \int_{B_1'} u(x')\langle f\rangle dx'. \tag{3.1.10}$$

In particular, the homogenized PDE associated with energy minimizing solutions of (3.1.2) is the standard Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ \partial_1 u = \langle f\rangle & \text{on } B_1'. \end{cases} \tag{3.1.11}$$

See Lemma 3.3.2 below for a precise statement and proof.

The homogenization for the extremal solutions $u^\varepsilon_{\max}$ and $u^\varepsilon_{\min}$ is trickier as there is no Γ-convergence type theory for them. In our second main theorem, we show that the homogenization limits of the extremal solutions are exactly extremal solutions to the general homogenized equation (3.1.7). We write the results in a non-technical way, see Section 3.4 for a more detailed statement.

Theorem 3.1.3. *The extremal solutions of* (3.1.2) *converge as* $\varepsilon \to 0$ *to the extremal solutions of* (3.1.7).

We summarize the homogenization results for the steady states in Figure 3.2.

One essential question, which we alluded to already, is: are the extremal solutions of (3.1.8) actually distinct from the energy minimizing solution satisfying the standard Neumann boundary condition $\partial_1 u = \langle f\rangle$? We will establish that, indeed, these solutions are distinct for many choices of Dirichlet boundary datum. In fact, this phenomenon also demonstrates the existence of the degenerate regions, which we call free facets/contact sets

$$\mathcal{C}(u) := \{\partial_1 u \neq \langle f\rangle\} \subset \{\nabla' u = 0\} \subset B_1'.$$

See (3.7.2) for a precise definition of $\mathcal{C}(u)$ in terms of viscosity solution theory. We prove the following result in Section 3.8.

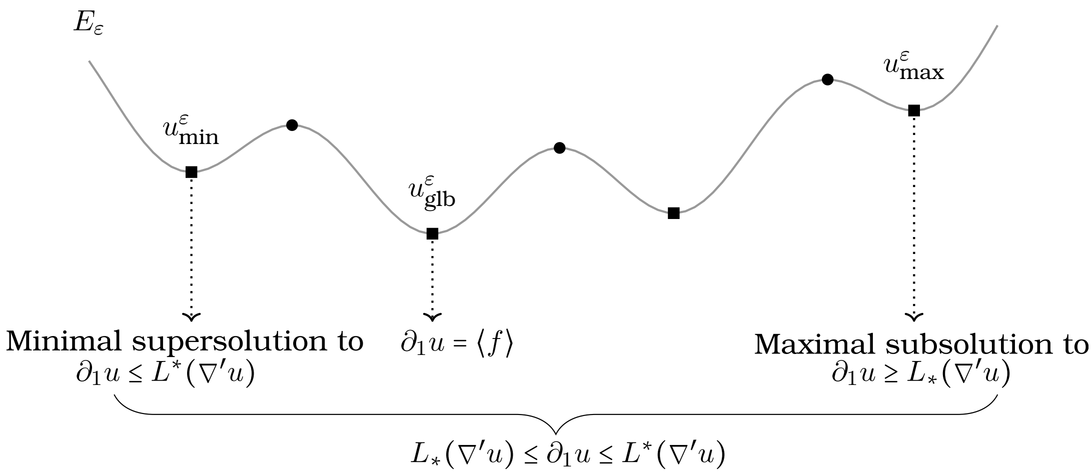

Figure 3.2: This figure formally illustrates the critical points of the energy E_ε as defined in (3.1.9) and their homogenization. On the left and right are the extremal solutions, and they homogenize exactly to the extremal solutions of $L_*(\nabla' u) \le \partial_1 u \le L^*(\nabla' u)$. The global energy minimizers $u^\varepsilon_{\text{glb}}$ homogenizes exactly to the standard Neumann problem $\partial_1 u = \langle f \rangle$.

Proposition 3.1.4. *There is a non-empty set of boundary data $\mathcal{F} \subset C(\partial B_1 \cap \{x_1 \ge 0\})$, open in the uniform topology, such that, for all $g \in \mathcal{F}$, the unique minimal supersolution u_g to* (3.1.7) *with $u_g = g$ on $\partial B_1 \cap \{x_1 \ge 0\}$ has nontrivial relatively open contact set $\mathcal{C}(u_g) \neq \emptyset$.*

Note that, in combination with the homogenization result for extremal solutions in Theorem 3.1.3, this shows that $u^\varepsilon_{\text{max}}$, $u^\varepsilon_{\text{min}}$, and $u^\varepsilon_{\text{glb}}$ are all distinct for a fixed boundary data g as in Proposition 3.1.4. In other words our result allows to understand the non-uniqueness of the ε problem (3.1.2) in terms of the non-uniqueness of the homogenized problem (3.1.7), which is often more accessible.

We also show in the same section that the homogenized boundary condition in the parabolic evolution (3.1.3) is, also, non-trivially distinct from the standard Neumann boundary condition. Specifically we show that when the Dirichlet boundary data satisfies a certain strong monotonicity property in time, the solution to the homogenized problem (3.1.3) will converge to the extremal steady states as time goes to infinity. As just discussed, these extremal steady states are not, in general, solutions to the standard Neumann problem. See Theorem 3.8.8

for the precise description and proof.

3.1.1 Pinning, contact set and rate-independent system

The parabolic homogenization in Theorem 3.1.1 shows the emergence of some novel phenomena from a macroscopic viewpoint:

1. *Pinning and rate-independent motion*: Formally there is no motion when the slope

$$\partial_1 u \in (L_*(\nabla' u), L^*(\nabla' u)),$$

that is, when

$$\nabla' u = 0 \text{ and } \partial_1 u \in (\min f, \max f).$$

When the interface moves, it follows a rate-independent principle: the state of the contact slope $\partial_1 u$ does not depend on the magnitude of $\partial_t u$, i.e. the rate of motion. Note that the initial problem (3.1.1) has no intrinsic frictional hysteresis, this phenomenon arises in the homogenization limit.

2. *Singular anisotropy and free contact set*: The free contact set

$$\mathcal{C}(u) = \{\partial_1 u \neq \langle f \rangle\} \subset \{\nabla' u = 0\},$$

plays a central role in the homogenized problem. As described in [82] this free region is related to and generalizes the role of the contact set in the thin obstacle problem. Thus a thin free boundary arises in the limiting homogenized problems (3.1.3) and (3.1.7).

3.1.2 Literature

Although our techniques in this paper are primarily based on comparison principle, the problem (3.1.1) has a natural gradient flow structure. In fact the phenomenon of rate independent hysteresis is most commonly studied in the energetic context, see the book [130] for more background and references. In order to explain further this connection, we introduce some of the underlying notations and ideas from the theory of gradient systems. We will be somewhat imprecise, in particular we will not detail the functional spaces or the forcing by boundary data. General gradient systems take the form

$$0 \in \partial_{\dot{u}} R(u(t), \dot{u}(t)) + \partial_u E(t, u(t)) + \partial_t E(t, u(t))$$

where E is the energy functional and R is the dissipation rate functional. In the energetic formulation of (3.1.1) the energy is E_ε, as defined in (3.1.9), and the dissipation takes the following L^2-form

$$R_\varepsilon(\partial_t u) := \frac{1}{2} \int_{B_1^+} |\partial_t u|^2 \, dx.$$

The homogenized system (3.1.3) also has a formal energetic structure. The homogenized energy E_* takes the form (3.1.10), simply the usual Γ-limit of the E_ε. However something much more subtle happens in the dissipation rate functional, formally speaking the dissipation rate functional is given by

$$R(\partial_t u; \nabla' u) = \frac{1}{2} \int_{B_1^+} |\partial_t u|^2 \, dx + \int_{B_1'} \mathcal{R}(\partial_t u(x'); \nabla' u(x')) dx',$$

where $\mathcal{R}(\tau; p)$ is the homogenized quantity defined in (3.1.4). The appearance of the L^1-type term in the dissipation rate is indicative of rate independent pinning. The origin of such rate independent pinning terms from homogenization of wiggly energies has been expected via analysis of simple ODE models since [1, 99, 131] and even earlier works on dry friction. In fact this microscopic origin of macroscopic hysteresis is one of the motivations for the theory of gradient systems with rate independent dissipation. What is very unusual in this problem is the singularly anisotropic dependence on the tangential gradient in the rate functional.

Various energetic convergence theories for gradient systems, including with rate independent / L^1 type dissipation, have been established in the literature [129, 129, 158]. However the rigorous passage to the limit from wiggly energy microscopic model to macroscopic model with L^1-type dissipation rate has only been fully addressed in one-dimensional ODE models [35, 71, 129]. We establish such a limit theorem in a multi-dimensional PDE setting for the first time. Our techniques, however, do not strongly use the formal energetic structure of the limit problem. In fact, due to the discontinuity of $\mathcal{R}(\tau; p)$ in p, we have been unable to rigorously establish an energetic formulation of the limit problem (3.1.3).

This is one of the main reasons we don't use energetic techniques to analyze the homogenization limit. We are very interested in whether it is possible to give a rigorous energetic interpretation of the homogenized system (3.1.3) and study the homogenization limit using evolutionary Γ-convergence techniques.

As previously mentioned, a central element in our proof of Theorem 3.1.1 is the comparison principle (Theorem 3.1.2) for solutions of (3.1.3). Similar homogenization challenges arise in other models studied in the literature. For instance, in the context of anisotropic Bernoulli-type problems, elliptic comparison principles were established in [80, 81, 83], enabling homogenization results for inhomogeneous one-phase Bernoulli problems [45, 49, 50, 81]. These results are in elliptic / stationary settings. In [47, 48], Caffarelli, Lee and Mellet analyzed a reaction-diffusion equation modeling flame propagation, which, under a certain singular limit, converges to the parabolic version of the Bernoulli problem

$$u_t = \Delta u \ \text{ in } \{u > 0\} \ \text{ and } \ |\nabla u| = Q(x/\varepsilon) \ \text{ on } \partial\{u > 0\}, \tag{3.1.12}$$

which is also known as the flame propagation free boundary problem. They consider several scalings, but in the case where the interface width is thinner than the heterogeneity and pinning occurs their work applies only to the one-dimensional case. The challenges in these studies are closely analogous to those in our time-dependent model. In fact, the equation (3.1.1) can be obtained via a formal flat asymptotic expansion of (3.1.12) near the free boundary points $x \in \partial\{u > 0\}$ in the case that $Q(x) = Q(x_1)$ depending only on one variable (which corresponds to the laminar media; see [82] for more details). In comparison to these previous results, this paper is the first to handle the parabolic setting in multiple dimensions. The comparison principle Theorem 3.1.2 is really the key new tool that allows for the homogenization result in higher dimensions.

As a PDE problem, the homogenized equation (3.1.8) can be viewed as an elliptic PDE with gradient degeneracy. Such problems have attracted interest, especially spurred by a result of Silvestre and Imbert [98], with further developments for nonlocal PDE in [14, 143] and others. There is also a connection with the Signorini or thin obstacle problem, which was described in more detail in the previous work of the authors [82]. The contact set in the elliptic problem (3.1.8) is closely related to the contact set in the thin obstacle problem. It is a kind of "unconstrained" analogue of the thin obstacle problem in the language of [86]. This connection was exploited to study the regularity of solutions to (3.1.8) in [82]. In this work we have derived (3.1.8) and its parabolic analogue (3.1.3) via a natural homogenization procedure, and this invites further possible investigation into connections with thin obstacle problems and gradient degenerate elliptic problems.

Pinning and the hysteresis caused by pinning are central important phenomena in the study of propagation of interfaces in heterogeneous media. Among others this includes models of capillary contact lines, domain boundaries in random magnetic materials, adhesion of thin films, and others. In particular, the model (3.1.1) is motivated by the dynamics of capillary droplets on rough surfaces [3, 5, 50, 89, 144, 177]. A simple dynamical model decreasing the capillary energy is the mean curvature flow with prescribed capillary contact angle condition [77, 91, 172, 173]. The PDE (3.1.1) arises as a linearization of these capillary models, similar to the connection with the flame propagation free boundary problem discussed above in (3.1.12). We aim to study the origin of rate independent contact angle hysteresis in interaction with gradient flow dynamics of the free surface. In the linearized setting of (3.1.1) we can begin to understand these complex phenomena in a slightly simpler scenario.

There have been many works studying pinning and de-pinning of interface motions in heterogeneous media, to name a few examples in slightly related PDE models [28, 44, 59, 68, 69, 72, 73, 81, 102, 108, 166, 175]. Generally speaking the study of stationary (pinned) interfaces in heterogeneous media can be more challenging than that of moving (de-pinned) interfaces. When the front is de-pinned the interface moves through the medium "seeing" the entire medium, and the large-scale averaging behavior is more accessible. The additional challenges in our work compared to much of the literature discussed above are (1) the large scale forcing $\partial_1 u$ depends nonlocally on the shape, especially the tangential gradient $\nabla' u$ of the interface causing the singular anisotropy of the limiting PDE in contrast to works like [59, 68, 69, 73] where the forcing is an external constant. This nonlocal geometric dependence is also one of the reasons for the existence of facets/contact sets that specify the free regions having distinct pinning phenomena; (2) the boundary condition is both quasistatic and rate independent so the comparison principle and uniqueness present a new challenge as compared to finite velocity models as in [108].

3.1.3 Organization of the paper

In Section 3.2 we introduce some notations, viscosity touching and crossing and the notions of half relaxed limits. In Section 3.3 we start with a Γ-convergence result for the homogenization of energy minimizers. Then we study the corrector problem. We introduce the homogenized pinning interval and prove the existence of plane-like correctors satisfying the strong Birkhoff property. In Section 3.4 we show the homogenization of extremal solutions of the elliptic problem (3.1.2). In Section 3.5 we make precise the notion of viscosity solution to the homogenized evolution (3.1.3). In Section 3.6 we prove the homogenization of (3.1.1) using the half-relaxed limit method. In Section 3.7 we prove the comparison principle of (3.1.3). Finally in Section 3.8 we show the existence of facets / contact sets for certain boundary conditions.

3.1.4 Acknowledgments

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3.2 Preliminaries

3.2.1 Notations

- If not particularly defined, $\alpha \in (0,1)$ will be a constant that represents the Hölder exponents that may change from line to line.
- For a $\mathbb{Z}^d$-periodic function $f = f(x', u)$ on $\mathbb{R}^{d-1} \times \mathbb{R}$, we denote the average as $\langle f \rangle := \int_{(0,1]^{d-1}} \int_0^1 f(y', r) dr dy'$. In particular, if $f = f(u)$ we have $\langle f \rangle = \int_0^1 f(r) dr$.

- Define

$$\mathbb{R}^d_+ := \mathbb{R}^d \cap \{x_1 > 0\} \quad \text{and} \quad \partial\mathbb{R}^d_+ = \mathbb{R}^d \cap \{x_1 = 0\}.$$

- Suppose $\Omega \subset \mathbb{R}^d$ is open (or $\Omega \subset \overline{\mathbb{R}^d_+}$ is relatively open), we write

$$\Omega' := \Omega \cap \{x_1 = 0\} \quad \text{and} \quad \Omega^+ := \{x_1 > 0\}.$$

In particular, we write $B_1^+ := B_1 \cap \{x_1 > 0\}$ and $B_1' := B_1 \cap \{x_1 = 0\}$. We write the "exterior boundary" as

$$\partial^+\Omega = \partial^+\Omega^+ := (\partial\Omega^+) \setminus \Omega'.$$

- For a space-time cylindrical domain

$$U = \Omega \times (t_1, t_2) \text{ (or } \Omega \times (t_1, t_2]) \subset \mathbb{R}^d \times \mathbb{R}$$

with $t_1 < t_2$, we write

$$U' := \Omega' \times (t_1, t_2] \quad \text{and} \quad U^+ := \Omega^+ \times (t_1, t_2],$$

and we define the following two types of parabolic boundaries

$$\partial_{\mathrm{p}} U := \overline{U} \setminus (\Omega \times (t_1, t_2]) \quad \text{and} \quad \partial_{\mathrm{p}}^+ U = \partial_{\mathrm{p}}^+ U^+ := \partial_{\mathrm{p}} U^+ \setminus U'.$$

- We write $D_T := B_1 \times (0, T]$ for $T \in (0, \infty)$ and $D_\infty := B_1 \times (0, \infty)$. In particular, we have

$$D_T' = B_1' \times (0, T] \quad \text{and} \quad D_T^+ = B_1^+ \times (0, T].$$

We also write

$$D_\infty' = B_1' \times (0, \infty) \quad \text{and} \quad D_\infty^+ = B_1^+ \times (0, \infty).$$

- For two sets A, B we denote the symmetric difference

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

- A function $u : \mathbb{R}^d \to [-\infty, \infty)$ is upper semicontinuous if for every $x \in U$

$$\limsup_{y \to x} u(y) \le u(x).$$

A lower semicontinuous function is defined symmetrically. We emphasize here that upper semicontinuous functions are allowed to take negative infinity values.

3.2.2 Touching, crossing and half relaxed limits

We introduce here the notion of touching and crossing, and their behavior under half relaxed limits. Since our problem involves a boundary condition, we mainly consider in a relatively open domain $U \subset \mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+$, and denote $U^+ = U \cap \mathbb{R}^d_+$, $U' = U \cap \partial\mathbb{R}^d_+$.

Definition 3.2.1. We say that a smooth function ψ *touches* an upper semicontinuous function $u : \overline{U} \to [-\infty, \infty)$ (strictly) from above at $x_0 \in U$ if there is an open domain $V \subset \mathbb{R}^d$ that contains x_0 so that $u - \psi$ attains its (strict) maximal value 0 at x_0 in $V \cap U$. We say ψ *touches* a lower semicontinuous function v (strictly) from below at x_0 if $-\psi$ touches $-v$ (strictly) from above at x.

It is often more appropriate to consider the notion of crossing, or parabolic touching, especially in parabolic equations with non-proper zeroth order terms [60].

Definition 3.2.2. In a cylindrical domain $U \times (t_1, t_2)$, we say that a smooth function ψ *crosses* an upper semicontinuous function $u : \overline{U} \times [t_1, t_2] \to [-\infty, \infty)$ (strictly) from above at $(x_0, t_0) \in U \times (t_1, t_2)$ if there is an open domain $V \subset \mathbb{R}^d$ that contains x_0 and a small $r > 0$ so that $u - \psi$ attains its (strict) maximal value 0 at (x_0, t_0) in the space-time domain $(V \cap U) \times (t_0 - r, t_0]$. The *crossing* from below is defined symmetrically for lower semicontinuous functions.

For a sequence of upper semicontinuous functions u_n that are bounded from above in a closed set $K \subset \mathbb{R}^d$, we define the upper half relaxed limit

$$\limsup_{n\to\infty}{}^* u_n(x) = \inf_{n\geq 1} \max_{|y-x|\leq 1/n, y\in K} u_n(y), \tag{3.2.1}$$

and symmetrically the lower half relaxed limit for a sequence of lower semicontinuous functions v_n that are bounded from below in K

$$\liminf_{n\to\infty}{}_* v_n(x) = \sup_{n\geq 1} \min_{|y-x|\leq 1/n, y\in K} v_n(y). \tag{3.2.2}$$

The application of half relaxed limits in viscosity solution limit problems was first introduced by Barles and Perthame [25, 26].

In the following we prove two technical lemmas about the perturbation properties of touching and crossing under the notion of half relaxed limits.

Lemma 3.2.3. *Let u^* be the upper half relaxed limit of u_n in $\overline{U}$, then u^* is upper semicontinuous. If u^* reaches a strict maximum at x_0 in $B_r(x_0) \cap U$ for some radius $r > 0$, then there is a subsequence u_{n_j} of u_n and a sequence of maximum points x_{n_j} of u_{n_j} in $B_r(x_0) \cap U$ that converge to x_0 as $j \to \infty$, and*

$$\lim_{j\to\infty} u_{n_j}(x_{n_j}) = u^*(x_0).$$

Similar results also hold for the lower half relaxed limits by symmetry.

Proof. See [164, Lemma 3.5].

□

Lemma 3.2.4. *Let u^* be the upper half relaxed limit of u_n in $\overline{U} \times [t_1, t_2]$. If u^* reaches a strict maximum at (x_0, t_0) in $(B_r(x_0) \cap U) \times (t_0 - r, t_0]$ for $(x_0, t_0) \in U \times (t_1, t_2)$ and some radius $r > 0$, then there is a subsequence u_{n_j} of u_n and a sequence of maximum points (x_{n_j}, t_{n_j}) of u_{n_j} in $(B_r(x_0) \cap U) \times (t_0 - r, t_{n_j}]$ that converge to (x_0, t_0) as $j \to \infty$, and*

$$\lim_{j\to\infty} u_{n_j}(x_{n_j}, t_{n_j}) = u^*(x_0, t_0).$$

Similar results also hold for the lower half relaxed limits by symmetry.

Proof. We assume that $u^*(x_0, t_0) = 0$. By definition of upper half relaxed limit, we can find a subsequence u_n, not relabeled, and a sequence of points $(y_n, s_n) \in U \times (t_1, t_2)$ such that

$$\lim_{n\to\infty} (y_n, s_n) = (x_0, t_0) \text{ and } \lim_{n\to\infty} u_n(y_n, s_n) = u^*(x_0, t_0).$$

We define for each n, (x_n, t_n) to be a maximum point of u_n in

$$\overline{(B_r(x_0) \cap U)} \times [t_0 - r, s_n].$$

By compactness, there is a limit point (x_0', t_0') of the sequence (x_n, t_n), and therefore

$$u^*(x_0', t_0') \ge \limsup_{n\to\infty} u_n(x_n, t_n) \ge \limsup_{n\to\infty} u_n(y_n, s_n) = u^*(x_0, t_0) = 0.$$

This shows that $t_0' \ge t_0$ and because $t_n \le s_n \to t_0$, we know that $t_0' = t_0$. Because u^* is strictly negative in $(B_r(x_0) \cap U) \times (t_0 - r, t_0]$ except for (x_0, t_0), we know that $x_0' = x_0$. This shows that (x_n, t_n) has to converge to (x_0, t_0) and the proof is complete.

□

3.3 General steady states

In this section we study the homogenization of the following more general steady state problem

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } B_1^+ \\ \partial_1 u^\varepsilon = f(x'/\varepsilon, u^\varepsilon/\varepsilon) & \text{on } B_1'. \end{cases} \tag{3.3.1}$$

Here $f(y,z)$ belongs to $C^\alpha(\mathbb{R}^{d-1}\times\mathbb{R})$ for some $\alpha > 0$ and is $\mathbb{Z}^{d-1}\times\mathbb{Z}$-periodic. We remark here that although we include more general inhomogeneity in the steady state problem, the parabolic homogenization remains open due to the lack of comparison principles similar to Theorem 3.1.2.

Remark 3.3.1. There are two main types of solutions to (3.3.1): distributional weak solutions and viscosity solutions. Under the assumption $f \in C^\alpha(\mathbb{R}^{d-1}\times\mathbb{R})$ the two notions are equivalent and are both classical solutions by applying the regularity results in [85, 133, 138] (see also Appendix 3.9.1).

First, in Section 3.3.1, we show the homogenization of energy minimizers by a Γ-convergence type argument. Then we consider the homogenization of general solutions of the PDE. This requires the introduction and classification of the plane-like correctors and the pinning interval, which is covered in Section 3.3.2 and Section 3.3.3. In the special case of laminar media the pinning interval takes a particularly simple form, and can be exactly computed. We carry this computation out in Section 3.3.4.

3.3.1 The global energy minimizers

In this subsection we consider the energy minimizing solutions of (3.3.1). Instead of considering $\mathbb{Z}^d$-periodicity, we allow f to be periodic with respect to a general lattice $\mathcal{L}\subset\mathbb{R}^d$. Specifically, we consider the energy functional

$$E_\varepsilon(u, B_1^+) := \int_{B_1^+} \frac{1}{2}|\nabla u|^2 + \int_{B_1'}\int_0^{u(x')} f_\varepsilon(x', r)\,dr\,dx', \tag{3.3.2}$$

where $f_\varepsilon(x', r) = f(x'/\varepsilon, r/\varepsilon)$ for some $\mathcal{L}$-periodic and Hölder continuous function f on $\mathbb{R}^d$. For any $\mathcal{L}$-periodic function f, we denote the average

$$\langle f\rangle := \fint_{P_{\mathcal{L}}} f(x)\,dx, \tag{3.3.3}$$

where $P_{\mathcal{L}}$ is a fundamental region of $\mathcal{L}$.

Because the homogenization for the periodic oscillations in a general lattice is not the major concern of this paper, we will return to the case $\mathcal{L} = \mathbb{Z}^d$ after this subsection.

For each $g \in H^1(B_1^+)$ we denote $H_g^1(B_1^+)$ the subspace of $H^1(B_1^+)$ that share the same trace as g on

$$\partial^+ B_1 = \partial B_1 \cap \{x_1 \geq 0\}.$$

Lemma 3.3.2. *As $\varepsilon \to 0$ the energies E_ε over $H_g^1(B_1^+)$, are equi-coercive and Γ-converge in the weak topology of $H_g^1(B_1^+)$ (see definitions in* [41]*) to*

$$E_0(u, B_1^+) := \int_{B_1^+} \frac{1}{2}|\nabla u|^2 + \int_{B_1'} \langle f \rangle u(x')dx'.$$

Moreover, if g is continuous then the corresponding global energy minimizers u^ε of E_ε over H_g^1, converge uniformly on $\overline{B_1^+}$ to the unique minimizer u of E_0 on $H_g^1(B_1^+)$, which solves

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ u = g & \text{on } \partial^+ B_1 \\ \partial_1 u = \langle f \rangle & \text{on } B_1'. \end{cases} \tag{3.3.4}$$

Proof. Let us first show that E_ε Γ-converges to E_0 as $\varepsilon \to 0$ in the weak topology of $H_g^1(B_1^+)$. By testing against finite linear combinations of indicator functions of lattice fundamental regions and a standard density argument, see for example [66, Chapter 2], we conclude that f_ε converge weak* in L^∞ to $\langle f \rangle$ on $B_1' \times \mathbb{R}$. This implies that for every fixed $u \in H_g^1(B_1^+)$

$$\begin{aligned}\int_{B_1'} \int_0^{u(x')} f_\varepsilon(x', r)drdx' &= \int_{B_1' \times \mathbb{R}} f_\varepsilon(x', r)\left(1_{\{0<r<u\}} - 1_{\{u<r<0\}}\right)dx'dr \\ &\to \langle f \rangle \int_{B_1'} u(x')dx', \text{ as } \varepsilon \to 0^+.\end{aligned}$$

In particular, for every $u \in H_g^1(B_1^+)$ we always have the recovery sequence $\hat{u}_\varepsilon = u$ and

$$E_0(u) = \lim_{\varepsilon \to 0} E_\varepsilon(u). \tag{3.3.5}$$

To prove the Γ-convergence of E_ε we also need to show that for $u_\varepsilon \rightharpoonup u$ in $H^1(B_1^+)$, one has

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq E_0(u). \tag{3.3.6}$$

Because the Dirichlet energy $\int_{B_1^+} |\nabla u|^2$ is lower semi-continuous under weak convergence of the gradients, it then suffice to show that

$$\liminf_{\varepsilon\to 0} \int_{B_1'} \int_0^{u_\varepsilon(x')} f_\varepsilon(x',r)drdx' \geq \langle f\rangle \int_{B_1'} u(x')dx'.$$

Because the trace operator $H^1(B_1^+) \to L^2(B_1')$ is compact (see [65, Theorem 3.85]), we know that u_ε converges strongly to u in $(L^1 \cap L^2)(B_1')$. This implies that

$$\left| \int_{B_1'} \int_0^{u_\varepsilon(x')} f_\varepsilon(x',r)drdx' - \int_{B_1'} \int_0^{u(x')} f_\varepsilon(x',r)drdx' \right| \leq \|f\|_{L^\infty(\mathbb{R}^d)} \|u_\varepsilon - u\|_{L^1(B_1')}$$

uniformly in $\varepsilon > 0$. In particular,

$$\liminf_{\varepsilon\to 0} \int_{B_1'} \int_0^{u_\varepsilon(x')} f_\varepsilon(x',r)drdx' = \liminf_{\varepsilon\to 0} \int_{B_1'} \int_0^{u(x')} f_\varepsilon(x',r)drdx' = \langle f\rangle \int_{B_1'} u(x')dx',$$

where the last equality uses the same argument in proving (3.3.5). This shows (3.3.6) and hence proves that the energies E_ε Γ-converge to E_0.

To show that E_ε are equi-coercive, it suffices to show that there are positive constants C_1 and C_2 so that

$$\|u\|^2_{H^1(B_1^+)} \leq C_1 E_\varepsilon(u) + C_2\left(\|f\|^2_{L^\infty} + \|g\|^2_{H^1(B_1^+)}\right). \tag{3.3.7}$$

We first observe that by Poincaré inequality there is a constant $C > 0$ such that for $u \in H^1_g(B_1^+)$

$$\|u\|_{H^1(B_1^+)} \leq \|u-g\|_{H^1(B_1^+)} + \|g\|_{H^1(B_1^+)} \leq C\left(\left(\int_{B_1^+} |\nabla u|^2\right)^{1/2} + \|g\|_{H^1(B_1^+)}\right).$$

Therefore for some constant $C > 0$ and arbitrary $\delta > 0$

$$\begin{aligned} \|u\|^2_{H^1(B_1^+)} &\leq C\left(E_\varepsilon(u) + \|f\|_{L^\infty} \|u\|_{L^1(B_1')} + \|g\|^2_{H^1(B_1^+)}\right) \\ &\leq C\left(E_\varepsilon(u) + \frac{1}{4\delta}\|f\|^2_{L^\infty} + \delta\|u\|^2_{L^1(B_1')} + \|g\|^2_{H^1(B_1^+)}\right). \end{aligned}$$

By trace theorem, for sufficiently small $\delta > 0$, one obtains for all $u \in H^1_g(B_1^+)$

$$C\delta\|u\|^2_{L^1(B_1')} \leq \frac{1}{2}\|u\|^2_{H^1(B_1^+)}.$$

This proves the inequality (3.3.7).

By standard theory of Γ-convergence [41, Theorem 2.10], we obtain that the global minimizers u^ε of E_ε converge weakly in $H^1_g(B_1^+)$ to the unique global minimizer u of E_0, which solves the Neumann problem (3.3.4). To show the uniform convergence of u^ε on $\overline{B_1^+}$ we consider the classical solution h satisfying

$$\begin{cases} \Delta h = 0 & \text{in } B_1^+ \\ h = g & \text{on } \partial^+ B_1^+ \\ \partial_1 h = 0 & \text{on } B_1'. \end{cases}$$

The existence of $h \in C(\overline{B_1^+}) \cap C^{2,\alpha}(B_1^+ \cup B_1')$ is obtained by using Perron's method and standard elliptic regularity theory. Now $u^\varepsilon - h$ satisfies the conditions in Lemma 3.9.1 and $u^\varepsilon - h = 0$ on $\partial^+ B_1^+$, therefore $u^\varepsilon - h$ has uniform bounded $C^\alpha(B_1^+)$-norm, which implies that u^ε converges uniformly on $\overline{B_1^+}$ to u by Arzela-Ascoli theorem.

□

3.3.2 Plane-like correctors

The global minimizer theory does not capture the homogenization of all the local minimizers or steady states nor does it capture the macroscopic evolution in parabolic flow. Instead of a single large-scale slope specified by the average value of f, there is an interval of pinned slopes. The pinning interval is described via a (global) corrector problem which we describe next.

Definition 3.3.3. Call $v \in C(\mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+)$ a *corrector* if it solves

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^d_+ \\ \partial_1 v = f(x', v) & \text{on } \mathbb{R}^d_+, \end{cases} \tag{3.3.8}$$

and

$$\sup_{\mathbb{R}^d_+} |v(x) - (\mu, p)\cdot x| < \infty,$$

for some $(\mu, p) \in \mathbb{R} \times \mathbb{R}^{d-1}$. We call (μ, p) the *effective slope* of v.

Correctors are one of the fundamental concepts in the theory of homogenization. In classical homogenization theory the corrector equation is invariant with respect to "vertical" translations – i.e. adding a constant. In interface problems, such as we consider here, the vertical translation invariance is lost and many new challenges arise. Such challenges have been encountered and addressed before in several related models on homogenization of moving interfaces [44, 46, 67, 69, 101, 136, 145]. The techniques and ideas trace back to the the fundamental contributions of Aubry and LeDaeron [21] and Mather [120], which is now often called Aubry-Mather theory.

It actually fits better with the philosophy of viscosity solutions to split the notion of corrector into two, a subsolution and a supersolution notion.

Definition 3.3.4. Call $v \in \mathrm{LSC}(\mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+; (-\infty, +\infty])$ a *supercorrector* with effective slope $(\mu, p) \in \mathbb{R} \times \mathbb{R}^{d-1}$ if

(i) v is a viscosity supersolution to (3.3.8)

(ii) There exists $C > 0$ so that

$$v \geq (\mu, p) \cdot x - C \ \text{ in } \ \mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+.$$

(iii) $v(0) < +\infty$.

We also call the triple (v, μ, p) a supercorrector.

Definition 3.3.5. Call $v \in \mathrm{USC}(\mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+; [-\infty, +\infty))$ a *subcorrector* with effective slope $(\mu, p) \in \mathbb{R} \times \mathbb{R}^{d-1}$ if

(i) v is a viscosity subsolution to (3.3.8)

(ii) There exists $C > 0$ so that

$$v \leq (\mu, p) \cdot x + C \ \text{ in } \ \mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+.$$

(iii) $v(0) > -\infty$.

We also call the triple (v, μ, p) a subcorrector.

Remark 3.3.6. These definitions are exactly suited to the perturbed test function type argument for homogenization, found below in Theorem 3.3.17. The finiteness at the origin and the upper / lower linear bounds will come naturally from the touching test function.

We define the homogenized coefficients

$$Q_*(p; f) := \inf\{\mu_+ \ ; (u, \mu_+, p) \text{ is a subcorrector to (3.3.8)}\}, \tag{3.3.9}$$

and

$$Q^*(p; f) := \sup\{\mu_- \ ; (u, \mu_-, p) \text{ is a supercorrector to (3.3.8)}\}. \tag{3.3.10}$$

We will usually drop the dependence on f and write $Q^*(p)$ or $Q_*(p)$. The interval $[Q_*(p), Q^*(p)]$ is called the pinning interval.

Let us make a couple of remarks about invariance properties of these definitions.

Remark 3.3.7. We remark here that the homogenized coefficients $Q_*(p; f)$ and $Q^*(p; f)$ are invariant under the translation $f(x', v) \mapsto f(x'+n', v+n_1)$ for some vector $(n', n_1) \in \mathbb{R}^{d-1} \times \mathbb{R}$. This is because given a subcorrector (u, μ, p) that corresponds to f, we can define $v(x_1, x') = u(x_1, x' + n') - n_1$ and obtain (v, μ, p) a subcorrector that corresponds to $f(x' + n', v + n_1)$. The supercorrector case is symmetrical.

Remark 3.3.8. The homogenized coefficients satisfy $Q^*(p; f - a) = Q^*(p; f) - a$ and $Q_*(p; f - a) = Q_*(p; f) - a$ for any constant a because there is a 1-1 correspondence between the semi-correctors corresponding to f and those corresponding to $f - a$ by subtracting ax_1. This fact can be used to normalize $\langle f \rangle = 0$, as we will do for convenience later.

Our main result of this section establishes some important properties of the pinning interval and shows the existence of correctors. The correctors, by construction, will satisfy a certain periodicity property.

Definition 3.3.9. We call a function v defined on the strip $\{0 \le x_1 \le T\}$ for some $T \in (0, \infty]$ to satisfy the *Birkhoff property* if for any $(k, s_1), (k, s_2) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$ such that $k \cdot p - s_1 \le 0 \le k \cdot p - s_2$, we always have

$$v(x + k) - s_1 \le v(x) \le v(x + k) - s_2.$$

In particular, we have for any $k \in \mathbb{R}^{d-1}$

$$v(x + k) - \lceil k \cdot p \rceil \le v(x) \le v(x + k) - \lfloor k \cdot p \rfloor. \tag{3.3.11}$$

Theorem 3.3.10. *The functions Q_* and Q^* as defined in* (3.3.9) *and* (3.3.10) *are lower and upper semicontinuous respectively on $\mathbb{R}^{d-1}$, and for all $p \in \mathbb{R}^{d-1}$*

$$\min f \le Q_*(p) \le \langle f \rangle \le Q^*(p) \le \max f.$$

Moreover, for any $\mu \in [Q_*(p), Q^*(p)]$, there exists a corrector (u, μ, p) such that u also satisfies the Birkhoff property (3.3.11) *and there is a constant $C > 0$ so that*

$$\sup_{x_1 \ge 0} |u(x) - \mu x_1 - p \cdot x'| \le C, \tag{3.3.12}$$

where C depends only on dimension d and f.

We decompose the proof of Theorem 3.3.10 into four technical lemmas. We first establish the following bound for the homogenized coefficients Q_* and Q^*.

Lemma 3.3.11. *For every tangent vector $p \in \mathbb{R}^{d-1}$ both $Q_*(p)$ and $Q^*(p)$ are contained in the interval $[\min f, \max f]$.*

Proof. Note that, $p \cdot x + (\max f)x_1$ is a subcorrector and $p \cdot x + (\min f)x_1$ is a supercorrector. Thus the sets in the definitions (3.3.9) and (3.3.10) are non-trivial, and the upper bound $Q_*(p) \le \max f$ and the lower bound $\min f \le Q^*(p)$ are established.

For the other side we only show that $Q^*(p) \le \max f$ as the case for Q_* is symmetrical. Suppose there is a supercorrector (v, μ, p) such that $\mu > \max f$. We pick $T > 0$ large and $\mu - \max f \gg \eta > 0$ small, to be specified below, and consider for some real number s the following auxiliary harmonic function

$$\phi_s(x_1, x') := (\max f + \eta)x_1 + p \cdot x' - \eta|x'|^2 + \eta(d-1)x_1^2 + s + v(0).$$

Notice that v is a supercorrector so for some constant C we have

$$v(x_1, x') \ge \mu x_1 + p \cdot x' + C.$$

Choosing $T > 0$ large enough, we have

$$(\max f)T + p \cdot x' + v(0) < \mu T + p \cdot x' + C \le v(T, x').$$

This implies that for small $\eta > 0$

$$\phi_0(T, x') = (\max f + \eta)T + \eta(d-1)T^2 + p \cdot x' - \eta|x'|^2 + v(0) < v(T, x').$$

Also because v is lower bounded by $\mu x_1 + p \cdot x' + C$, there is a large radius $R > 0$ independent of s such that

$$\phi_s(x_1, x') \le \phi_0(x_1, x') < v(x_1, x')$$

for all $0 \le x_1 \le T$, $s \le 0$ and $|x'| \ge R$. Let s_* be the supremum of s such that $\phi_s < v$ on $\{0 \le x_1 \le T, |x'| \le R\}$, then $s_* \le 0$ because $\phi_0(0) = v(0)$ and $s_* > -\infty$ because v is lower bounded in $\{0 \le x_1 \le T, |x'| \le R\}$. On the other hand, by the harmonicity of ϕ_{s_*} and superharmonicity of v, the function ϕ_{s_*} will touch v from below at a point $x_0' \in \{x_1 = 0, |x'| \le R\}$. This establishes a contradiction to the supercorrector condition of v at $x = x_0'$ as $\partial_1 \phi_{s_*}(x_0') > \max f$.

□

Lemma 3.3.12. *The functions Q_* and Q^* as defined in* (3.3.9) *and* (3.3.10) *are lower and upper semicontinuous respectively on $\mathbb{R}^{d-1}$.*

Proof. We only focus on Q_* as the case of Q^* is symmetrical. Let $p_n \to p \in \mathbb{R}^{d-1}$ be a converging sequence of tangential vectors and $(V_n, Q_*(p_n) + \varepsilon_n, p_n)$ a corresponding sequence of subcorrectors to (3.3.13) with $0 < \varepsilon_n \to 0$. Notice that for any $C \in \mathbb{Z}$, we have

$$(V_n + C, Q_*(p_n) + \varepsilon_n, p_n)$$

is still a subcorrector, and therefore for each n, we can choose appropriate $C_n \in \mathbb{Z}$ such that $\tilde{V}_n := V_n + C_n$ are bounded from above by $p_n \cdot x + (Q_*(p_n) + \varepsilon_n)\, x_1 + 2$, which means that $\tilde{V}_n$ are locally uniformly bounded from above. This shows that if we write

$$W := \limsup_{n\to\infty}{}^* \tilde{V}_n \quad \text{and} \quad \tau = \liminf_{n\to\infty} Q_*(p_n),$$

we obtain a triple (W, τ, p) that is a subcorrector By Lemma 3.2.3. By definition of Q_* this shows that

$$\liminf_{n\to\infty} Q_*(p_n) = \tau \ge Q_*(p).$$

□

Next we show that the energy minimizing slope is always pinned. The proof uses the Γ-convergence established in Lemma 3.3.2.

Lemma 3.3.13. *For all* $p \in \mathbb{R}^{d-1}$

$$Q_*(p) \le \langle f \rangle \le Q^*(p).$$

Proof. We without loss normalize $\langle f \rangle = 0$ by applying Remark 3.3.8. It then suffices to show that $0 \in [Q_*(p), Q^*(p)]$. We first find the global minimizer u_ε to the energy (3.3.2) at scale $\varepsilon > 0$ with boundary data $g = q \cdot x$ on $\partial^+ B_1^+ = \partial B_1 \cap \{x_1 \ge 0\}$ for some tangent vector $q \in \mathbb{R}^{d-1}$. By Lemma 3.3.2, we know that u_ε converges locally uniformly to u_q solving

$$\begin{cases} \Delta u_q = 0 & \text{in } B_1^+ \\ u_q = q \cdot x & \text{on } \partial^+ B_1^+ \\ \partial_1 u_q = \langle f \rangle = 0 & \text{on } B_1'. \end{cases}$$

By standard elliptic theory we know that $u_q(x) \equiv q \cdot x$.

We now define for any small $\delta > 0$ the following harmonic function

$$\phi(x) = q \cdot x + \delta x_1 + \frac{\delta^2}{d-1}|x'|^2 - \delta^2 x_1^2.$$

This function touches u_q strictly from above at $x = 0$ with $\partial_1 \phi(0) = \delta > 0$ and by local uniform convergence of u_ε and Lemma 3.2.3, we can find small constants $C_\varepsilon = o_\varepsilon(1)$ such that $\phi_\varepsilon = \phi + C_\varepsilon$ touches u_ε strictly from above at some $y_\varepsilon \in B_1'$ with $|y_\varepsilon| = o_\varepsilon(1)$. Consider

$$v_\varepsilon(y) = \frac{u_\varepsilon(\varepsilon y + y_\varepsilon) - u_\varepsilon(y_\varepsilon)}{\varepsilon},$$

and

$$\psi_\varepsilon(y) = \frac{\phi_\varepsilon(\varepsilon y + y_\varepsilon) - \phi_\varepsilon(y_\varepsilon)}{\varepsilon}.$$

By smoothness of ϕ, we know that

$$v_\varepsilon(y) \le \psi_\varepsilon(y) \le \nabla\phi(0) \cdot y + o_\varepsilon(1)$$

is locally bounded from above. Therefore for a choice of subsequence ε_j so that

$$s := \lim_{j \to \infty} \left(\frac{u_{\varepsilon_j}(x_{\varepsilon_j})}{\varepsilon_j} - \left\lfloor \frac{u_{\varepsilon_j}(x_{\varepsilon_j})}{\varepsilon_j} \right\rfloor \right) \in [0, 1]$$

exists, we obtain

$$v_0 := \limsup_{j \to \infty}{}^* v_{\varepsilon_j} - s$$

is a well-defined subcorrector to (3.3.8) with upper bound $q \cdot y + \delta y_1$, which also touches v_0 from above at $x = 0$. Since $\partial_1 \phi(0) = \delta$, we know that $\delta \geq Q_*(q)$. As $\delta > 0$ can be chosen arbitrarily small we have shown that

$$Q_*(q) \leq 0.$$

Similar arguments show that $0 \leq Q^*(q)$.

□

Now we proceed to the construction of the correctors. It is convenient to consider the following equation truncated in $\{0 \leq x_1 \leq T\}$ for some $T > 0$

$$\begin{cases} \Delta v = 0 & \text{in } \{0 < x_1 < T\} \\ v = p \cdot x & \text{on } \{x_1 = T\} \\ \partial_1 v = f(x', v) & \text{on } \{x_1 = 0\}. \end{cases} \tag{3.3.13}$$

We take v_+^T to be the maximal subsolution defined as

$$v_+^T(x) = \sup\{v(x) \ ; \ v \text{ is a subsolution to (3.3.13)}\}. \tag{3.3.14}$$

and similarly

$$v_-^T(x) = \inf\{v(x) \ ; \ v \text{ is a supersolution to (3.3.13)}\}. \tag{3.3.15}$$

Remark 3.3.14. In general, the solutions v_+^T and v_-^T are not periodic functions but they do satisfy the Birkhoff property as in Definition 3.3.9. This is because when $k \cdot p - s \leq 0$ with $(k, s) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$, the function $v(x + k) - s$ is a subsolution to (3.3.13) whenever v is a subsolution, which shows that

$$v_+^T(x + k) - s \leq v_+^T(x).$$

Symmetrically we have $v_+^T(x) \leq v_+^T(x + k) - s$ whenever $k \cdot p - s \geq 0$. An analogous argument shows that v_-^T also satisfies the Birkhoff property.

For the convenience of the following discussions, we denote $\mu_+^T \geq -\|f\|_{L^\infty}$ as follows

$$\mu_+^T := \sup\{\mu \ ; \ p \cdot x + \mu(x_1 - T) \geq v_+^T\}, \tag{3.3.16}$$

and $\mu_-^T \leq \|f\|_{L^\infty}$ to be

$$\mu_-^T := \inf\{\mu \ ; \ p \cdot x + \mu(x_1 - T) \leq v_-^T\}. \tag{3.3.17}$$

By a similar argument in the proof of Lemma 3.3.11, both $|\mu_\pm^T|$ are bounded uniformly by $\|f\|_{L^\infty}$ for all $T > 0$.

Notice that by (3.3.16), the functions $v_+^T + \lfloor \mu_+^T T \rfloor$ are uniformly bounded from above by the linear function $p \cdot x + \mu_+^T x_1$. We define

$$V_+ := \limsup_{T\to\infty}{}^* (v_+^T + \lfloor \mu_+^T T \rfloor) \quad \text{and} \quad m_+ := \limsup_{T\to\infty} \mu_+^T, \tag{3.3.18}$$

and symmetrically

$$V_- := \liminf_{T\to\infty}{}_* (v_-^T + \lceil \mu_-^T T \rceil) \quad \text{and} \quad m_- := \liminf_{T\to\infty} \mu_-^T. \tag{3.3.19}$$

By Lemma 3.2.3, (V_+, m_+, p) is a subcorrector and (V_-, m_-, p) is a supercorrector to (3.3.8).

Lemma 3.3.15. *For any $\mu \in [Q_*(p), Q^*(p)]$, there exists a corrector (u, μ, p) such that u also satisfies the Birkhoff property* (3.3.11) *and there is a constant $C > 0$ so that*

$$\sup_{x_1 \geq 0} |u(x) - \mu x_1 - p \cdot x'| \leq C, \tag{3.3.20}$$

where C depends only on dimension d and f.

The proof of this lemma also shows the following result.

Corollary 3.3.16. *The truncated solutions $v_+^T + \lfloor \mu_+^T T \rfloor$ and $v_-^T + \lceil \mu_-^T T \rceil$ in* (3.3.18) *and* (3.3.19) *converges locally uniformly on $\mathbb{R}_+^d \cup \partial \mathbb{R}_+^d$ to V_+, V_- respectively.*

Proof of Lemma 3.3.15. Let us first show the special case when $\mu = Q_*(p)$. The case for $\mu = Q^*(p)$ is symmetrical. We make the following two claims on (V_+, m_+, p) as defined in (3.3.18)

(A) $m_+ = Q_*(p)$;

(B) (V_+, m_+, p) is a corrector to (3.3.8) that satisfies the bound (3.3.20).

To show claim (A), we observe that for a subcorrector (u, μ, p) to (3.3.8) and all $C \in \mathbb{Z}$ such that

$$C > \sup_{\mathbb{R}_+^d} u(x) - (\mu, p) \cdot x,$$

we have, by maximality of v_+^T, for some $s \in [0,1)$

$$v_+^T \geq u - C - \mu T - s,$$

where s is taken so that $s + \mu T$ is an integer. This implies, by (3.3.16), that for every $T > 0$ and $0 \leq x_1 \leq T$

$$p \cdot x + \mu_+^T x_1 \geq u - C - \mu T + \mu_+^T T - s.$$

At $x_1 = 0$ we have

$$\mu_+^T \leq \mu + \frac{C - u(0) + 1}{T},$$

which implies that

$$m_+ = \limsup_{T \to \infty} \mu_+^T \leq \mu.$$

for every subcorrector (u, μ, p). Combining the fact that (V_+, m_+, p) is also a subcorrector, we have finished the proof of Claim (A).

To show claim (B), we first observe that $V_+ \leq p \cdot x + \mu_+^T x_1$ is defined by the upper half relaxed limit of the truncated subcorrectors $v_+^T + \lfloor \mu_+^T T \rfloor$. For notational convenience, we denote

$$U_T(x) := v_+^T + \lfloor \mu_+^T T \rfloor - p \cdot x - \mu_+^T x_1 \quad \text{and} \quad c_T = \lfloor \mu_+^T T \rfloor - \mu_+^T T.$$

By Lemma 3.9.1 and Lemma 3.9.3, U_T is a classical solution of

$$\begin{cases} \Delta U_T = 0 & \text{in } \{0 < x_1 < T\} \\ U_T = c_T & \text{on } \{x_1 = T\} \\ \partial_1 U_T = f(x', U_T + p \cdot x) - \mu_+^T & \text{on } \{x_1 = 0\}. \end{cases} \tag{3.3.21}$$

Notice that $U_T \leq 0$ according to the definition of μ_+^T in (3.3.16).

To show that V_+ is also a supercorrector, we just need to show a uniform lower bound on U_T independent of $T > 0$.

By Remark 3.3.14 and the Birkhoff property (3.3.11), we know that for $k \in \mathbb{Z}^{d-1}$

$$U_T(x+k) + k \cdot p - \lceil k \cdot p \rceil \leq U_T(x) \leq U_T(x+k) + k \cdot p - \lfloor k \cdot p \rfloor, \tag{3.3.22}$$

and so $|U_T(x+k) - U_T(x)| \leq 1$ for all $x \in \{0 \leq x_1 \leq T\}$, $k \in \mathbb{Z}^{d-1}$. We claim that the uniform boundedness follows if there is a constant $C > 0$ depending only on dimension d and f so that for all $T > 0$ there is a point $x'_T \in \{x_1 = 0\}$ such that

$$\operatorname*{osc}_{x_1 = 0, |x' - x'_T| \leq 2} U_T \leq C. \tag{3.3.23}$$

This is because U_T is harmonic in $\{0 < x_1 < T\}$, bounded on $\{x_1 = T\}$, and if (3.3.23) holds then combining the Birkhoff property (3.3.22) we know that U_T is uniformly bounded on $\{0 \le x_1 \le T\}$ independent of $T > 0$.

Now we return to prove (3.3.23). By the definition μ_+^T, for all $T > 0$ there exists a point $x'_T \in \{x_1 = 0\}$ such that

$$U_T(x'_T) \ge -3.$$

Now we solve for an auxiliary function η satisfying

$$\begin{cases} \Delta\eta = 0 & \text{in } \{0 < x_1 < 2, |x' - x'_T| < 5\} \\ \eta = 0 & \text{on } \partial^+\{0 < x_1 < 2, |x' - x'_T| < 5\} \\ \partial_1\eta = \partial_1 U_T & \text{on } \{x_1 = 0, |x' - x'_T| < 5\}. \end{cases}$$

Notice that $|\partial_1 U_T| \le 5\,\|f\|_{L^\infty}$, which means that η is uniformly bounded independent of T. On the other hand we have by Harnack inequality there is a constant $C > 0$ depending on dimension such that

$$\max_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} (\eta - U_T) \le C \min_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} (\eta - U_T),$$

as $\eta - U_T$ is a positive harmonic function on $\{0 < x_1 < 2, |x' - x'_T| < 5\}$ satisfying zero Neumann boundary condition on $\{x_1 = 0\}$. This implies that

$$\operatorname*{osc}_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} U_T \le C\left[\max_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} U_T + \max_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} |\eta|\right],$$

on left hand side of which we have

$$\operatorname*{osc}_{\{x_1=0, |x'-x'_T|\le 2\}} U_T \le \operatorname*{osc}_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} U_T,$$

which implies that

$$\operatorname*{osc}_{\{x_1=0, |x'-x'_T|\le 2\}} U_T \le C\left[3 + \max_{\{0\le x_1\le 1, |x'-x'_T|\le 2\}} |\eta|\right] \le C'$$

for some positive constant $C' > 0$ independent of T. Combining Lemma 3.9.1 and 3.9.3, this shows that the limit V_+ is a classical solution to (3.3.8) with the distance from $Q_*(p)x_1 + p \cdot x$ uniformly bounded by a constant $C > 0$ depending only on dimension d and f.

For $\mu \in [Q_*(p), Q^*(p)]$, we consider for some constant $c \in \mathbb{Z}$ the supercorrector

$$W^\mu(x) = V_-(x) - (Q^*(p) - \mu)x_1 + c$$

and the subcorrector $W_\mu \leq W^\mu$ defined as

$$W_\mu(x) = V_+(x) + (\mu - Q_*(p))x_1.$$

The existence of a corrector (V_μ, μ, p) with $W_\mu \leq V_\mu \leq W^\mu$ follows from the Perron's method. As V_μ is defined by taking maximal values of subsolutions (or minimal values of supersolutions) in between W_μ, W^μ, we know that V_μ also satisfies the Birkhoff property as in Definition 3.3.9 by a similar argument in Remark 3.3.14. □

Proof of Theorem 3.3.10. By Lemma 3.3.11, Lemma 3.3.12, Lemma 3.3.13 and Lemma 3.3.15, the proof is complete. □

3.3.3 Homogenization of general viscosity solutions

Using the correctors and the pinning interval defined in the previous subsection, we can now present a homogenization result for general viscosity solutions.

Theorem 3.3.17. *Let $\varepsilon_k \to 0$ and u_k be a sequence of solutions to*

$$\begin{cases} \Delta u_k = 0 & \text{in } B_1^+ \\ \partial_1 u_k = f(x'/\varepsilon_k, u_k/\varepsilon_k) & \text{on } B_1' \end{cases} \tag{3.3.24}$$

such that $u_k \to u$ locally uniformly in $B_1^+ \cup B_1'$. Then u solves

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ Q_*(\nabla' u) \leq \partial_1 u \leq Q^*(\nabla' u) & \text{on } B_1', \end{cases} \tag{3.3.25}$$

where the pinning interval $[Q_*(p;f), Q^*(p;f)]$ is defined in (3.3.9) *and* (3.3.10).

Proof of Theorem 3.3.17. We only show the supersolution condition, the subsolution condition is proved by a symmetrical argument.

Let ϕ be any smooth function that touches the limit function u from below at some $x_0 \in B_1'$, $\Delta\phi(x_0) > 0$ and $u \geq \phi$ in $\overline{B_\delta^+(x_0)}$ for some $\delta < 1 - |x_0|$. Then by Lemma 3.2.3 there exists a sequence of points $x_k \to x_0$ and constants $c_k \to 0$ as $k \to \infty$ such that $\phi_k := \phi + c_k$ touches u_k from below at $x_k \in B_1'$ for each k, and $u_k \geq \phi_k$ in $\overline{B_{\delta/2}^+(x_k)}$.

By choosing subsequences, we can further assume the existence of

$$s = \lim_{k\to\infty} (u_k(x_k)/\varepsilon_k - \lfloor u_k(x_k)/\varepsilon_k \rfloor) \in [0,1].$$

Let

$$v_k(x) := \frac{u_k(\varepsilon_k x + x_k) - u_k(x_k)}{\varepsilon_k} \quad \text{and} \quad \psi_k(x) := \frac{\phi_k(\varepsilon_k x + x_k) - \phi_k(x_k)}{\varepsilon_k}.$$

By smoothness of ϕ_k near x_0, we know that for any fixed $R > 0$, the sequence of functions v_k is bounded from below in $\overline{B_R^+}$ because

$$v_k(x) \geq \psi_k(x) \geq \nabla\phi_k(0) \cdot x + o_k(1).$$

Therefore the lower half relaxed limit of v_k

$$v_*(x) := \liminf_{k\to\infty}{}_* v_k(x)$$

exists as a lower-semicontinuous function $\mathbb{R}^d_+ \cup \partial\mathbb{R}^d_+ \to (-\infty, +\infty]$. Notice that for sufficiently large k, v_k is by assumption a supersolution to

$$\partial_1 v_k \leq f(v_k + u_k(x_k)/\varepsilon_k) \text{ on } B'_R$$

which by Lemma 3.2.3 implies that

$$\partial_1 v_* \leq f(v_* + s)$$

in the viscosity sense on $\{x_1 = 0\}$. On the other hand, we know that the linear function $l(x) := \nabla\phi(x_0) \cdot x$ touches v_* from below at exactly $x = 0$. This shows that $(v_*, \partial_1\phi(x_0), \nabla'\phi(x_0))$ is a supercorrector to (3.3.8), and the proof is complete by applying Theorem 3.3.10.

□

3.3.4 The homogenized coefficients in the laminar case

Let us now compute the precise coefficients Q_* and Q^* in the special case that f is laminar, i.e. $f(x', u) = f(u)$ is a 1-periodic C^α function only of $u \in \mathbb{R}$. The corrector equation becomes

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^d_+, \\ \partial_1 v = f(v) & \text{on } \mathbb{R}^{d-1}, \end{cases} \tag{3.3.26}$$

satisfying

$$\sup_{\mathbb{R}^d_+} |v(x) - (\mu, p) \cdot x| < \infty,$$

for some $(\mu, p) \in \mathbb{R} \times \mathbb{R}^{d-1}$. By Remark 3.3.8, we assume

$$\langle f \rangle = \int_0^1 f(v)dv = 0.$$

Lemma 3.3.18. *Suppose* $f(x', v) \equiv f(v)$ *is 1-periodic in* v *and* $f \in C^{\alpha}(\mathbb{R})$ *for some* $\alpha > 0$*, then*

$$Q_*(p) = L_*(p) := \begin{cases} \min f & \text{if } p = 0, \\ \langle f \rangle = 0 & \text{if } p \neq 0, \end{cases} \tag{3.3.27}$$

and symmetrically

$$Q^*(p) = L^*(p) := \begin{cases} \max f & \text{if } p = 0, \\ \langle f \rangle = 0 & \text{if } p \neq 0, \end{cases} \tag{3.3.28}$$

where $\langle f \rangle = \int_0^1 f(r)dr$*.*

By Corollary 3.3.16, it suffices to analyze the truncated versions (3.3.13), which in the current case takes the form

$$\begin{cases} \Delta v = 0 & \text{in } \{0 < x_1 < T\} \\ v = p \cdot x & \text{on } \{x_1 = T\} \\ \partial_1 v = f(v) & \text{on } \{x_1 = 0\}. \end{cases} \tag{3.3.29}$$

Let v_+^T and v_-^T be the maximal subsolutions and minimal supersolutions respectively. We notice that if $p = 0$, the extremal solutions v_+^T and v_-^T are tangentially invariant linear functions. It suffices to solve for 1-variable solution $v = v(x_1)$ that satisfies

$$\begin{cases} v''(x_1) = 0 & \text{for } x_1 \in (0, T) \\ v(T) = 0 & v'(0) = f(v(0)). \end{cases}$$

Notice that a solution will take the form $v(x_1) = -\frac{v(0)}{T}(x_1 - T)$, where $v(0)$ satisfies

$$-\frac{v(0)}{T} = f(v(0)). \tag{3.3.30}$$

Let $v(0) = w_+^T$ be the maximal solution and $v(0) = w_-^T$ be the minimal solution to the above equation, then we know that

$$v_\pm^T(x_1, x') = -\frac{w_\pm^T}{T}(x_1 - T).$$

On the other hand, we have the following characterization of $w_\pm^T$.

Lemma 3.3.19. *Suppose* f *is continuous and 1-periodic in* (3.3.30)*, then as* $T \to \infty$*, we have*

$$\lim_{T\to\infty} f(w_+^T) = \lim_{T\to\infty} -\frac{w_+^T}{T} = \min f,$$

and

$$\lim_{T\to\infty} f(w_-^T) = \lim_{T\to\infty} -\frac{w_-^T}{T} = \max f.$$

Proof. We only show the w_+^T case as the w_-^T case is symmetrical. On one hand, we know that

$$-\frac{w_+^T}{T} = f(w_+^T) \geq \min f,$$

and hence

$$w_+^T \leq -(\min f)T.$$

Because $-\frac{v}{T}$ decays linearly and f remains bounded, this shows that when $v > -(\min f)T$, we always have $-\frac{v}{T} < f(v)$. On the other hand, when $-(\min f)T - 3 \leq v \leq -(\min f)T - 1$, we have

$$\min f + \frac{1}{T} \leq -\frac{v}{T} \leq \min f + \frac{3}{T}.$$

By the 1-periodicity of f there is a $v_* \in [-(\min f)T - 3, -(\min f)T - 1]$ such that $f(v_*) = \min f$. Now, we obtain the inequality

$$-\frac{v_*}{T} \geq \min f + \frac{1}{T} > f(v_*).$$

By using the intermediate value theorem, we know that there must be a number $\tilde{v} \geq v_* \geq -(\min f)T - 3$ such that (3.3.30) holds for $v(0) = \tilde{v}$. By maximality of w_+^T we have

$$w_+^T \geq \tilde{v} \geq -(\min f)T - 3.$$

This shows that

$$f(w_+^T) = -\frac{w_+^T}{T} \leq \min f + \frac{3}{T} \to \min f$$

as $T \to \infty$. □

For $p \neq 0$, because f is 1-periodic in u, $v_\pm^T(x - p/|p|^2) + 1 = v_\pm^T(x)$, with tangential translation invariance:

$$v_\pm^T(x - q) = v_\pm^T(x) \quad \text{for} \quad q \perp p \text{ and } e_1.$$

This implies $\eta := v_\pm^T - p \cdot x$ is a $\frac{1}{|p|}$-periodic 2-variable function, with continuity derived from $v_\pm^T(x) - v_\pm^T(x - sp/|p|^2) = s$. The problem (3.3.29) is thus reduced to:

$$\begin{cases} \partial_1^2 \eta + \partial_2^2 \eta = 0 & \text{in } \{0 < x_1 < T\}, \\ \eta = 0 & \text{on } \{x_1 = T\}, \\ \partial_1 \eta = f(\eta + |p| x_2) & \text{on } \{x_1 = 0\}, \end{cases} \tag{3.3.31}$$

with $\eta \in C(\{0 \le x_1 \le T\})$, $1/|p|$-periodic in x_2. By the C^α-regularity of f, we know that η is a classical solution to (3.3.31) according to Lemma 3.9.1 and 3.9.3.

Now we show that η has bounded distance (independent of T) from $\mu_+^T(x_1 - T)$ with

$$\lim_{T\to\infty} \mu_+^T = \langle f \rangle = 0,$$

where μ_+^T is defined in (3.3.16).

By Corollary 3.3.16 η has uniform distance independent of T from an affine function $\mu_+^T(x_1 - T)$ with μ_+^T having a limit $\mu_+^\infty = \lim_{T\to\infty} \mu_+^T$. Let

$$\langle \eta \rangle (x_1) := |p| \int_0^{1/|p|} \eta(x_1, x_2) dx_2,$$

we have, by (3.3.31),

$$\langle \eta \rangle (x_1) = |p| \int_0^{1/|p|} f(\eta(0, x_2) + |p| x_2) dx_2 (x_1 - T).$$

By Lemma 3.3.15 and Corollary 3.3.16, we have

$$\mu_+^\infty = \lim_{T\to\infty} \mu_+^T = \lim_{T\to\infty} |p| \int_0^{1/|p|} f(\eta(0, x_2) + |p| x_2) dx_2.$$

Lemma 3.3.20. *For all $T > 0$ and any viscosity solution η to* (3.3.31)*, we have*

$$|p| \int_0^{1/|p|} f(\eta(0, x_2) + |p| x_2) dx_2 = \langle f \rangle = 0,$$

where $\langle f \rangle := \int_0^1 f(z) dz = 0$.

Proof. By applying $\partial_2 \eta$ to (3.3.31) we have

$$\begin{aligned}
0 &= \int_0^T \int_0^{1/|p|} \partial_2 \eta(x_1, x_2) \Delta \eta(x_1, x_2) dx_2 dx_1 \\
&= -\int_0^T \int_0^{1/|p|} \nabla \partial_2 \eta(x_1, x_2) \cdot \nabla \eta(x_1, x_2) dx_2 dx_1 - \int_0^{1/|p|} \partial_1 \eta(0, x_2) \partial_2 \eta(0, x_2) dx_2 \\
&= -\int_0^T \int_0^{1/|p|} \frac{1}{2} \partial_2 |\nabla \eta|^2 (x_1, x_2) dx_2 dx_1 - \int_0^{1/|p|} \partial_1 \eta(0, x_2) \partial_2 \eta(0, x_2) dx_2 \\
&= -\int_0^{1/|p|} \partial_1 \eta(0, x_2) \partial_2 \eta(0, x_2) dx_2.
\end{aligned}$$

On the other hand, by making the substitution

$$z = \eta(0, x_2) + |p| x_2 \quad \text{and} \quad dz = (\partial_2 \eta(0, x_2) + |p|) dx_2$$

we get

$$\begin{aligned}\langle f\rangle &= \int_0^1 f(z)dz\\ &= \int_0^{1/|p|} f(\eta(0,x_2)+|p|x_2)(\partial_2\eta(0,x_2)+|p|)dx_2\\ &= \int_0^{1/|p|} \partial_1\eta(0,x_2)\partial_2\eta(0,x_2)dx_2 + |p|\int_0^{1/|p|} f(\eta(0,x_2)+|p|x_2)dx_2\\ &= |p|\int_0^{1/|p|} f(\eta(0,x_2)+|p|x_2)dx_2.\end{aligned}$$

□

Proof of Lemma 3.3.18. The result follows from Lemma 3.3.19, Lemma 3.3.20. □

3.4 Homogenization of extremal solutions in laminar media

In this section, we study the homogenization of the extremal solutions, that is, the minimal supersolutions and maximal subsolutions of

$$\begin{cases}\Delta u = 0 & \text{in } B_1^+\\ \partial_1 u = f(u/\varepsilon) & \text{on } B_1'.\end{cases} \tag{3.4.1}$$

Definition 3.4.1. Given a boundary data $g \in C(\partial B_1 \cap \{x_1 \geq 0\})$ we define the *extremal solutions* of (3.4.1): the *minimal supersolution*

$$u^\varepsilon_{g,\min}(x) := \inf\{u(x) \ : \ u \text{ is a supersolution to (3.4.1) and } u \geq g \text{ on } \partial B_1 \cap \{x_1 \geq 0\}\},$$

and the *maximal subsolution*

$$u^\varepsilon_{g,\max}(x) := \sup\{u(x) \ : \ u \text{ is a subsolution to (3.4.1) and } u \leq g \text{ on } \partial B_1 \cap \{x_1 \geq 0\}\}.$$

We remark here that for each fixed ε the extremal solutions $u^\varepsilon_{g,\min/\max}$ are classical solutions to (3.4.1) according to Perron's method, Lemma 3.9.1 and Lemma 3.9.3.

We show that extremal solutions to (3.4.1) converge, respectively, to extremal solutions of

$$\begin{cases}\Delta u = 0 & \text{in } B_1^+\\ \partial_1 u \in [L_*(\nabla' u), L^*(\nabla' u)] & \text{on } B_1'.\end{cases} \tag{3.4.2}$$

Here the homogenized coefficients, as derived in Section 3.3, are

$$L_*(p) = (\min f)\,1_{\{p=0\}} + \langle f\rangle 1_{\{p\neq 0\}} \quad \text{and} \quad L^*(p) = (\max f)\,1_{\{p=0\}} + \langle f\rangle 1_{\{p\neq 0\}}. \tag{3.4.3}$$

To reduce notation, for the rest of the section we will keep the normalization

$$\langle f \rangle = 0.$$

This normalization can be achieved by adding an affine function as explained in Remark 3.3.8.

Definition 3.4.2. An upper semicontinuous function u is called a *subsolution* to (3.4.2) if for any smooth function ϕ touching u from above at some $x \in B_1^+ \cup B_1'$, either $\Delta\phi(x) \geq 0$ or $x \in B_1'$ and

$$\partial_1 \phi(x) \geq L_*(\nabla' \phi(x)).$$

Definition 3.4.3. A lower semicontinuous function u is called a *supersolution* to (3.4.2) if for any smooth function ϕ touching u from below at some $x \in B_1^+ \cup B_1'$, either $\Delta\phi(x) \leq 0$ or $x \in B_1'$ and

$$\partial_1 \phi(x) \leq L^*(\nabla' \phi(x)).$$

Remark 3.4.4. The following three viscosity inequality constraints are equivalent for a lower semicontinuous function u (the arguments are symmetric for upper semicontinuous functions):

(i) $\partial_1 u \leq L^*(\nabla' u)$.

(ii) $\partial_1 u \leq \max f,\ \min\{\partial_1 u, |\nabla' u|\} \leq 0$.

(iii) $\partial_1 u \leq \max f,\ |\nabla' u| \partial_1 u \leq 0$.

Definition 3.4.5. Given a boundary data $g \in C(\partial B_1 \cap \{x_1 \geq 0\})$ we define the *extremal solutions* of (3.4.2): the *minimal supersolution*

$$u_{g,\min}(x) := \inf\{u(x) \ :\ u \text{ is a supersolution to (3.4.2) and } u \geq g \text{ on } \partial B_1 \cap \{x_1 \geq 0\}\},$$

and the *maximal subsolution*

$$u_{g,\max}(x) := \sup\{u(x) \ :\ u \text{ is a subsolution to (3.4.2) and } u \leq g \text{ on } \partial B_1 \cap \{x_1 \geq 0\}\}.$$

Our main result of this section is the homogenization: extremal solutions of (3.4.1) converge to extremal solutions of (3.4.2).

Theorem 3.4.6. *Let $g \in C(\partial B_1 \cap \{x_1 \geq 0\})$ and $u^\varepsilon = u^\varepsilon_{g,\min}$ are minimal supersolutions to* (3.4.1) *with $u^\varepsilon = g$ on $\partial^+ B_1 = \partial B_1 \cap \{x_1 \geq 0\}$. Then*

$$u^\varepsilon \to u_{g,\min} \quad \textit{uniformly on } \overline{B_1^+},$$

where $u_{g,\min}$ is the minimal supersolution of (3.4.2) *with boundary data g on $\partial^+ B_1$. The analogous statement for maximal subsolutions holds by symmetry.*

Proof of Theorem 3.1.3. The proof is done by combining Remark 3.4.4, Theorem 3.4.6. □

Notice that it is difficult to directly check the minimality of the limit of $u^\varepsilon_{g,\min}$. To overcome this difficulty we propose an alternative characterization of minimal supersolutions to (3.4.2) by local viscosity solution testing properties. Such characterization appears also in [81, 82] and is a suitable criterion for using the test function type argument in nonlinear homogenization as discussed in [78].

3.4.1 A characterization of the extremal homogenized steady states in the laminar case

In this subsection, we introduce an equivalent characterization for the minimal supersolutions as defined in Definition 3.4.5. This characterization is for preparation of the nonlinear homogenization argument that will appear in Section 3.4.2

Theorem 3.4.7. *Let $\xi \in C(\overline{B_1^+})$ be a harmonic function on B_1^+ such that the following conditions hold on B_1':*

(1) $\min\{\partial_1 \xi, |\nabla' \xi|\} = 0$

(2) $\partial_1 \xi \leq \max f$

(3) If a smooth function of the form $\eta(x) \equiv \eta(x_1)$ touches ξ from above at $z \in B_1'$ and for some open domain $\Omega \subset\subset \mathbb{R}^d$ containing z such that $\Omega \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$ we have $\eta \geq \xi$ in $\Omega \cap \overline{B_1^+}$ and $\eta > \xi$ on $\partial\Omega \cap \overline{B_1^+}$, then $\partial_1 \eta(z) = \partial_1 \eta(0) \geq \max f$.

Then ξ is equal to the minimal supersolution $u_{\xi,\min}$ to (3.4.2) *that satisfies $u_{\xi,\min} = \xi$ on $\partial B_1 \cap \{x_1 \geq 0\}$.*

Before showing the theorem, let us introduce the following auxiliary lemma.

Lemma 3.4.8. *Given any supersolution v to* (3.4.2)*, there is a partition*

$$B_1' = \mathcal{C}_v \sqcup \mathcal{N}_v \sqcup \Gamma_v,$$

where

$$\mathcal{C}_v := \{x \in B_1' \;;\; \exists \text{ smooth } \phi \text{ touching } v \text{ from below at } x \text{ such that } \partial_1\phi(x) > 0\}$$

is relatively open in B_1', $\mathcal{N}_v := B_1' \setminus \overline{\mathcal{C}_v}$ and Γ_v the common boundary of $\mathcal{C}_v$ and $\mathcal{N}_v$.

Proof. Any such supersolution v is also a supersolution to

$$\min\{\partial_1 v, |\nabla' v|\} \le 0,$$

and hence we can apply the results in Section 2 in [82]. □

Proof of Theorem 3.4.7. We only discuss the minimal supersolution case, as the maximal subsolution case is symmetrical. We also denote by g the restriction of ξ on $\partial B_1 \cap \{x_1 \ge 0\}$.

Step 1: Minimal supersolution satisfies the three conditions. By Remark 3.4.4 and the standard Perron's method, the minimal supersolution u satisfies conditions (1) and (2). The condition (3) is carried out by a similar argument, but due to the unusual form, we still perform the argument here. To verify condition (3), assume there exists a smooth function $\eta_0(x) = \eta_0(x_1)$ such that:

$$\partial_1\eta_0(0) < \max f,$$

and η_0 touches $u = u_{g,\min}$ from above at a point $z \in \Omega$ in $\Omega \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$, with $\eta_0 > u$ on $\partial\Omega \cap \overline{B_1^+}$. Consider the function:

$$(\max f)x_1 + \eta_0(0) - \tau,$$

which is a supersolution to (3.4.2) for each constant τ. When $\tau = 0$, since $\partial_1\eta_0(0) < \max f$, there exists a small $\delta > 0$ such that:

$$(\max f)x_1 + \eta_0(0) > \eta_0(x_1) \quad \text{on} \quad 0 < x_1 \le \delta.$$

In particular, on $\partial\Omega \cap \{0 \le x_1 \le \delta\}$, we have:

$$(\max f)x_1 + \eta_0(0) \ge \eta_0(x_1) > u(x).$$

This implies:

$$(\max f)x_1 + \eta_0(0) > u \quad \text{on} \quad \partial\big(\Omega \cap \{x_1 < \delta\}\big) \cap \{x_1 \geq 0\}.$$

By continuity, we can choose $\tau_0 > 0$ small enough so that:

$$(\max f)x_1 + \eta_0(0) - \tau_0 > u \quad \text{on} \quad \partial\big(\Omega \cap \{x_1 < \delta\}\big) \cap \{x_1 \geq 0\}.$$

Taking the minimum of u and $(\max f)x_1 + \eta_0(0) - \tau_0$ inside $\Omega \cap \overline{B_1^+}$ yields a strictly smaller supersolution to (3.4.2) with boundary data g, contradicting the minimality of u. Thus, condition (3) holds.

Step 2: the three conditions imply minimality. To prove the reverse, we show the comparison principle between a supersolution v to (3.4.2) such that $v \geq g$ on $\partial B_1 \cap \{x_1 \geq 0\}$ and a subsolution ξ satisfying conditions (1) and (3). Following the argument in [82, Section 8.2], we use tangential convolutions and harmonic lifts (also see Appendix 3.9.2). These methods allow us to assume, v to be a supersolution, semiconvex on B_1', and harmonic in B_1^+, while ξ to be a subsolution, semiconcave on B_1', and harmonic in B_1^+. Specifically we know that v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } B_1^+ \\ v \geq g & \text{on } \partial B_1 \cap \{x_1 \geq 0\} \\ \partial_1 v \leq L^*(\nabla' v) = (\max f)1_{\{\nabla' v = 0\}} & \text{on } B_1', \end{cases} \tag{3.4.4}$$

and ξ satisfies condition (3) and

$$\begin{cases} \Delta \xi = 0 & \text{in } B_1^+ \\ \xi \leq g & \text{on } \partial B_1 \cap \{x_1 \geq 0\} \\ \partial_1 \xi \geq 0 & \text{on } B_1'. \end{cases} \tag{3.4.5}$$

We denote for $h > 0$ the strict perturbation

$$v_h := v - hx_1 + 2h.$$

Suppose v_h touches ξ from above at some point $x_0 \in B_1'$. By semi-convexity of v_h and semi-concavity of ξ, both v_h and ξ are differentiable at x_0. By condition (1), we have:

$$\partial_1 \xi(x_0) \geq 0,$$

which implies that

$$x_0 \notin \mathcal{N}_v \sqcup \Gamma_v \tag{3.4.6}$$

because in this case $\partial_1 v_h(x_0) < 0$ by (3.4.4) and Lemma 3.4.8. Let $\Omega \subset \mathcal{C}_v$ be the component containing x_0. We claim by condition (3), there must exist another touching point $y \in \partial'\Omega \cap B_1' \subset \Gamma_v \cap B_1'$. For $\delta, \varepsilon > 0$, define:

$$\Omega_{\delta,\varepsilon} := \left\{ x \in \Omega\,;\, \text{dist}(x, \partial'\Omega) > \varepsilon,\, \text{dist}(x, \partial' B_1') > \delta \right\}.$$

Because $v_h > \xi$ on $\partial B_1 \cap \{x_1 \geq 0\}$, we know that when $\delta, \varepsilon > 0$ are small we always have

$$\max_{\partial'\Omega_{\delta,\varepsilon}} (\xi - v_h) = \max_{(\partial'\Omega_{0,\varepsilon}) \cap B_{1-\delta}'} (\xi - v_h).$$

If $\max_{\partial'\Omega_{\delta,\varepsilon}} (\xi - v_h) < 0$ (i.e., no touching point y exists on $(\partial'\Omega_{0,\varepsilon}) \cap B_{1-\delta}'$), then because

$$\partial_1 v_h = \partial_1 v - h < \max f$$

and v is constant on Ω, we have for sufficiently small $\tau > 0$:

$$(\max f - \tau)x_1 + v(x_0) \geq v \quad \text{on} \quad \Omega_{\delta,\varepsilon} \times [0, \tau].$$

This produces a 1-variable function $(\max f - \tau/2)x_1 + v(x_0)$ that violates condition (3) on $\Omega_{\delta,\varepsilon} \times [0, \tau]$. Thus, for fixed small $\delta > 0$ and each small $\varepsilon > 0$, there exists

$$y = y_\varepsilon \in (\partial'\Omega_{0,\varepsilon}) \cap B_{1-\delta}'$$

such that ξ touches v from below at y_ε. Taking a subsequence as $\varepsilon \to 0$, we obtain a point $y_0 \in (\partial'\Omega) \cap B_{1-\delta}'$ where ξ touches v from below, which is a contradiction since we already showed above in (3.4.6) that there are no touching points in Γ_v. □

3.4.2 Homogenization of the minimal supersolutions

According to Theorem 3.3.17 and Theorem 3.4.7, to prove Theorem 3.4.6, we need to show that any locally uniform limit u of a subsequence $u_k := u^{\varepsilon_k}$ on $B_1^+ \cup B_1'$ is harmonic in B_1^+ and satisfies the three viscosity solution conditions in Theorem 3.4.7. We split the proof into two lemmas.

We first show that a minimal supersolution u satisfies the condition (1) in Theorem 3.4.7. This is essentially proving that

$$\partial_1 u \geq \langle f \rangle = 0,$$

which incorporates the effects of global minimizers to minimal supersolutions.

Lemma 3.4.9. *Let* $u_k := u^{\varepsilon_k}$ *be a sequence of minimal supersolutions to* (3.4.1) *with* $\varepsilon_k \to 0$ *and* $u_k \to u$ *locally uniformly in* $B_1^+ \cup B_1'$. *Then* u *satisfies condition (1) in Theorem 3.4.7.*

Remark 3.4.10. The proof of this lemma also shows that in the general $f(x', v)$ case (instead of merely this laminar case $f(v)$), the minimal supersolution u to the homogenized equation (3.3.1) satisfies

$$\langle f \rangle \leq \partial_1 u \leq Q^*(\nabla' u; f).$$

Proof. To show condition (1), by Theorem 3.3.17, Theorem 3.3.10, Lemma 3.3.18 and Remark 3.4.4, it suffices to show the subsolution condition

$$\partial_1 u \geq 0.$$

Suppose, for contradiction, that this is not the case. Then there exists a smooth function ϕ that touches u from above at some point $x_0 \in B_1'$, satisfying

$$\Delta\phi(x_0) < 0 \quad \text{and} \quad \partial_1\phi(x_0) < 0,$$

and $u \leq \phi$ in $\overline{B_\delta^+(x_0)}$ for some $0 < \delta < 1 - |x_0|$. By the smoothness of ϕ, we may choose δ small enough so that

$$\Delta\phi(x) < 0 \quad \text{and} \quad \partial_1\phi(x) < 0$$

for all $x \in \overline{B_\delta^+(x_0)}$.

Without loss of generality, we can make the touching to be strict by applying a small perturbation (e.g., $\phi + \tau|x - x_0|^2$ for a small $\tau > 0$) so that

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi > u \text{ in } \overline{B_\delta^+(x_0)} \setminus \{x_0\}. \tag{3.4.7}$$

By Lemma 3.2.3, there exist sequences of points $x_k \to x_0$ and constants $C_k \to 0$ as $k \to \infty$ such that

$$\phi_k := \phi + C_k$$

touches u_k from above at $x_k \in B_1'$ for each k, and $u_k \geq \phi_k$ in $\overline{B_{\delta/2}^+(x_k)}$.

Now, we solve the following global energy minimization problem:

$$\psi_k := \operatorname*{argmin}_{\psi} \left\{ E_{\varepsilon_k}\left(\psi, B_{\delta/2}^+(x_k)\right) \; ; \; \psi = \phi_k \text{ on } \partial B_{\delta/2}(x_k) \cap \{x_1 \geq 0\} \right\},$$

where E_{ε_k} is defined in (3.3.2). By Lemma 3.3.2, global minimizers

$$\psi_k \in C(\overline{B^+_{\delta/2}(x_k)}) \cap C^{1,\alpha}_{\mathrm{loc}}(B^+_{\delta/2}(x_k) \cup B'_{\delta/2}(x_k))$$

converge uniformly to ψ_0, solving

$$\begin{cases} \Delta\psi_0 = 0 & \text{in } B^+_{\delta/2}(x_0), \\ \psi_0 = \phi & \text{on } \partial B^+_{\delta/2}(x_0) \cap \{x_1 \geq 0\}, \\ \partial_1\psi_0 = 0 & \text{on } B'_{\delta/2}(x_0). \end{cases} \tag{3.4.8}$$

Since ϕ is a strict supersolution of (3.4.8), we obtain

$$\psi_0 < \phi \ \text{ in } \ \overline{B'_{\delta/4}(x_k)}$$

for all sufficiently large $k > 0$. By the uniform convergence of ψ_k, this implies

$$\psi_k < \phi_k \text{ in } \overline{B'_{\delta/4}(x_k)},$$

for all sufficiently large $k > 0$. This leads to a contradiction to the minimality of u_k if we consider the minimum of u_k and ψ_k, which produces a strictly smaller supersolution to (3.1.2). □

Lemma 3.4.11. *Let $u_k := u^{\varepsilon_k}$ be a sequence of minimal supersolutions to* (3.4.1) *(the ε-problem $\partial_1 u = f(u/\varepsilon)$) with $\varepsilon_k \to 0$ and $u_k \to u$ locally uniformly in $B^+_1 \cup B'_1$. Then u satisfies condition (3) in Theorem 3.4.7.*

Proof. To show condition (3), we assume there exists a smooth one-variable function $\eta = \eta(x_1)$ that touches u from above at a point $z \in B'_1$, satisfying

$$\partial_1\eta(0) < \max f,$$

and such that for some small open domain $\Omega \subset\subset \mathbb{R}^d$ containing z with $\Omega \cap \overline{B^+_1} \subset\subset B^+_1 \cup B'_1$, we have

$$\eta \geq u \text{ in } \Omega \cap \overline{B^+_1}, \ \eta(0) = u(z) \ \text{ and } \ \eta > u \text{ on } \partial\Omega \cap \overline{B^+_1}.$$

Notice that, since $\partial_1\eta(0) < \max f$, for small $\tau > 0$, the function

$$(\max f)x_1 + \eta(0) - \tau$$

satisfies

$$(\max f)x_1 + \eta(0) - \tau > u \text{ on } \partial(\Omega \cap \{x_1 < \tau\}) \cap \overline{B_1^+} =: \partial\Omega^\tau \cap \overline{B_1^+}.$$

Let $\tau \gg h > 0$ be small, and define the fattened domain

$$\Omega_h^\tau := \bigcup_{x \in \Omega^\tau} B_h(x).$$

By the continuity of u and $(\max f)x_1 + \eta(0) - \tau$, we also have

$$(\max f)x_1 + \eta(0) - \tau > u \text{ in } \overline{\Omega_h^\tau \setminus \Omega^\tau} \cap \overline{B_1^+}.$$

By the local uniform convergence of u_k to u and Lemma 3.2.3, for large k, there exist points $z_k \in \Omega \cap B_1'$ with $z_k \to z$ and constants $c_k \to 0$ such that

$$(\max f)x_1 + \eta(0) + c_k$$

touches u_k from above at z_k in $\overline{\Omega_h^\tau} \cap \overline{B_1^+}$, and

$$(\max f)x_1 + \eta(0) + c_k > u_k \text{ in } \overline{\Omega_h^\tau \setminus \Omega^\tau} \cap \overline{B_1^+}.$$

Moreover, for all $0 < \nu \le \tau/2$, the inequality

$$(\max f)x_1 + \eta(0) + c_k - \nu > u_k \tag{3.4.9}$$

still holds in $\overline{\Omega_h^\tau \setminus \Omega^\tau} \cap \overline{B_1^+}$. Since $\tau > 0$ is chosen independently of ε_k, for large k, we have $\tau \gg \varepsilon_k$, where ε_k is the period of $f_{\varepsilon_k} := f(\cdot/\varepsilon_k)$. This implies that we can choose $0 < \nu_k \le \tau/2$ such that

$$f_{\varepsilon_k}(\eta(0) + c_k - \nu_k) = \max f,$$

making

$$(\max f)x_1 + \eta(0) + c_k - \nu_k$$

a supersolution to (3.1.2) for $\varepsilon = \varepsilon_k$. This contradicts the fact that u_k's are minimal supersolutions because (3.4.9) and

$$\begin{aligned} u_k(z_k) &= \eta(0) + c_k \\ &> \eta(0) + c_k - \nu_k \end{aligned}$$

shows that the minimum of u_k and $(\max f)x_1 + \eta(0) + c_k - \nu_k$ is a supersolution strictly smaller than u_k.

□

Let us now prove Theorem 3.4.6.

Proof of Theorem 3.4.6. We first show that any sequence $u^{\varepsilon_l} = u^{\varepsilon_l}_{g,\min}$ of minimal supersolutions contains a subsequence u_k that converges uniformly on $\overline{B_1^+}$ to $u \in C(\overline{B_1^+})$. To see this, we let h to be the solution to

$$\begin{cases} \Delta h = 0 & \text{in } B_1^+ \\ h = g & \text{on } \partial B_1 \cap \{x_1 \geq 0\} \\ \partial_1 h = 0 & \text{on } B_1'. \end{cases}$$

By Perron's method and standard elliptic regularity theory

$$h \in C(\overline{B_1^+}) \cap C^{2,\alpha}(B_1^+ \cup B_1').$$

Therefore, we know that, by Perron's method again, the functions $v_k := u_k - h$ are harmonic in B_1^+, $v_k = 0$ on $\partial^+ B_1^+ = \partial B_1 \cap \{x_1 \geq 0\}$ and

$$\partial_1 v_* \leq \max f \quad \text{and} \quad \partial_1 v^* \geq \min f.$$

By Lemma 3.9.1, we know that v_k's are uniformly bounded in $C^\alpha(B_1^+)$-norm. By using Arzela-Ascoli theorem, we know that v_k, after passage to a subsequence, uniformly converges to some $\tilde{v} \in C(\overline{B_1^+})$. This shows that u_k, also after passage to a subsequence, converges to $u = \tilde{v} + h \in C(\overline{B_1^+})$.

Now, observe that condition (2) follows from the fact that $\partial_1 u_k \leq \max f$ for all k and Lemma 3.2.3. Condition (1) and (3) are proved in Lemma 3.4.9 and Lemma 3.4.11 respectively. The proof is then complete by applying Theorem 3.3.17 and Theorem 3.4.7.

□

3.5 Singularly anisotropic pinned Neumann condition

In this section we make precise definition of the homogenized parabolic flow (3.1.3). For notational convenience, we will denote the homogenized coefficients as

$$L_*(\nabla' u) = m 1_{\{\nabla' u = 0\}} \quad \text{and} \quad L^*(\nabla' u) = M 1_{\{\nabla' u = 0\}},$$

for some real numbers $m < 0 < M$. In the case of the homogenized equation (3.1.3) we correspondingly take

$$m = \min f,\ M = \max f \text{ and } \langle f \rangle = 0.$$

Recall the space-time domain are defined as follows

$$D^+_\infty := B^+_1 \times (0,\infty) \quad \text{and} \quad D'_\infty := B'_1 \times (0,\infty).$$

We also write

$$U \subset\subset B^+_1 \cup B'_1$$

to be a relatively open domain, and we also denote

$$U' = U \cap B'_1,\ U^+ = U \cap B^+_1 \text{ and } \partial^+ U = \partial U \setminus U'.$$

Definition 3.5.1. An upper semicontinuous function $u : \overline{D^+_\infty} \to [-\infty,\infty)$ is called a viscosity subsolution to (3.1.3) if for any smooth function ϕ crossing u from above (see Definition 3.2.2) at $(x_0,t_0) \in D^+_\infty \cup D'_\infty$ in the cylindrical neighborhood $U \times (t_0 - r, t_0 + r)$, we have either (i)

$$\partial_t \phi(x_0,t_0) \le \Delta\phi(x_0,t_0)$$

or (ii) $\partial_t\phi(x_0,t_0) > \Delta\phi(x_0,t_0)$, $(x_0,t_0) \in D'_\infty$, and the following conditions hold:

(a) (*Local stability*) The inner normal derivative satisfies

$$\partial_1\phi(x_0,t_0) \ge L_*(\nabla'\phi(x_0,t_0)).$$

(b) (*Dynamic slope condition: transversal case*) If $\partial_t\phi(x_0,t_0) > 0$, then

$$\partial_1\phi(x_0,t_0) \ge 0.$$

(c) (*Dynamic slope condition: laminar case*) If $\phi > u$ on $\partial^+U \times \{t_0\}$ and satisfies

$$\nabla'\phi \equiv 0 \text{ and } \partial_t\phi(x_0,t_0) > 0$$

then

$$\partial_1\phi(x_0,t_0) \ge M.$$

A viscosity supersolution is defined symmetrically by reversing the crossing and inequalities, and replacing M and L_* with m and L^*, respectively. A viscosity solution is defined as a continuous function on $\overline{D_\infty^+}$ that is both a viscosity subsolution and a viscosity supersolution.

Remark 3.5.2. The dynamic slope conditions (b) and (c) form a rate-independent motion law (see, for example, [130]), which is crucial in proving the comparison principle. Heuristically the motion law for a subsolution means that if $\partial_t u(x_0, t_0) > 0$ for some $(x_0, t_0) \in B_1' \times (0, \infty)$ then with some strictness condition the maximal inner normal slope is saturated *somewhere* on the component of the contact set containing (x_0, t_0).

The proof of our main theorem on parabolic homogenization, Theorem 3.6.1, follows the idea of half-relaxed limits. An essential piece of the half-relaxed limit strategy is a semicontinuous comparison principle. Thus, one of the central results of this work, is a comparison principle / uniqueness property for (3.1.3).

Let us reformulate Theorem 3.1.2 here.

Theorem 3.5.3. *Let u be a subsolution and v a supersolution of* (3.1.3) *in the sense of Definition 3.5.1. If $u \le v$ on the parabolic boundary $\partial_{\mathrm{p}}^+ D_\infty^+$ then $u \le v$ on the whole space-time domain $\overline{D_\infty^+}$.*

Remark 3.5.4. This comparison principle justifies that the viscosity solution conditions in Definition 3.5.1 are sufficient to characterize the homogenized limit problem (3.1.3).

The proof of Theorem 3.5.3 is postponed to Section 3.7. First we will apply Theorem 3.5.3 to establish the homogenization result Theorem 3.6.1 in the following section.

3.6 Homogenization of the parabolic flow in the laminar case

In this section, we consider the viscosity theoretic homogenization of the parabolic flow (3.1.1) as $\varepsilon \to 0^+$

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon & \text{in } B_1^+ \times (0, \infty) \\ \partial_1 u^\varepsilon = f(u^\varepsilon/\varepsilon) & \text{on } B_1' \times (0, \infty), \end{cases} \tag{3.1.1}$$

where $f \in C^\alpha(\mathbb{R})$ is a 1-periodic function.

We normalize f by using Remark 3.3.8 so that

$$\langle f \rangle = \int_0^1 f(v)dv = 0,$$

and we take

$$m = \min f \quad \text{and} \quad M = \max f$$

in the Definition 3.5.1.

Recall in Section 3.2.1 we denote for all $T \in (0, \infty)$ the following space-time domains

$$D_T^+ := B_1^+ \times (0, T] \quad \text{and} \quad D_T' := B_1' \times (0, T]$$

and we call

$$\partial_{\mathrm{p}}^+ D_T^+ := \overline{D_T^+} \setminus \left(D_T^+ \cup D_T' \right)$$

the (positive) parabolic boundary.

Let us restate Theorem 3.1.1 in a more precise way.

Theorem 3.6.1. *Let $T \in (0, \infty)$, $f \in C^\alpha(\mathbb{R})$ be a 1-periodic function for some $\alpha > 0$ and $g \in C(\partial_{\mathrm{p}}^+ D_\infty^+)$. If u^ε are solutions to* (3.1.1) *that satisfy $u^\varepsilon = g$ on $\partial_{\mathrm{p}}^+ D_\infty^+$, then u^ε converge uniformly on $\overline{D_T^+}$ to the unique solution u to* (3.1.3) *in the sense of Definition 3.5.1 that shares the same boundary data g on $\partial_{\mathrm{p}}^+ D_\infty^+$.*

We show that the upper and lower half relaxed limits of u^ε, u^* and u_*, are, respectively, a sub and supersolution of (3.1.3). Then we apply the comparison principle Theorem 3.5.3 to show that $u^* \le u_*$ and conclude that $u^* = u_*$.

Remark 3.6.2. For any $g \in C(\partial_{\mathrm{p}}^+ D_\infty^+)$ and $\varepsilon > 0$, by Perron's method and standard parabolic regularity theory, there exists a classical solution of (3.1.1)

$$u^\varepsilon \in C(\overline{D_\infty^+}) \cap C_{\mathrm{loc}}^{2+\alpha, 1+\alpha/2}(D_\infty^+) \cap C_{\mathrm{loc}}^{1+\alpha, 1/2+\alpha/2}(D_\infty^+ \cup D_\infty')$$

satisfying the boundary condition $u^\varepsilon = g$ on $\partial_{\mathrm{p}}^+ D_\infty^+$. See Lemma 3.9.4 and Lemma 3.9.5 for more details.

Let us first consider an L^∞ bound for u^ε, which guarantees the existence of the half relaxed limits.

Lemma 3.6.3. *Suppose u^ε is a viscosity solution to* (3.1.1) *and $u^\varepsilon = g(x,t)$ on the parabolic boundary for some $g \in C\left(\partial_{\mathrm{p}}^+ D_T^+\right)$, then there is a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{L^\infty\left(\overline{D_T^+}\right)} \le C\left(\|g\|_{L^\infty(\partial_{\mathrm{p}}^+ D_T^+)} + \|f\|_{L^\infty(\mathbb{R})}\right).$$

Proof. This follows by comparison principle using supersolutions of (3.1.1) of the form

$$\|g\|_{L^\infty} + \|f\|_{L^\infty}(1 - x_1)$$

and symmetrically defined subsolutions. □

Given a fixed Dirichlet boundary data g on the parabolic boundary $\partial_{\mathrm{p}}^+ D_T^+$, we define the following two functions, the upper half relaxed limit of u^ε

$$u^* := \limsup_{\varepsilon \to 0}{}^* u^\varepsilon \tag{3.6.1}$$

and the lower half relaxed limit of u^ε

$$u_* := \liminf_{\varepsilon \to 0}{}_* u^\varepsilon. \tag{3.6.2}$$

We now show that on the parabolic boundary $\partial_{\mathrm{p}}^+ D_T^+$ we have $u^* = u_* = g$. See Section 3.2 for the precise definitions of half relaxed limits.

Lemma 3.6.4. *Suppose u^ε are solutions as described in Lemma 3.6.3 with boundary data $g \in C(\partial_{\mathrm{p}}^+ D_T^+)$, then*

$$u^* = g = u_* \text{ on } \partial_{\mathrm{p}}^+ D_T^+.$$

Proof. By using standard parabolic theory, we can find classical solutions $H_\pm$ to the following heat equation

$$\begin{cases} \partial_t H_\pm = \Delta H_\pm & \text{in } D_T^+ \\ \partial_1 H_\pm = \pm \|f\|_{L^\infty} & \text{on } D_T' \\ H_\pm = g & \text{on } \partial_{\mathrm{p}}^+ D_T^+. \end{cases} \tag{3.6.3}$$

By comparison principle

$$H_+ \le u^\varepsilon \le H_-,$$

and therefore $u_* = u^* = g$ by the continuity of $H_\pm$. □

In the following three subsections we show that the half relaxed limits u^*, u_* are respectively the viscosity sub/supersolutions to (3.1.3) in the sense of Definition 3.5.1. In Section 3.6.4, we finish this section by proving Theorem 3.6.1 assuming the comparison principle in Theorem 3.5.3.

3.6.1 Local stability condition

Lemma 3.6.5. *The function* $u^* = \limsup^* u^\varepsilon$ *solves*

$$\partial_1 u^*(x,t) \geq L_*(\nabla' u^*) \text{ on } B_1' \times (0,T)$$

in the sense of condition (a) from Definition 3.5.1. Similarly, $u_* = \liminf_* u^\varepsilon$ *satisfies the symmetrical supersolution condition.*

Remark 3.6.6. The proof of this lemma also works for the general semilinear case $f(x',v)$ that is periodic with respect to $\mathbb{Z}^{d-1} \times \mathbb{Z}$. In this case similar arguments show

$$\partial_1 u^* \geq Q_*(\nabla' u^*; f) \quad \text{and} \quad \partial_1 u_* \leq Q^*(\nabla' u_*; f)$$

in the viscosity sense, where Q_* and Q^* are homogenized coefficients as defined in (3.3.9) and (3.3.10). It is an interesting open question to characterize similar conditions to Definition 3.5.1 (b) and (c) that provide a comparison principle for the general semilinear case.

In order to prove Lemma 3.6.5 we need to use the following technical lemma, which we prove after the proof of Lemma 3.6.5.

Lemma 3.6.7. *Suppose* $v : \overline{\mathbb{R}^d_+} \times \mathbb{R} \to [-\infty, \infty)$ *is a viscosity subsolution to*

$$\begin{cases} \partial_t v \leq \Delta v & \text{in } \mathbb{R}^d_+ \times \mathbb{R} \\ \partial_1 v \geq f(v+c) & \text{on } \partial\mathbb{R}^d_+ \times \mathbb{R} \\ v(y,t) \leq p \cdot y & \text{in } \overline{\mathbb{R}^d_+} \times \mathbb{R} \\ v(0,0) = 0 \text{ and } v(y,t) = -\infty \text{ for } t > 0, \end{cases} \tag{3.6.4}$$

where c *is an arbitrary constant, then*

$$p_1 \geq L_*(p') = (\min f) 1_{\{p'=0\}}.$$

Proof of Lemma 3.6.5. We prove that u^* satisfies the subsolution condition (a) for u^*, the supersolution condition (a) for u_* follows from a symmetric argument. Suppose that ψ is a smooth test function that crosses u^* strictly from above (by using standard perturbations) at (x_0, t_0) in $U \times (t_0 - r, t_0 + r)$ with

$$\partial_t \psi(x_0, t_0) > \Delta\psi(x_0, t_0). \tag{3.6.5}$$

By the smoothness of ψ, we may constrain to smaller domain U and radius $r > 0$ so that this strict supersolution property (3.6.5) holds also in $U \times (t_0 - r, t_0 + r)$ that contains (x_0, t_0), where $U \subset\subset B_1^+ \cup B_1'$ is relatively open.

By Lemma 3.2.4, we can find a subsequence (not relabelled) of $\varepsilon \to 0^+$, $c_\varepsilon \to 0$ and a sequence of points

$$(x_\varepsilon, t_\varepsilon) \in U \times (t_0 - r, t_0 + r)$$

that converge to (x_0, t_0) so that for each ε the function $\psi + c_\varepsilon$ crosses u^ε from above at $(x_\varepsilon, t_\varepsilon)$ in $U \times (t_0 - r, t_0 + r)$. Because ψ is a strict supersolution and u^ε is a subsolution to the heat equation, the strong comparison principle implies that $(x_\varepsilon, t_\varepsilon) \in U' \times (t_0 - r, t_0 + r)$. This shows that $x_0 \in U' = U \cap B_1'$.

Now we aim to show that

$$\partial_1 \psi(x_0, t_0) \geq L_*(\nabla' \psi(x_0, t_0)).$$

Consider the (parabolic) rescalings:

$$v_\varepsilon(y,t) := \frac{u^\varepsilon(\varepsilon y + x_\varepsilon, \varepsilon^2 t + t_\varepsilon) - u^\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon} \quad \text{and} \quad \psi_\varepsilon(y,t) := \frac{\psi(\varepsilon y + x_\varepsilon, \varepsilon^2 t + t_\varepsilon) - \psi(x_\varepsilon, t_\varepsilon)}{\varepsilon}.$$

Note that

$$\psi_\varepsilon(y,t) \leq p \cdot y + O(\varepsilon |y|^2 + \varepsilon |t|),$$

where $p := \nabla\psi(x_0, t_0)$. So, for any $R > 0$, there exists a small $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the v_ε solve in the classical sense

$$\begin{cases} \partial_t v_\varepsilon = \Delta v_\varepsilon & \text{in } B_R^+ \times (-R^2, 0] \\ \partial_1 v_\varepsilon = f\left(v_\varepsilon + \frac{u^\varepsilon(x_\varepsilon, t_\varepsilon)}{\varepsilon}\right) & \text{on } B_R' \times (-R^2, 0] \\ v_\varepsilon(y,t) \leq p \cdot y + O(\varepsilon R^2) & \text{in } \overline{B_R^+} \times [-R^2, 0] \\ v_\varepsilon(0,0) = \psi_\varepsilon(0,0) = 0, \end{cases} \tag{3.6.6}$$

where the final two properties follow since ψ_ε crosses v_ε from above at $(0,0)$.

Choose a subsequence $\varepsilon_j \to 0$ so that the limit exists

$$\lim_{j\to\infty}\left[\frac{u^{\varepsilon_j}(x_{\varepsilon_j},t_{\varepsilon_j})}{\varepsilon_j} - \left\lfloor \frac{u^{\varepsilon_j}(x_{\varepsilon_j},t_{\varepsilon_j})}{\varepsilon_j}\right\rfloor\right] = c \in [0,1].$$

For each fixed radius $R > 0$ the following upper half-relaxed limit, by the third condition in (3.6.6), is well-defined on $\overline{B_R^+} \times [-R^2, 0]$

$$v^* := \limsup_{j\to\infty}{}^* v_{\varepsilon_j}.$$

Because $R > 0$ is arbitrarily chosen, the limit function v^* is well-defined on all $\overline{\mathbb{R}^d_+} \times (-\infty, 0]$. Note that $p \cdot y$ touches v^* from above at $(0,0)$ in $\overline{\mathbb{R}^d_+} \times (-\infty, 0]$ so $v^*(y) \in [-\infty, \infty)$ on $\overline{\mathbb{R}^d_+} \times (-\infty, 0]$ and $v^*(0,0) = 0$. By extending $v^* = -\infty$ on $\overline{\mathbb{R}^d_+} \times (0,\infty)$, we can make v^* an upper semicontinuous function on $\overline{\mathbb{R}^d_+} \times \mathbb{R}$. Moreover, if we also extend $v_\varepsilon = -\infty$ on $\overline{B_R^+} \times (0,\infty)$, then

$$v^* = \limsup_{j\to\infty}{}^* v_{\varepsilon_j}$$

on the whole $\overline{\mathbb{R}^d_+} \times \mathbb{R}$.

By stability of touching in Lemma 3.2.3 and that v_ε are classical solutions to (3.6.6), v^* solves (3.6.4) with $p = \nabla\psi(x_0, t_0)$. The proof is done by applying Lemma 3.6.7.

□

Proof of Lemma 3.6.7. We prove this lemma by constructing a subcorrector V to Definition 3.3.5 having slope p. To that end, we consider the following sequence of functions with $r > 0$

$$w_r(y,t) := v^*(y, t/r),$$

and their upper half relaxed limit (which is well-defined because $v^*(y,t) \le p \cdot y$)

$$w(y,t) := \limsup_{r\to 0^+}{}^* w_r(y,t).$$

The function w is upper semicontinuous and takes the form

$$w(y,t) = \begin{cases} V_0(y) & t = 0 \\ V_1(y) & t < 0 \\ -\infty & t > 0, \end{cases}$$

with $V_0 \geq V_1$.

We claim that $V = V_0$ is a subcorrector to Definition 3.3.5 with slope p. By the bound $V_0(0) \geq 0$ and $V_0(y) \leq p \cdot y$, it suffices to show that V_0 is a subsolution. Suppose $\eta(y)$ is a smooth function touching V_0 strictly from above at $y_0 \in \overline{\mathbb{R}^d_+}$, then we can extend

$$\eta(y,t) = \eta(y) + t^2$$

and know that $\eta(y,t)$ touches w strictly from above at $(y_0, 0)$. By Lemma 3.2.3 again, there are

$$\text{constants } c_r \to 0 \text{ and points } (y_r, t_r) \to (y_0, 0)$$

such that $\eta + c_r$ touches w_r strictly from above at (y_r, t_r). Note that the functions w_r satisfies

$$\begin{cases} r\partial_t w_r \leq \Delta w_r & \text{in } \mathbb{R}^d_+ \times \mathbb{R} \\ \partial_1 w_r \geq f(w_r + c) & \text{on } \partial\mathbb{R}^d_+ \times \mathbb{R}. \end{cases}$$

This implies that either

$$r\partial_t \eta(y_r, t_r) = 2rt_r \leq \Delta\eta(y_r, t_r)$$

or $y_r \in \partial\mathbb{R}^d_+$ and

$$\partial_1 \eta(y_r, t_r) \geq f(\eta(y_r, t_r) + c_r + c).$$

Sending $r \to 0^+$ shows that V_0 is indeed a subcorrector to Definition 3.3.5 with slope p. Note that the translating constant c does not change the homogenized coefficients due to Remark 3.3.7.

□

3.6.2 Dynamic slope condition: transversal case

In this subsection we prove that the half relaxed limits u^*, u_* satisfy the condition (b) in Definition 3.5.1.

Lemma 3.6.8. *The function* $u^* = \limsup^* u^\varepsilon$ *solves*

$$\text{if } \partial_t u^*(x,t) > 0 \text{ then } \partial_1 u^*(x,t) \geq 0 \text{ on } B_1' \times (0,T)$$

in the sense of condition (b) from Definition 3.5.1. Similarly, $u_* = \liminf_* u^\varepsilon$ *satisfies the symmetrical supersolution condition.*

Proof. As before we only prove that u^* satisfies the subsolution condition (b) for u^*, and the supersolution condition (b) for u_* follows from a symmetric argument. By a similar argument to the proof of the previous lemma, we may start with ψ that is a smooth function crossing u^* strictly (by standard perturbations) from above at (x_0, t_0) and satisfies the following conditions in $U_r(x_0) \times (t_0 - r, t_0 + r)$

$$\partial_t \psi > \Delta\psi, \; \partial_1 \psi < 0 \quad \text{and} \quad \partial_t \psi > 0. \tag{3.6.7}$$

Here $U_r(z) := B_r(z) \cap \overline{\mathbb{R}^d_+}$. By the local stability condition (a)

$$\nabla\psi(x_0, t_0) = -\nu e_1 \;\text{ for some } \nu > 0. \tag{3.6.8}$$

In the following we do a sequence of modifications on ψ and r so that all the above conditions are preserved and in $U_r(x_0) \times (t_0 - r, t_0 + r)$

(i) $0 < -\partial_1\psi \ll \partial_t\psi$,

(ii) and ψ is strictly superharmonic.

We obtain (i) by adding ψ with $-cx_1$ for

$$c = r + \sup_{U_r(x_0)\times(t_0-r,t_0+r)} \partial_1\psi.$$

As $r \to 0^+$, by smoothness of ψ, the modified functions $\psi - cx_1$ satisfy all the previous properties and also condition (i) above.

We obtain (ii) by replacing ψ by

$$\psi^\mu := \psi + \mu x_1 - \frac{1}{\mu} x_1^2$$

for a small $\mu > 0$. When $\mu > 0$ is sufficiently small, ψ^μ is strictly superharmonic and all previous properties are preserved whenever $r < \mu^2/2$. In the following we omit the symbol μ.

By Lemma 3.2.4, there is a subsequence, not relabeled, of $\varepsilon \to 0^+$, $c_\varepsilon \to 0$ and a sequence of points $(x_\varepsilon, t_\varepsilon) \in U'_r(x_0) \times (t_0 - r, t_0 + r)$ so that $\psi_\varepsilon := \psi + c_\varepsilon$ crosses u^ε from above at $(x_\varepsilon, t_\varepsilon)$ in $U_r(x_0) \times (t_0 - r, t_0 + r)$. Moreover, $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$. By using (3.6.8) and condition (i) above, we obtain for some $a > 0$ and all $\varepsilon, r > 0$ that are sufficiently small

$$\max_{x\in\overline{U_r(x_0)}} \psi_\varepsilon(x, t_0 - r) + ar < \min_{x\in\partial^+ U_r(x_0)} \psi_\varepsilon(x, t_\varepsilon). \tag{3.6.9}$$

In the following context, we without loss assume that ψ_ε crosses u^ε at $(x_\varepsilon, t_\varepsilon)$ strictly from above by using standard perturbations.

Now we use energy minimization to construct a supersolution of the ε problem. First, we define $u = \xi^\varepsilon$ to be the global energy minimizer to the following energy for some $\delta > 0$ to be chosen:

$$\int_{U_r^+(x_0)} \frac{1}{2}|\nabla u|^2 + \int_{U_r'(x_0)} \int_0^{u(x')} (f(v/\varepsilon) - \delta)\, dv dx',$$

subject to the boundary condition $u(x) = \psi_\varepsilon(x, t_\varepsilon)$ on $\partial^+ U_r(x_0)$. The minimizer ξ^ε then satisfy the following problem

$$\begin{cases} \Delta \xi^\varepsilon = 0 & \text{in } U_r^+(x_0), \\ \partial_1 \xi^\varepsilon = f\left(\frac{\xi^\varepsilon}{\varepsilon}\right) - \delta & \text{on } U_r'(x_0), \\ \xi^\varepsilon(x) = \psi_\varepsilon(x, t_\varepsilon) & x \in \partial^+ U_r(x_0). \end{cases} \tag{3.6.10}$$

We claim that for small $\varepsilon,\ \delta > 0$ the function ξ^ε crosses u^ε from above at some $(z_\varepsilon, s_\varepsilon) \in U_r'(x_0) \times (t_0 - r, t_\varepsilon)$. This will finish the proof because by the viscosity solution condition of u^ε and the smoothness of ξ^ε

$$\partial_1 \xi^\varepsilon(z_\varepsilon, s_\varepsilon) \geq f(u^\varepsilon(z_\varepsilon, s_\varepsilon)) = f(\xi^\varepsilon(z_\varepsilon, s_\varepsilon)),$$

contradicting the boundary condition of ξ^ε at $(z_\varepsilon, s_\varepsilon)$ according to (3.6.10).

To prove the crossing property, we first observe that according to Lemma 3.3.2, we see that ξ^ε converges uniformly to ξ in $\overline{U_r(x_0)}$ solving

$$\begin{cases} \Delta \xi = 0 & \text{in } U_r^+(x_0), \\ \partial_1 \xi = -\delta & \text{on } U_r'(x_0), \\ \xi(x) = \psi(x, t_0) & x \in \partial^+ U_r(x_0). \end{cases} \tag{3.6.11}$$

Combining (3.6.9) and (3.6.11), we obtain

$$\begin{aligned} \max_{x \in \overline{U_r(x_0)}} (u^\varepsilon(x, t_0 - r) - \xi^\varepsilon(x)) &\leq \max_{x \in \overline{U_r(x_0)}} (\psi_\varepsilon(x, t_0 - r) - \xi^\varepsilon(x)) \\ &< \min_{x \in \partial^+ U_r(x_0)} \psi_\varepsilon(x, t_\varepsilon) - \min_{x \in \overline{U_r(x_0)}} \xi^\varepsilon(x) - ar \\ &< \min_{x \in \partial^+ U_r(x_0)} \psi(x, t_0) - \min_{x \in \overline{U_r(x_0)}} \xi(x) - ar + o_\varepsilon(1) \\ &\leq -ar + o_\varepsilon(1) \\ &< 0 \end{aligned}$$

when $\varepsilon > 0$ is small. This shows that

$$\xi^\varepsilon(x) > u^\varepsilon(x,t) \text{ for } (x,t) \in \partial_{\mathrm{p}}^+ \left(U_r(x_0) \times (t_0 - r, t_\varepsilon)\right). \tag{3.6.12}$$

On the other hand, when $\delta > 0$ is small we have for $x \in U'_r(x_0)$

$$\partial_1 \xi(x) = -\delta > \partial_1 \psi(x, t_0),$$

and because ξ is harmonic and ψ is superharmonic we obtain by strong comparison principle that $\xi < \psi$ on $U_r(x_0)$ and therefore for small $\varepsilon > 0$

$$\xi^\varepsilon(x_\varepsilon, t_\varepsilon) < \psi_\varepsilon(x_\varepsilon, t_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon). \tag{3.6.13}$$

Let s_ε be the infimum of $s \in (t_0 - r, t_\varepsilon)$ so that $u^\varepsilon(x,s) - \xi^\varepsilon(x,s)$ is positive for some $x \in \overline{U_r(x_0)}$. By (3.6.12) and (3.6.13), $s_\varepsilon \in (t_0 - r, t_\varepsilon)$. By continuity, ξ^ε will cross u^ε from above at some $(z_\varepsilon, s_\varepsilon) \in U_r(x_0) \times (t_0 - r, t_\varepsilon)$, but by strong comparison principle, z_ε will not belong to $U_r^+(x_0)$ and hence $z_\varepsilon \in U'_r(x_0)$ as we desired.

□

3.6.3 Dynamic slope condition: laminar case

In this subsection we prove that the half relaxed limits u^*, u_* satisfy Definition 3.5.1 (c).

Lemma 3.6.9. *The function u^* defined in* (3.6.1) *satisfies Definition 3.5.1 (c). Similarly, u_* defined in* (3.6.2) *satisfies the supersolution version of Definition 3.5.1 (c).*

Proof. As before we only prove that u^* satisfies the subsolution condition Definition 3.5.1 (c) for u^*. The supersolution condition Definition 3.5.1 (c) for u_* follows from a symmetric argument. By a similar argument to the proof of the previous two lemmas, we may start with $\psi = \psi(x_1, t)$ that is a smooth spatially one-variable function, crossing u^* from above and satisfies the following conditions in $U \times (t_0 - r, t_0 + r)$

$$\partial_t \psi > \partial_1^2 \psi, \ \partial_1 \psi < \max f \ \text{ and } \ \partial_t \psi > 0. \tag{3.6.14}$$

By using Lemma 3.2.4 we also have $\varepsilon \to 0^+, c_\varepsilon \to 0$ and a sequence of points $(x_\varepsilon, t_\varepsilon) \in U' \times (t_0 - r, t_0 + r)$ so that $\psi_\varepsilon := \psi + c_\varepsilon$ crosses u^ε from above at $(x_\varepsilon, t_\varepsilon)$ in $U \times (t_0 - r, t_0 + r)$. The domain $U \subset\subset B_1^+ \cup B'_1$ is relatively open.

Furthermore, by the assumption $\psi > u^*$ on $\partial^+ U \times \{t = t_0\}$, we also have, combining $\partial_t \psi > 0$ in (3.6.14), that for some $c > 0$ independent of ε

$$\psi_\varepsilon(x_1, t_\varepsilon) \geq c + u^\varepsilon(x,t) \text{ for } (x,t) \in \partial_{\mathrm{p}}^+ (U \times (t_0 - r, t_\varepsilon)). \tag{3.6.15}$$

For a small $\mu > 0$, we replace U by

$$U_\mu := U \cap \{x_1 < \mu\}.$$

This replacement preserves all the previous properties and when $\mu > 0$ is chosen sufficiently small we also have

$$\psi(x_1, t) \leq \psi(0,t) + (\max f - \beta) x_1, \tag{3.6.16}$$

for some small $\beta > 0$ and all $x \in U_\mu$, $t \in [t_0 - r, t_0 + r]$. In the following we will omit the symbol μ.

Let $r_\varepsilon \in \mathbb{R}$ be the maximal number such that

$$r_\varepsilon \leq u^\varepsilon(x_\varepsilon, t_\varepsilon) \quad \text{and} \quad f\left(\frac{r_\varepsilon}{\varepsilon}\right) = \max f =: M.$$

By the periodicity of f, we have

$$|r_\varepsilon - u^\varepsilon(x_\varepsilon, t_\varepsilon)| \leq \varepsilon.$$

We construct

$$\phi_\varepsilon(x_1, t) \equiv \phi_\varepsilon(x_1) := (M - \varepsilon)x_1 - \varepsilon x_1^2 + r_\varepsilon - \varepsilon.$$

Then we have by (3.6.15) and (3.6.16)

$$\phi_\varepsilon(x_1, t) \geq \psi_\varepsilon(x_1, t_\varepsilon) - 2\varepsilon > u^\varepsilon(x,t) \text{ for } (x,t) \in \partial_{\mathrm{p}}^+ (U \times (t_0 - r, t_\varepsilon))$$

when $\varepsilon > 0$ is sufficiently small. On the other hand, we know that

$$\phi_\varepsilon(0, t_\varepsilon) = r_\varepsilon - \varepsilon < u^\varepsilon(x_\varepsilon, t_\varepsilon).$$

By a similar argument to the last part of the previous lemma, we see that ϕ_ε crosses u^ε from above at some $(z_\varepsilon, s_\varepsilon) \in U' \times (t_0 - r, t_\varepsilon)$, which is impossible because otherwise by the viscosity solution condition of u^ε

$$\partial_1 \phi_\varepsilon(0, s_\varepsilon) \geq f(u^\varepsilon(z_\varepsilon, s_\varepsilon)/\varepsilon) = f(\phi_\varepsilon(0, s_\varepsilon)/\varepsilon) = f(r_\varepsilon/\varepsilon + 1) = \max f,$$

which contradicts the definition of ϕ_ε.

□

3.6.4 Proof of the parabolic homogenization

Now we can combine the elements above to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. Let u^* and u_* be the upper and lower half-relaxed limits of u_ε, as defined in (3.6.1) and (3.6.2) respectively. By Lemma 3.6.5, Lemma 3.6.8 and Lemma 3.6.9, u^* is a subsolution and u_* is a supersolution of (3.1.1) in the sense of Definition 3.5.1. By Lemma 3.6.4 and the comparison principle Theorem 3.5.3, $u_* \geq u^*$. Since $u_* \leq u^*$ by definition, then $\bar{u} := u_* = u^*$ is continuous on $\overline{D^+_\infty}$, is a viscosity solution of (3.1.3), and u^ε converge locally uniformly on $\overline{D^+_\infty}$ to $\bar{u}$, .

□

3.7 The parabolic comparison principle

In this section, we discuss the proof of comparison principle Theorem 3.5.3 for (3.1.3). In Section 3.7.1, we prove the openness of the facets/contact sets, which is a crucial step because these sets consist of points where the Neumann condition becomes degenerate. In Section 3.7.2, we first give a sketch of proof for Theorem 3.5.3 due to the length, and then provide a whole proof.

3.7.1 Contact sets

Because of the gradient degeneracy of the boundary condition

$$\partial_1 u \in [m1_{\{\nabla' u=0\}}, M1_{\{\nabla' u=0\}}],$$

it turns out to be useful to consider the points $x \in D'_\infty$, at which the solution u satisfies $\partial_1 u(x) \neq 0$. As is often the case in viscosity solution theory, we make this precise for weak solutions via sub and supersolution touching conditions.

Definition 3.7.1. Given a subsolution u to (3.1.3) as defined in Definition 3.5.1, we partition D'_∞ as

$$D'_\infty = \mathcal{C}_-(u) \sqcup \mathcal{N}_-(u) \sqcup \Gamma_-(u), \tag{3.7.1}$$

where

$$\mathcal{C}_-(u) := \left\{ (x_0, t_0) \in D'_\infty \ ; \begin{array}{c} \text{there is a smooth function } \phi \\ \text{crossing } u \text{ from above at } (x_0,t_0), \\ \text{and} \quad \partial_1\phi(x_0,t_0) < 0. \end{array} \right\} \tag{3.7.2}$$

and

$$\mathcal{N}_-(u) := D'_\infty \setminus \overline{\mathcal{C}_-(u)} \quad \text{and} \quad \Gamma_-(u) := D'_\infty \setminus (\mathcal{C}_-(u) \cup \mathcal{N}_-(u)) \,. \tag{3.7.3}$$

The sets $\mathcal{C}_+(v), \mathcal{N}_+(v)$ and $\Gamma_+(v)$ corresponding to a supersolution v are defined symmetrically. We will call $\mathcal{C}_-(u)$ and $\mathcal{C}_+(v)$ contact sets (or facets) for u and v respectively.

Remark 3.7.2. The terminology contact set is used due to (a somewhat distant) relationship with the contact set in the thin obstacle problem. See also [82] for more details on this connection.

Notice that if the contact set $\mathcal{C}$ is open, then the interface Γ will be the common boundary of the two disjoint open sets $\mathcal{C}$ and $\mathcal{N}$. Unfortunately, in contrast to the elliptic case [82, Lemma 2.8], for general subsolutions (or supersolutions) we are unable to show that the contact sets defined above are open in D'_∞. This is due to the lack of regularity in time. In the following lemma we show that $\mathcal{C}_-(u)$ is open when u is Lipschitz in time.

In the following we write $U \subset\subset B_1^+ \cup B_1'$ a relatively open domain and we also write

$$U^+ = U \cap B_1^+ \quad \text{and} \quad U' = U \cap B_1'.$$

Lemma 3.7.3. *Suppose that u is a subsolution of* (3.1.3) *on $D_\infty^+ \cup D'_\infty$ with*

$$\|\partial_t u\|_{L^\infty(U \times (t_1,t_2))} < +\infty.$$

Then $\mathcal{C}_-(u) \cap U' \times (t_1, t_2)$ is relatively open in D'_∞ and $\nabla' u = 0$ on $\mathcal{C}_-(u) \cap U' \times (t_1, t_2)$. A symmetric result holds for supersolutions.

In the proof we will show that for any $(x_0 + \varepsilon z, t_0 + \varepsilon h)$ near enough to $(x_0, t_0) \in \mathcal{C}_-(u)$ we can create another test function with negative inward normal derivative touching u from above *exactly* at $(x_0 + \varepsilon z, t_0 + \varepsilon h)$. Similar ideas have appeared before in [82, Lemma 2.8], [53, Lemma 3.1] and [80, Proof of Theorem 5.3, Step 3]. All of those examples were elliptic, the parabolic analogue is trickier since the time variable needs to be treated in a distinct way from the spatial variables. In order to force the touching test function to touch at a specific space-time point we need to use the Lipschitz hypothesis. The test function is created by bending upwards in the tangential and temporal directions the linearization of u at (x_0,t_0). By a Lipschitz bending in time we can force the touching time to be exactly $t_0+\varepsilon h$. By the subsolution condition the new test function can only touch u from above where its tangential derivative is zero, this will force the spatial location of the touching point to be $x_0+\varepsilon z$.

Proof. Let $(x_0,t_0)\in U'\times(t_1,t_2)$ be a point in $\mathcal{C}_-(u)$. By the definition there is a smooth function ϕ that crosses u from above at (x_0,t_0) with $\beta:=\partial_1\phi(x_0,t_0)<0$. By the subsolution condition of u, specifically Definition 3.5.1 (a), we have $\nabla'\phi(x_0,t_0)=0$. Also call $\alpha:=\partial_t\phi(x_0,t_0)$. Call $L>0$ to be a uniform upper bound for the time derivative $\partial_t u$, so that, in particular, $|\alpha|\le L$.

Consider

$$\tilde{\phi}(x,t):=\begin{cases}\phi(x,t) & \text{if } t\le t_0\\ \phi(x,t_0)+L(t-t_0) & \text{if } t>t_0.\end{cases}$$

Then $\tilde{\phi}$ touches u from above at (x_0,t_0). In the following we replace ϕ by $\tilde{\phi}$.

For convenience we consider the following rescalings for small $\varepsilon>0$

$$u^\varepsilon(y,s):=\frac{u(x_0+\varepsilon y,t_0+\varepsilon s)-u(x_0,t_0)}{\varepsilon}\quad\text{and}\quad\phi^\varepsilon(y,s):=\frac{\phi(x_0+\varepsilon y,t_0+\varepsilon s)-\phi(x_0,t_0)}{\varepsilon}.$$

When $\varepsilon>0$ is chosen small, we have the following differential inequality (in the sense of viscosity solutions) and upper half flatness condition with $\alpha(s):=\alpha\min\{s,0\}+L\max\{s,0\}$

$$\begin{cases}\varepsilon\partial_s u^\varepsilon\le\Delta_y u^\varepsilon & \text{in } B_1^+\times(-1,1)\\ \text{subsolution condition as in Definition 3.5.1} & \text{on } B_1'\times(-1,1)\\ u^\varepsilon(y,s)\le\beta y_1+\alpha(s)+\omega(\varepsilon) & \text{on } \overline{B_1^+}\times[-1,1],\end{cases}$$

where $\omega(\varepsilon)\to0^+$ as $\varepsilon\to0^+$.

In the following we construct a family of smooth test functions $v_{\tau,h,z}$ that, when $0<\varepsilon\ll\delta$, touch u^ε from above at exactly $(z,h)\in B_1^+\times(-1,1)$ for some appropriate parameter $\tau\in[0,1]$. We define

$$\begin{aligned}v_{\tau,h,z}(y,s):=\alpha(s)+\frac{\beta}{2}y_1+\delta|y'-z|^2-d\delta y_1^2+3L|s-h|\\ -\left(\delta|z|^2+3L|h|\right)\tau+\omega(\varepsilon)(1-\tau),\end{aligned}$$

where $\delta<\min\{1,|\beta|\}/(100d)$. Notice that for $|s|\le1$, $|y|\le1$ and when $0<\varepsilon\ll\delta$

$$\Delta_y v_{\tau,h,z}=-2\delta<-4L\varepsilon\le\varepsilon\partial_s v_{\tau,h,z},$$

in the sense of viscosity solutions. For $(y,s) \in \partial_{\mathrm{p}}^{+}(B_1^+ \times (-1,1)) \cup \overline{B_1^+} \times \{s=1\}$, if $|z| < 1/4$ and $|h| < \frac{\delta}{32L}$ we have $v_{1,h,z} > u^\varepsilon$ because

$$\begin{aligned} v_{1,h,z} - (\beta y_1 + \alpha(s) + \omega(\varepsilon)) &= (3L|s-h| - 3L|h|) - \frac{\beta}{2} y_1 + \delta|y' - z|^2 - d\delta y_1^2 - \delta|z|^2 - \omega(\varepsilon) \\ &\geq 3L|s| - 6L|h| + \left(-\frac{\beta}{2} - (d+1)\delta\right) y_1 + \delta|y'|^2 \\ &\quad - 2\delta y' \cdot z + \delta y_1^2 - \omega(\varepsilon) \\ &\geq \min\{3L, \delta\} - 6L|h| - 2\delta|z| - \omega(\varepsilon) \\ &\geq \frac{\delta}{4} - 6L|h| \\ &> 0. \end{aligned}$$

On the other hand we have $v_{1,h,z}(0,0) = 0$, which means that the graph of $v_{\tau,h,z}$ intersects with u^ε as τ approaches 1 from the negative side. By the half flatness condition we know that

$$v_{0,h,z} \geq \alpha(s) + \frac{3\beta}{4} y_1 + \omega(\varepsilon)$$

stays above u^ε, which combining the strong comparison principle and the above two properties implies that there exists a maximal $\tau^* = \tau^*(z,h) \in [0,1]$ such that $v_{\tau^*,h,z}$ touches u^ε from above at some

$$(y_0, s_0) \in B_1' \times (-1,1).$$

We claim that the touching point can only be

$$(y_0, s_0) = (z, h).$$

If $s_0 \neq h$ then

$$|\partial_s v_{\tau^*,h,z}(y_0, s_0)| \geq 2L > L,$$

where this inequality is interpreted as a supersolution condition in the viscosity sense, in other words the lower bound inequality holds on the temporal component of elements of the subdifferential. This violates that u^ε is L-Lipschitz regular in time. On the other hand, by the strong comparison principle we have $y_0 \in B_1'$ and if $y_0 \neq z$, then

$$\nabla' v_{\tau^*,h,z}(y_0, h) = 2\delta(y_0 - z) \neq 0,$$

which contradicts the subsolution condition of u^ε at (y_0,h) (see condition (a) in Definition 3.5.1) since $\partial_1 v_{\tau^*,h,z}(y_0,h) = \frac{\beta}{2} < 0$. This implies that $\mathcal{C}_-(u) \cap U \times (t_1,t_2)$ is open.

To show that $\nabla' u = 0$ in $\mathcal{C}_-(u) \cap U \times (t_1,t_2)$, we observe that $v_{\tau^*,h,z}(\cdot,h)$ touches $u(\cdot,h)$ from above exactly at z, which is the spatial minimum of a parabola and by arbitrariness of $z \in \mathcal{C}_-(u) \cap U \times \{h\}$, we see that $u(\cdot,h)$ is $C^{1,1}$ from one side and has (tangential) gradient 0 everywhere on $\mathcal{C}_-(u) \cap U \times \{h\}$, which forces $\nabla' u = 0$ on $\mathcal{C}_-(u) \cap U \times \{h\}$. □

3.7.2 The comparison principle for the homogenized parabolic problem

In order to exhibit the core ideas we present the sketch of proof of Theorem 3.5.3. In order to present a clear sketch we will need to allow some slightly incorrect statements, which will then be clarified in the detailed proof.

Sketch of proof of Theorem 3.5.3. We give a sketch of proof under the assumption that the subsolution u and the supersolution v are smooth. First we can perturb u to

$$u_\mu(x,t) := u(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t}, \tag{3.7.4}$$

for an arbitrary end time $T > 0$ and a small $\mu > 0$. In particular this forces the maximum of $u_\mu - v$ on $\overline{D_T^+}$ to occur on the interior D_T^+ and not at the end time $t = T$. By the maximum principle for strict subsolutions of the heat equation, the maximum point can only occur on $B_1' \times (0,T)$.

Let $(x_0,t_0) \in B_1' \times (0,T)$ be a maximum point of $u_\mu - v$, then by a proper vertical translation of u_μ we may assume that u_μ touches v from below at $(x_0,t_0) \in B_1' \times (0,T)$. Derivative tests imply

$$\partial_1(u_\mu - v)(x_0,t_0) \le 0 \quad \text{and} \quad \partial_t(u_\mu - v)(x_0,t_0) = 0.$$

Using Lemma 3.7.3, there are only three cases:

Case 1: $(x_0,t_0) \notin \mathcal{C}_-(u) \cup \mathcal{C}_+(v)$.

Case 2: $(x_0,t_0) \in \mathcal{C}_-(u) \cap \mathcal{C}_+(v)$.

Case 3: $(x_0,t_0) \in \mathcal{C}_-(u) \Delta \mathcal{C}_+(v)$.

Case 1 can be excluded since at the touching point $(x_0, t_0) \in D'_\infty \setminus (\mathcal{C}_-(u) \cup \mathcal{C}_+(v))$

$$\partial_1 u + \mu = \partial_1 u_\mu \le \partial_1 v \le 0,$$

which shows that $\partial_1 u(x_0, t_0) < 0$, contradicting the assumption that $(x_0, t_0) \notin \mathcal{C}_-(u)$, which is defined in (3.7.2).

Now we show that Case 2 is impossible by using the (transversal) dynamic slope condition Definition 3.5.1 (b). Observe that by the definition of $\mathcal{C}_-(u)$ and $\mathcal{C}_+(v)$ respectively, see (3.7.2),

$$\partial_1 u(x_0, t_0) < 0 \quad \text{and} \quad \partial_1 v(x_0, t_0) > 0. \tag{3.7.5}$$

By the (transversal) dynamic slope condition Definition 3.5.1 (b), we must have $\partial_t v(x_0, t_0) \ge 0$ because

$$\text{if } \partial_t v(x_0, t_0) < 0 \text{ then } \partial_1 v(x_0, t_0) \le 0$$

which is a contradiction of (3.7.5). On the other hand, if

$$0 \le \partial_t v(x_0, t_0) = \partial_t u_\mu(x_0, t_0)$$

then

$$\partial_t u(x_0, t_0) = \partial_t u_\mu(x_0, t_0) + \frac{\mu}{(T - t_0)^2} \ge \frac{\mu}{(T - t_0)^2} > 0$$

and the (transversal) dynamic slope condition Definition 3.5.1 (b) implies

$$\partial_1 u(x_0, t_0) \ge 0$$

again contradicting (3.7.5). This finishes Case 2.

We claim that Case 3 can be reduced to Case 1 and 2 by using the conditions in Definition 3.5.1. We only argue with the case $(x_0, t_0) \in \mathcal{C}_-(u) \setminus \mathcal{C}_+(v)$ as the other one is symmetrical. By Lemma 3.7.3, $\mathcal{C}_-(u)$ is open in D'_∞. Let Ω, open in B'_1, be the (relatively open) connected component of $\mathcal{C}_-(u) \cap \{t = t_0\}$ containing (x_0, t_0). We claim that there is another point

$$(\tilde{x}_0, t_0) \in \partial' \Omega \cap B'_1 \subset \Gamma_-(u)$$

such that u touches v from below also at $(\tilde{x}_0, t_0)$. Otherwise $u < v$ on $\partial' \Omega \cap B'_1 \times \{t_0\}$.

Now we argue imprecisely for the sketch and think of $u = u(x_1, t)$ as a spatially one-dimensional function near Ω. Notice that under this assumption $u_\mu = u_\mu(x_1, t)$ is also spatially 1D.

If

$$\partial_t u_\mu(x_0, t_0) = \partial_t v(x_0, t_0) < 0,$$

then by the condition Definition 3.5.1 (c) of v as a supersolution we have

$$\partial_1 u_\mu(x_0, t_0) = \partial_1 u(0, t_0) + \mu \le m$$

contradicting the strict subsolution condition Definition 3.5.1 (a) of u. If otherwise

$$\partial_t u_\mu(x_0, t_0) = \partial_t v(x_0, t_0) \ge 0$$

then

$$\partial_t u(x_0, t_0) = \partial_t u_\mu(x_0, t_0) + \frac{\mu}{(T - t_0)^2} > 0,$$

which, by the condition Definition 3.5.1 (b) of the subsolution u we know that

$$\partial_1 u(x_0, t_0) = \partial_1 u(0, t_0) \ge 0$$

contradicting the assumption $\partial_1 u(x_0, t_0) < 0$ because $(x_0, t_0) \in \mathcal{C}_-(u)$. This finishes the proof of the existence of $\tilde{x}_0$.

Now to prove the claim that Case 3 can be reduced to Case 1 and 2, we observe that $(\tilde{x}_0, t_0) \in \Gamma_-(u)$ is either contained in $\mathcal{C}_+(v)$ or outside $\mathcal{C}_+(v)$. If $(\tilde{x}_0, t_0) \notin \mathcal{C}_+(v)$ then we are in Case 1. If $(\tilde{x}_0, t_0) \in \mathcal{C}_+(v) \cap \Gamma_-(u)$ then we can reduce to Case 2 because $\mathcal{C}_+(v)$ is open by Lemma 3.7.3 and by definition of $\Gamma_-(u)$ we can choose points $(\hat{x}_0, t_0)$ in $\mathcal{C}_-(u)$ that has small distance to $(\tilde{x}_0, t_0)$ and

$$u_\mu(\hat{x}_0, t_0) = u_\mu(\tilde{x}_0, t_0) = v(\tilde{x}_0, t_0) = v(\hat{x}_0, t_0),$$

where we have used the fact that u and v depend only on time on each component of $\mathcal{C}_-(u)$ and $\mathcal{C}_+(v)$ respectively, which implies that u touches v from below at $(\hat{x}_0, t_0)$ and

$$(\hat{x}_0, t_0) \in \mathcal{C}_-(u) \cap \mathcal{C}_+(v).$$

□

Proof of Theorem 3.5.3. Let us first introduce the strategy of the proof. The proof begins by regularizing the sub/supersolutions u and v via parabolic tangential sup/inf-convolutions to ensure Lipschitz regularity in time and tangential spatial variables. These regularized functions are replaced by their caloric lifts (heat equation solutions with the same boundary data), preserving sub/supersolution properties. Next we consider the maximum value M of the regularized $u - v$, with a penalizing term $-2\mu + \mu x_1 - \frac{\mu}{T-t}$ to force strict ordering of derivatives and ensure the maximum point exists and only occurs on $B_1' \times (0,T)$. At potential maximum points, we show that the regularized u and v are differentiable and hence we can test the boundary condition of u and v in different cases (which were described above in the proof sketch) depending on whether the maximum point is in the contact set of u and/or v.

Step 1: Regularization of sub and supersolutions

Since constants are solutions to (3.1.3), we may without loss assume that u and v are bounded by considering $\min\{v, K\}$ and $\max\{u, -K\}$ for a large $K > 0$. For $\delta > 0$ and $\overline{U} = \overline{D^+_\infty}$ we consider the parabolic tangential sup-convolution $\mathcal{T}^\delta u$ of u and inf-convolution $\mathcal{T}_\delta v$ of v as defined in Definition 3.9.6. For any fixed end time $T > 0$, and a small parameter $\theta > 0$ we define

$$\mathcal{O} := B^+_{1-\theta/2} \times (\theta/2, T] \quad \text{and} \quad \mathcal{O}' := B'_{1-\theta/2} \times (\theta/2, T].$$

The functions $\mathcal{T}^\delta u$ and $\mathcal{T}_\delta v$ satisfy the following properties:

(i) *Viscosity solution conditions*: By Lemma 3.9.8 and Corollary 3.9.9, for any small $\theta > 0$ there is a small $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$, $\mathcal{T}^\delta u$ is a subsolution and $\mathcal{T}_\delta v$ is a supersolution of (3.1.3) on

$$\mathcal{O} \cup \mathcal{O}' = \left(B^+_{1-\theta/2} \cup B'_{1-\theta/2}\right) \times (\theta/2, T].$$

(ii) *Lipschitz regularity and semi-convexity in time and tangential variables*: By Lemma 3.9.7 and boundedness of u and v, both $\mathcal{T}_\delta v$ and $\mathcal{T}^\delta u$ are Lipschitz in time t and in the tangential variable x' for all $(x,t) = (x_1, x', t) \in \overline{\mathcal{O}}$. The Lipschitz constant is independent of $(x,t) \in \overline{\mathcal{O}}$. Moreover, by the definition of sup/inf-convolutions, the function $\mathcal{T}_\delta v(x_1, x', t)$ is $\frac{1}{\delta}$-semi-convex in (x', t) and $\mathcal{T}^\delta u$ is $\frac{1}{\delta}$-semi-concave in (x', t) for any fixed $x_1 \geq 0$.

(iii) *Boundary ordering*: Define the fattened boundary

$$J_\theta := \overline{B_1^+} \times [0,T] \setminus \left(B_{1-\theta}^+ \cup B_{1-\theta}'\right) \times (\theta, T].$$

By Lemma 3.9.10 and the upper semicontinuity of $u - v$, we have

$$\limsup_{\theta\to 0,\delta\to 0} \max_{J_\theta} \left(\mathcal{T}^\delta u - \mathcal{T}_\delta v\right) \leq \max_{\partial_{\mathrm{p}}^+ D_T^+} (u - v).$$

In particular, if

$$\max_{\partial_{\mathrm{p}}^+ D_T^+} (u - v) < 0,$$

then for sufficiently small $\theta, \delta > 0$ we have

$$\max_{J_\theta} \left(\mathcal{T}^\delta u - \mathcal{T}_\delta v\right) < 0. \tag{3.7.6}$$

(iv) *Caloric lifts*: By Lemma 3.9.11 and Definition 3.9.12, we consider on the domain $\mathcal{O}$ the caloric lifts of $\mathcal{T}^\delta u$ and $\mathcal{T}_\delta v$:

$$\tilde{u}^\delta := \mathcal{H}\mathcal{T}^\delta u \quad \text{and} \quad \tilde{v}_\delta := \mathcal{H}\mathcal{T}_\delta v, \text{ on } \mathcal{O} = B_{1-\theta/2}^+ \times (\theta/2, T].$$

By Lemma 3.9.13, both $\tilde{u}$ and $\tilde{v}$ are also sub and supersolutions to (3.1.3) respectively.

By Lemma 3.9.14 the lifts $\tilde{u}^\delta$ and $\tilde{v}_\delta$ are continuous on $\mathcal{O} \cup \mathcal{O}'$. Moreover, because the restrictions of $\tilde{u}$ and $\tilde{v}$ on $\mathcal{O}'$ are Lipschitz, by Lemma 3.9.16, both $\tilde{u}$ and $\tilde{v}$ are Lipschitz near $\mathcal{O}'$.

In general $\tilde{u}$ and $\tilde{v}$ may not be continuous on the parabolic boundary $\partial_{\mathrm{p}}^+\mathcal{O}$. However, by Lemma 3.9.11, the upper semicontinuous envelope of $\tilde{u}^\delta$ on $\overline{\mathcal{O}}$ coincides with $\mathcal{T}^\delta u$ on $\partial_{\mathrm{p}}^+\mathcal{O}$. Similar holds for $\tilde{v}$. This implies that

$$\max_{\partial_{\mathrm{p}}^+\mathcal{O}} (\tilde{u}^\delta - \tilde{v}_\delta) = \max_{\partial_{\mathrm{p}}^+\mathcal{O}} (\mathcal{T}^\delta u - \mathcal{T}_\delta v) \leq \max_{J_\theta} \left(\mathcal{T}^\delta u - \mathcal{T}_\delta v\right). \tag{3.7.7}$$

(v) *Contact sets*: Denote the contact sets for the subsolution $\tilde{u}^\delta$ and for the supersolution $\tilde{v}_\delta$, respectively,

$$\mathcal{C}_- := \mathcal{C}_-(\tilde{u}^\delta) \subset \mathcal{O}' \quad \text{and} \quad \mathcal{C}_+ := \mathcal{C}_+(\tilde{v}_\delta) \subset \mathcal{O}'. \tag{3.7.8}$$

By property (ii), (iv) and Lemma 3.7.3, both $\mathcal{C}_-$ and $\mathcal{C}_+$ are open relative to $\mathcal{O}'$. Similar to above in (3.7.3), we also consider the non-contact sets and free boundaries: define $\mathcal{N}_\pm := \mathcal{O}' \setminus \overline{\mathcal{C}_\pm}$ and $\Gamma_\pm := \mathcal{O}' \setminus (\mathcal{C}_\pm \cup \mathcal{N}_\pm)$ so that $\mathcal{O}'$ is decomposed as a disjoint union in two ways

$$\mathcal{O}' = \mathcal{C}_\pm \sqcup \mathcal{N}_\pm \sqcup \Gamma_\pm.$$

Step 2: Comparison with auxiliary perturbations

In this step we make perturbations to allow strictness of some of the inequalities arising from derivative tests. It suffices to show, for parameters satisfying $0 < \delta \ll \theta \ll \mu \ll 1 \ll T$, that

$$M := \max_{(x,t)\in\overline{\mathcal{O}}} \left(\tilde{u}^\delta(x,t) - \tilde{v}_\delta(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t} \right) \le 0, \tag{3.7.9}$$

Indeed, if the above inequality holds then because

$$\tilde{u}^\delta - \tilde{v}_\delta \ge \mathcal{T}^\delta u - \mathcal{T}_\delta v \ge u - v \text{ on } \overline{\mathcal{O}},$$

we also have

$$\max_{(x,t)\in\overline{\mathcal{O}}} \left(u(x,t) - v(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t} \right) \le 0. \tag{3.7.10}$$

On the other hand, by property (iii) we have for sufficiently small $\theta > 0$

$$\sup_{(x,t)\in\overline{D_T^+}\setminus\overline{\mathcal{O}}} \left(u(x,t) - v(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t} \right) \le 0.$$

Combining (3.7.10), we obtain

$$\max_{(x,t)\in\overline{D_T^+}} \left(u(x,t) - v(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t} \right) \le 0.$$

Sending $\mu \to 0$ implies that $u \le v$ for $t < T$. Then sending $T \to \infty$ we conclude that $u \le v$ on the whole $\overline{D_\infty^+}$.

To prove (3.7.9), we argue by contradiction and assume that $M > 0$. The maximum is achieved at some point $(x_0, t_0) \in \overline{\mathcal{O}} = \overline{B_{1-\theta/2}^+} \times [\theta/2, T]$. By applying (3.7.6) and (3.7.7), we obtain that for any small $\mu > 0$ there are small $\theta_0, \delta_0 > 0$ such that for all $0 < \theta < \theta_0$ and $0 < \delta < \delta_0$

$$\tilde{u}^\delta - \tilde{v}_\delta < 0 \text{ on } \partial_{\mathrm{p}}^+ \mathcal{O}.$$

On the other hand, we have

$$\tilde{u}^\delta(x,t) - \tilde{v}_\delta(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t} \to -\infty$$

as $t \to T^-$, and therefore the maximum point

$$(x_0, t_0) \in \left(B^+_{1-\theta/2} \cup B'_{1-\theta/2}\right) \times (\theta/2, T).$$

Since $\tilde{u}^\delta - 2\mu + \mu x_1 - \frac{\mu}{T-t}$ is a strict subsolution to the heat equation and $\tilde{v}_\delta$ is a supersolution to the heat equation in the interior $\mathcal{O}$ then

$$(x_0, t_0) \in B'_{1-\theta/2} \times (\theta/2, T).$$

Define

$$U(x,t) := \tilde{u}^\delta(x,t) - 2\mu + \mu x_1 - \frac{\mu}{T-t} - M \quad \text{and} \quad V(x,t) := \tilde{v}_\delta(x,t).$$

By previous arguments we have established that U touches V from below at

$$(x_0, t_0) \in B'_{1-\theta/2} \times (\theta/2, T).$$

Step 3: Differentiability at touching (maximum) points

In the previous steps we have regularized u and v to obtain U and V, however, the regularization is only in the tangential directions and we need some additional argument to establish differentiability in the normal direction. To address this, we first observe that by the construction, $U\big|_{\mathcal{O}'}$ and $V\big|_{\mathcal{O}'}$ are respectively semi-concave and semi-convex on $\mathcal{O}'$, and therefore both of them are $C^{1,1}$ in (x', t) variables at the touching point (x_0, t_0). As U and V differ from $\tilde{u}$ and $\tilde{v}$ by addition of a smooth function near (x_0, t_0), we have

$$\text{both } \tilde{u}\big|_{\mathcal{O}'} \text{ and } \tilde{v}\big|_{\mathcal{O}'} \text{ are } C^{1,1} \text{ at } (x_0, t_0).$$

We now apply Lemma 3.9.17 to $\tilde{u}$ and $\tilde{v}$ to obtain that both $\tilde{u}$ and $\tilde{v}$ are differentiable both in space and time at (x_0, t_0). By conditions (a) and (b) in Definition 3.5.1, the derivatives satisfy

$$\partial_1 \tilde{u}(x_0, t_0) \geq L_*(\nabla' \tilde{u}(x_0, t_0)), \tag{3.7.11}$$

and

$$\text{If } \partial_t \tilde{u}(x_0, t_0) > 0 \text{ then } \partial_1 \tilde{u}(x_0, t_0) \geq 0. \tag{3.7.12}$$

A symmetric result holds for $V = \tilde{v}$.

As U is a smooth perturbation of $\tilde{u}$ we know that U also differentiable at (x_0, t_0), and by (3.7.11) and (3.7.12), we have

$$\partial_1 U(x_0, t_0) \geq L_*(\nabla' U(x_0, t_0)) + \mu, \tag{3.7.13}$$

and

$$\text{If } \partial_t U(x_0, t_0) \geq -\frac{\mu}{(T - t_0)^2} \text{ then } \partial_1 U(x_0, t_0) \geq \mu. \tag{3.7.14}$$

On the other hand, because U touches V from below at (x_0, t_0), we have the following formulae

$$\partial_1 (U - V)(x_0, t_0) \leq 0, \ \nabla'(U - V)(x_0, t_0) = 0 \ \text{ and } \ \partial_t (U - V)(x_0, t_0) = 0. \tag{3.7.15}$$

Step 4: Case analysis on the location of the touching points

By the decompositions of $\mathcal{O}'$ described Step 1 item (v), one of the following holds:

Case 1. $(x_0, t_0) \notin \mathcal{C}_- \cup \mathcal{C}_+$.

Case 2. $(x_0, t_0) \in \mathcal{C}_- \cap \mathcal{C}_+$.

Case 3. $(x_0, t_0) \in \mathcal{C}_- \Delta \mathcal{C}_+$.

We rule out each possibility case by case, finally obtaining a contradiction of the existence of an interior maximum point and establishing (3.7.9). The arguments will follow the sketch presented earlier in the section, now filling in the missing technical details.

Case 1. In this case $(x_0, t_0) \notin \mathcal{C}_- \cup \mathcal{C}_+$. By definition of the contact sets $\mathcal{C}_-$ and $\mathcal{C}_+$, the functions $\tilde{u}$ and $\tilde{v}$ satisfy Neumann sub and supersolution conditions, respectively, at $(x_0, t_0) \notin \mathcal{C}_- \cup \mathcal{C}_+$. By the differentiability of $\tilde{u}$ and $\tilde{v}$ at (x_0, t_0), as discussed in Step 3,

$$\partial_1 \tilde{u}(x_0, t_0) \geq 0 \geq \partial_1 \tilde{v}(x_0, t_0).$$

This implies, by (3.7.13), that

$$\partial_1 U(x_0, t_0) \geq \mu > 0,$$

while, on the other hand, by (3.7.15),

$$\partial_1 U(x_0, t_0) \leq \partial_1 V(x_0, t_0) = \partial_1 \tilde{v}(x_0, t_0) \leq 0,$$

giving a contradiction.

Case 2. Now $(x_0,t_0)\in\mathcal{C}_-\cap\mathcal{C}_+$. Call

$$b := \partial_t U(x_0,t_0) = \partial_t V(x_0,t_0).$$

Either $b\geq 0$ or $b<0$. If $b\geq 0$, then by (3.7.14),

$$\partial_1 U(x_0,t_0)\geq \mu, \text{ or equivalently } \partial_1\tilde{u}(x_0,t_0)\geq 0,$$

contradicting the fact that $(x_0,t_0)\in\mathcal{C}_-$. Indeed, for any smooth function ϕ crossing $\tilde{u}$ from above at (x_0,t_0), we know that $\phi(\cdot,t_0)$ touches $\tilde{u}(\cdot,t_0)$ from above at (x_0,t_0) and therefore

$$\partial_1\phi(x_0,t_0)\geq\partial_1\tilde{u}(x_0,t_0)\geq 0,$$

which contradicts the definition of $\mathcal{C}_-$ in (3.7.2). If $b<0$ then by (3.7.12), or rather the symmetric supersolution statement for $V=\tilde{v}$,

$$\partial_1 V(x_0,t_0) = \partial_1\tilde{v}(x_0,t_0)\leq 0,$$

which contradicts $(x_0,t_0)\in\mathcal{C}_+$.

Case 3. In the following we finish the proof by showing that Case 3 can be reduced to Case 1 and Case 2. To see this we focus on the case that the touching point

$$(x_0,t_0)\in\mathcal{C}_-\cap(\mathcal{N}_+\cup\Gamma_+)$$

as the other case is symmetrical. It suffices to prove that there exists another touching point

$$(\tilde{x}_0,t_0)\in\partial\mathcal{C}_-\cap\left(B'_{1-\theta/2}\times\{t_0\}\right)\subset\Gamma_-, \tag{3.7.16}$$

at which U touches V from below. Let us first show that the existence of the point in (3.7.16) implies a contradiction. The new touching point $(\tilde{x}_0,t_0)$ either belongs to $\mathcal{N}_+\cup\Gamma_+$ or $\mathcal{C}_+$. By Case 1 above $(\tilde{x}_0,t_0)$ does not belong to $\mathcal{N}_+\cup\Gamma_+$. Next suppose that $(\tilde{x}_0,t_0)\in\mathcal{C}_+$. By property (ii) and Lemma 3.7.3, $\tilde{u}(\cdot,t_0)$, $\tilde{v}(\cdot,t_0)$ and therefore $U(\cdot,t_0)$, $V(\cdot,t_0)$ are constant on each component of $\mathcal{C}_-\cap\{t=t_0\}$ and $\mathcal{C}_+\cap\{t=t_0\}$ respectively. Therefore $V(\cdot,t_0)$ is constant in a (tangential) neighborhood of $\tilde{x}_0$, while in any small (tangential) neighborhood of $(\tilde{x}_0,t_0)$ there is a $(\hat{x}_0,t_0)\in\mathcal{C}_-$ such that

$$U(\hat{x}_0,t_0)=U(\tilde{x}_0,t_0).$$

This implies that U also touches V from below at $(\hat{x}_0, t_0) \in \mathcal{C}_- \cap \mathcal{C}_+$. This is Case 2 which we have already shown that it cannot occur.

Now we return to prove the existence of the touching point in (3.7.16). This is where Definition 3.5.1 (c) comes into play. In fact, we show a slightly stronger result that, if we denote $\Omega_{t_0} \subset \mathcal{C}_- \cap \{t = t_0\}$ as the component (which by Lemma 3.7.3 is relatively open in $B'_{1-\theta/2} \times \{t_0\}$) of the latter set that contains (x_0, t_0), then there must be a touching point

$$(\tilde{x}_0, t_0) \in \partial'\Omega_{t_0} \cap (B'_{1-\theta/2} \times \{t_0\}) \subset \Gamma_-. \tag{3.7.17}$$

To see this, we argue by contradiction and assume that such a point does not exist. Notice that the relative boundary satisfies the containment

$$\partial'\Omega_{t_0} \subset (\Gamma_- \cap \{t = t_0\}) \cup \partial' B'_{1-\theta/2} \times \{t_0\}.$$

By property (iii), (iv) in Step 1 and the strict perturbations in Step 2, we know that

$$U < V \text{ on } \partial' B'_{1-\theta/2} \times \{t_0\}.$$

Therefore, if $(\tilde{x}_0, t_0)$ does not exist as in (3.7.17), then we reduce to the following condition

$$U < V \text{ on } \partial'\Omega_{t_0}.$$

As we have pointed out in the sketch, the main difficulty here is to view U as a spatially 1D function, which is definitely not one in general. To avoid this issue we need to construct a 1D function by using the fact that $\nabla' U = 0$ on $\mathcal{C}_-$.

We first observe that since $\mathcal{C}_-$ is open in $\mathcal{O}'$ and $t_0 < T$, there exists a small $r_0 > 0$ and a modulus of continuity $\omega(r) \downarrow 0$ such that for any $r_0 > r > 0$,

$$\left\{x \ ; \ (x, t_0) \in \Omega_{t_0}, \ \mathrm{dist}((x, t_0), \partial'\Omega_{t_0}) > r\right\} \times (t_0 - \omega(r), t_0 + \omega(r)) \subset\subset \mathcal{C}_-.$$

We write for $r, h > 0$ the following spatial domains

$$\Omega^r := \left\{x \ : \ (x, t_0) \in \Omega_{t_0}, \ \mathrm{dist}((x, t_0), \partial'\Omega_{t_0}) > r\right\} \quad \text{and} \quad \Omega^r_h := \{0 < x_1 < h\} \times \Omega^r$$

When $r_0 > 0$ is sufficiently small, we still have $U < V$ on $\partial' (\Omega^r \times \{t_0\})$ by the continuity of $U - V$. By the strong comparison principle for the heat equation, $U < V$ in $\mathcal{O}$ and then, in particular, we have reduced (3.7.17) to

$$U < V \text{ on } \partial^+\Omega^r_h \times \{t_0\}. \tag{3.7.18}$$

Second, we show that U is continuously differentiable on

$$\Omega^r \times \{t_0\} \subset \Omega_{t_0},$$

for all small $r > 0$. Indeed, for a fixed $y \in \mathbb{R}^{d-1}$ such that

$$(x_0 + y, t_0) \in \Omega^r \times \{t_0\},$$

we have

$$U(x,t) - U(x+y,t) \equiv 0 \quad \text{for } x \in B'_r(x_0) \times (t_0 - \omega(r), t_0 + \omega(r)).$$

Moreover, $U(x,t) - U(x+y,t)$ solves the heat equation in the interior

$$B^+_r(x_0) \times (t_0 - \omega(r), t_0 + \omega(r)).$$

If we denote

$$F_r := B^+_r(x_0) \times (t_0 - \omega(r), t_0 + \omega(r)),$$

we have by the standard Dirichlet boundary regularity estimate of heat equation (or also interior estimate such as Lemma 3.9.15), that for some $C > 0$ independent of y

$$\|U(x,t) - U(x+y,t)\|_{C^{1,\alpha}_{x,t}(\overline{F_{r/2}})} \le C \|U(x,t) - U(x+y,t)\|_{L^\infty_{x,t}(F_r)}.$$

This implies that U is differentiable at $(x_0 + y, t_0)$ if and only if it is differentiable at (x_0, t_0) and the latter was already proved in Step 3. Moreover, the derivative of U at $(x_0 + y, t_0)$ is continuous with respect to y because U is continuous on $\mathcal{O}'$ by Lemma 3.9.14.

Third, we notice that because on Ω_{t_0} we have $\nabla' U = 0$, by (3.7.13)

$$\partial_1 U \ge L_*(\nabla' U) + \mu = m + \mu.$$

On the other hand, let $\varepsilon > 0$ small and consider the domain

$$K_\varepsilon := \Omega^r_\varepsilon \times (t_0 - \varepsilon, t_0 + \varepsilon).$$

By the continuous differentiability of U on $\Omega^r \times \{t_0\}$ and that $\nabla' U\big|_{\mathcal{C}_-} = 0$, we have

$$V \ge U \ge U(x_0, t_0) + (m + \mu)\, x_1 + b(t - t_0) - o(\varepsilon), \text{ on } \overline{K_\varepsilon}, \tag{3.7.19}$$

where

$$b = \partial_t U(x_0, t_0) = \partial_t V(x_0, t_0).$$

Now, we give the construction of the test functions in different cases of the time derivative b. If $b \geq 0$, then we obtain a contradiction to the assumption that $(x_0, t_0) \in \mathcal{C}_-$ by the same argument in Case 2. If $b < 0$, we claim that the following function

$$H(x,t) := U(x_0,t_0) + b(t-t_0) + (m+\mu/2)x_1 - \frac{|b|}{100}|t-t_0|$$

touches V from below at (x_0, t_0) in $\overline{K_\varepsilon}$ for sufficiently small $\varepsilon > 0$. By comparison principle of heat equations, it suffices to show that $H \leq V$ on the parabolic boundary $\partial_{\mathrm{p}} K_\varepsilon$, as H is a subsolution to the heat equation. Indeed:

- On $\partial K_\varepsilon \cap \{x_1 = 0\}$, by $\frac{1}{\delta}$-semi-concavity of U in the time variable,

  $$H = U(x_0,t_0) + b(t-t_0) - \frac{|b|}{100}|t-t_0| \leq U(x_0,t_0) + b(t-t_0) - \frac{1}{2\delta}|t-t_0|^2 \leq U(x,t)$$

  for $|t - t_0| \leq \varepsilon$, and $\varepsilon > 0$ smaller, if necessary, depending on $|b| > 0$ and on $\delta > 0$.

- On $K_\varepsilon \cap \{x_1 \geq \varepsilon/2\}$, by (3.7.19),

  $$H \leq U(x_0,t_0) + b(t-t_0) + (m+\mu)\, x_1 - \frac{1}{4}\mu\varepsilon \leq U(x,t) \leq V(x,t)$$

  when $\varepsilon > 0$ is small enough.

- When $t = t_0 - \varepsilon$, by (3.7.19) again,

  $$H = U(x_0,t_0) + b(t-t_0) + (m+\mu/2)x_1 - \frac{|b|}{100}\varepsilon \leq U(x,t) \leq V(x,t)$$

  when $\varepsilon > 0$ is small enough. Again we are using $|b| > 0$.

- On $\partial^+\Omega^r_\varepsilon \times (t_0 - \varepsilon, t_0 + \varepsilon)$, by (3.7.18) and the continuity of $U - V$,

  $$V(x,t) > U(x_0,t_0) + (m+\mu)\, x_1 + b(t-t_0) \geq H,$$

  for $\varepsilon > 0$ sufficiently small.

Combining the above discussions, we conclude that the one-dimensional test function H touches V from below at (x_0, t_0) in K_ε. By the laminar dynamic slope condition Definition 3.5.1 (c) of $V = \tilde{v}$,

$$m \geq \partial_1 H(x_0,t_0) = m + \mu/2,$$

which is a contradiction.

This concludes the proof of (3.7.16), which, also, concludes the proof that Case 3 cannot occur, showing (3.7.9) and concluding the proof of comparison.

□

3.8 Special cases exhibiting facets

In this section we study special solutions to (3.1.3) and its steady states that will certainly exhibit facets (or contact set). This justifies that the homogenized equation (3.1.3) does not reduce to the (trivial) standard Neumann problems.

First we show, in Section 3.8.1, that any minimal supersolution to the elliptic problem (3.1.7) satisfies a "boundary maximum principle" on an open subset of B_1'. Solutions of Neumann problems

$$\Delta u = 0 \text{ in } B_1^+ \text{ with } \partial_1 u = 0 \text{ on } B_1', \text{ and } u = g \text{ on } \partial^+ B_1^+$$

do not, in general, have this property. Specifically the set of $g \in C(\partial B_1^+)$ for which the Neumann solution fails the boundary maximum principle is non-empty and open in the uniform topology.

In Section 3.8.2, we study the viscosity solutions to (3.1.3) that satisfy a specific type of time-monotone Dirichlet boundary condition. We show that they converge to an extremal steady state to the elliptic problem (3.1.7) as time goes to infinity. By the discussions in Section 3.8.1, these steady states are generally not Neumann steady states. This indicates the general existence of facets/contact sets in the parabolic problem (3.1.3).

3.8.1 Strong subsolutions and boundary maximum principle

In this subsection, we study the strong subsolutions arising in condition (3) of Theorem 3.4.7. We show that if $\max f$ is sufficiently large, then the corresponding minimal supersolutions to (3.1.7) satisfies a *boundary maximum principle* on an open subset of B_1'. This property is not generally satisfied by Neumann solutions. This phenomenon indicates that facets exist generally instead of merely in some isolated examples.

Definition 3.8.1. For $K \in \mathbb{R} \cup \{+\infty\}$, say that u is a K-*strong subsolution* to the gradient degenerate problem (3.1.7) if u is a subsolution of

$$\Delta u \geq 0 \text{ in } B_1^+ \text{ with } \partial_1 u \geq 0 \text{ on } B_1', \tag{3.8.1}$$

and for any smooth function $\eta(x) = \eta(x_1)$ (depending only on x_1) that touches u from above at $z \in B_1'$, the following holds: If there exists a neighborhood $\Omega \subset \mathbb{R}^d$ containing z, with $\Omega \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$, such that

- $\eta \geq u$ in $\Omega \cap \overline{B_1^+}$,
- $\eta > u$ on $\partial\Omega \cap \overline{B_1^+}$,

then the derivative satisfies $\partial_1\eta(z) = \partial_1\eta(0) \geq K$. We simply call a $+\infty$-strong subsolution a strong subsolution. Define K-strong supersolutions symmetrically.

Definition 3.8.2. Say that $u \in \mathrm{USC}(\overline{B_1^+})$ satisfies the *boundary maximum principle* if for any $U \subset\subset B_1'$

$$\max_{\overline{U}} u = \max_{\partial' U} u.$$

Define, similarly, the boundary minimum principle for $u \in \mathrm{LSC}(\overline{B_1^+})$.

Remark 3.8.3. For a generic boundary data $g \in C(\partial B_1 \cap \{x_1 \geq 0\})$ we can find a continuum of solutions to (3.1.8) with only the gradient degenerate boundary condition

$$\partial_1 u |\nabla' u| = 0$$

that are homogenization limits. Indeed, we consider the homogenization limits of the semi-linear problem (3.1.2) with

$$f(u) := |K| \sin(2\pi u).$$

Then we define for each $K \in \mathbb{R}$ the solution u_K to be the maximal subsolution to (3.1.8) if $K > 0$, or the minimal supersolution if $K < 0$. Notice that as $\langle f \rangle = 0$, u_K is exactly the solution to the Neumann boundary condition $\partial_1 u_K = 0$ when $K = 0$.

Lemma 3.8.4 ([82, Section 8.1])**.** *A subsolution to (3.8.1) is further a $+\infty$-strong subsolution if and only if it satisfies the boundary maximum principle on B_1'.*

Let us discuss the implications of the K-strong subsolution property for $K < +\infty$.

Lemma 3.8.5. *Suppose a K-strong subsolution u is Lipschitz on B_1^+ with Lipschitz constant $0 < S < K$. Then u satisfies the boundary maximum principle on B_1'.*

Proof. It suffices to show that u is a $+\infty$-strong subsolution by Lemma 3.8.4.

Suppose there is a smooth function of the form $\eta(x) \equiv \eta(x_1)$ that touches u from above at $z \in B_1'$ and for some open domain $\Omega \subset\subset \mathbb{R}^d$ containing z such that $\Omega \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$, $\eta \geq u$ in $\Omega \cap \overline{B_1^+}$, $\eta > u$ on $\partial\Omega \cap \overline{B_1^+}$.

On the other hand, by the Lipschitz continuity, we have

$$u(x_1, x') \leq Sx_1 + u(0, x'), \quad \text{for } (x_1, x') \in \overline{\Omega'} \times [0, \delta],$$

for some small $\delta > 0$. Because $\eta > u$ on $\partial\Omega \cap \overline{B_1^+}$, we know that

$$u(0, x') < \eta(0) = u(z), \quad \text{for } (0, x') \in \partial'\Omega'.$$

This implies

$$u(x_1, x') < (S + \varepsilon)x_1 + u(z), \quad \text{for } (x_1, x') \in \partial\left(\Omega' \times (-1, \delta)\right) \cap \overline{B_1^+},$$

for any small $\varepsilon > 0$ such that $S + \varepsilon < K$, contradicting the K-strong subsolution property of u.

□

Corollary 3.8.6. *There is a finite nondecreasing function* $\ell : [0,1) \to [0, +\infty)$ *with* $\ell(1^-) = +\infty$ *so that for any* $r \in (0,1)$ *if* $\max f > \ell(r)$ *then any minimal supersolution* u *to* (3.1.7) *with* $\operatorname{osc}_{B_1^+} u \leq 1$ *satisfies the boundary maximum principle on* B_r'.

Proof. By Lemma 3.1 in [82], any minimal supersolution u of (3.1.7) with $\operatorname{osc}_{B_1^+} u \leq 1$ is Lipschitz in $B_r^+ \cup B_r'$ with Lipschitz constant at most $C(d, r)$ independent of $\max f$. Define $\ell(r) := C(d, r) + 1$. Then, assuming $\max f > \ell(r)$ as in the statement, then u is a $(\max f)$-strong subsolution in $\overline{B_r^+}$ having Lipschitz constant strictly smaller than $\ell(r) < \max f$. Then Lemma 3.8.5 implies that u satisfies the boundary maximum principle. □

Proof of Proposition 3.1.4. By Remark 3.3.8 we focus on the case $\langle f \rangle = 0$. We argue with $\operatorname{osc}_{\partial^+ B_1^+} g \leq \varepsilon$ for $\varepsilon > 0$ small and $\max f > 0$ fixed. Notice that we can replace solutions u by u/ε so that we can just focus on the case $\operatorname{osc}_{\partial^+ B_1^+} g < 2$ and $\max f/\varepsilon$ being large.

For each $g \in C(\partial^+ B_1^+)$ define v^g to be the unique solution of

$$\begin{cases} \Delta v^g = 0 & \text{in } B_1^+ \\ \partial_1 v^g = 0 & \text{on } B_1' \\ v^g = g & \text{on } \partial^+ B_1^+. \end{cases}$$

We start with an example where the boundary maximum principle fails for v^g. Let $h(x) := x_1$. Since $h(x) = x_1$ is a strict subsolution of the above Neumann problem $v^h(x) > x_1$ in B_1^+. In particular v^h attains its positive maximum value on B_1^+ at some $x' \in B_1'$ with $|x'| < 1$. In particular v^h fails to satisfy the boundary maximum principle in B_r' for $|x'| < r_0 < 1$ and r_0 sufficiently close to 1.

Define $\mathcal{F} \subset C(\partial^+ B_1^+) \cap \{\operatorname{osc}_{\partial^+ B_1^+} g < 2\}$ to consist of all data g such that v^g does not satisfy the boundary maximum principle on B_{r_0}' with r_0 defined as above. Notice that $\mathcal{F}$ is open, since failing the boundary maximum principle on some specific $U \subset\subset B_1'$ is an open condition in the uniform topology, and $g \mapsto v^g$ is continuous in uniform topology on $C(\partial^+ B_1^+)$ by comparison principle. The set $\mathcal{F}$ is nonempty we we have already established that $v^h \in \mathcal{F}$, note that $0 \leq v^h \leq 1$ by comparison principle so that the oscillation condition is satisfied.

By Corollary 3.8.6 the minimal supersolutions u_g of (3.1.7) with boundary data $g \in \mathcal{F}$ satisfy the boundary maximum principle on B_{r_0}' when $\max f/\varepsilon$ is sufficiently large. If $\mathcal{C}_{u_g} = \emptyset$, then, by Lemma 3.4.8, $\partial_1 u_g = 0$ on B_1' and hence by uniqueness $u_g = v^g$, violating the assumption that v^g does not satisfy the boundary maximum principle on B_{r_0}'. □

3.8.2 Extremal steady states and parabolic flows with monotone shift

In this subsection, we discuss the steady states of the homogenized flow having a specific monotone Dirichlet boundary condition.

Definition 3.8.7. On the parabolic boundary $\partial_{\mathrm{p}}^+ D_\infty^+$, we call a continuous function g to be a *monotone shift* if g is monotone in time and

$$\pi_g := \lim_{T \to \infty} g(\cdot, T)$$

exists in the sense of uniform convergence, and either

$$\min_{x \in \partial^+ B_1^+} \pi_g(x) - \max_{x \in \overline{B_1^+}} g(x, 0) > 0, \tag{3.8.2}$$

or

$$\min_{x\in\overline{B_1^+}} g(x,0) - \max_{x\in\partial^+ B_1^+} \pi_g(x) > 0. \tag{3.8.3}$$

We call g an up-shift if in the case (3.8.2) and a down-shift if in the case (3.8.3).

We show that the long-time limit of solutions to the parabolic flow (3.1.3) under monotone shift boundary data is an extremal solution of the elliptic equation (3.1.7).

Theorem 3.8.8. *Let $g \in C(\partial_{\mathrm{p}}^+ D_\infty^+)$ a monotone up-shift (resp. down-shift) as in Definition 3.8.7. Let u be the solution to* (3.1.3) *with boundary data g on $\partial_{\mathrm{p}}^+ D_\infty^+$. Then $u(\cdot,t)$ converges uniformly on $\overline{B_1^+}$ to u_0 the minimal supersolution of* (3.1.7) *with boundary condition $u_0 = \pi_g$ on $\partial^+ B_1^+$ (resp. maximal subsolution).*

Proof of Theorem 3.8.8. The case for down-shift is symmetrical, so it suffices to consider the case that g is an up-shift.

By comparison principle, we know that as g is a bounded function on $\partial_{\mathrm{p}}^+ D_\infty^+$, the functions $u(\cdot,t)$ are uniformly bounded as $t \to \infty$. We define for $x \in \overline{B_1^+}$ and $t \in [-1,1]$

$$v^*(x,t) \equiv v^*(x) := \limsup_{T\to\infty}{}^* u(x,t+T) \quad \text{and} \quad v_*(x,t) \equiv v_*(x) := \liminf_{T\to\infty}{}_* u(x,t+T).$$

Then v^* is upper semicontinuous and v_* is lower semicontinuous with $v^* \ge v_*$ in $\overline{B_1^+}$. Furthermore, by Lemma 3.6.4, $v^* = v_* = \pi_g$ on $\partial^+ B_1^+$.

Let $\psi(x,t)$ be a smooth function that touches v^* from above at $t = 0$ and $x_0 \in B_1^+ \cup B_1'$, then because v^* is constant in time, $\partial_t \psi(x_0,0) = 0$. If $\Delta\psi(x_0,0) < 0$, then in a small neighborhood of $(x_0,0)$, we have $\Delta\psi < \partial_t\psi$ making ψ a strict supersolution of the heat equation. By Lemma 3.2.3, we can find a sequence of large $T \to \infty$ and $C_T = o_T(1)$ such that $\psi(\cdot, t-T) + C_T$ touches u from above at $t_T \in [T-1, T+1]$ and some $x_T \in B_1^+ \times B_1'$ such that $|x_T - x_0| + |t_T - T| = o_T(1)$. By the strict supersolution property of ψ, we know that $x_T \in B_1'$, and hence we have the condition for all large T,

$$\partial_1\psi(x_T, t_T - T) \ge L_*(\nabla'\psi(x_T, t_T - T)),$$

which implies that v^* is a subsolution to the equation (3.1.7). Similarly, we can show that v_* is a supersolution to (3.1.7).

To show that $v_* = v^*$ is a minimal supersolution to (3.1.7), it suffices to show the condition (3) in Theorem 3.4.7. By Theorem 3.4.7, we know that

$$v_* \geq \tilde{v} \geq \min_{x \in \partial^+ B_1^+} \pi_g(x) > \max_{x \in \overline{B_1^+}} g(x,0),$$

where $\tilde{v}$ is the minimal supersolution to (3.1.7) with boundary data π_g. Moreover, for every fixed $x \in \partial^+ B_1^+$ we have by monotonicity in time

$$\pi_g(x) \geq g(x,t), t \geq 0.$$

It is not difficult to check that $\tilde{v}$ is itself a stationary solution to the parabolic flow (3.1.3). According to the comparison principle 3.5.3, this shows that $\tilde{v} \geq u$ on the whole space-time domain $\overline{D_\infty^+}$. In particular,

$$v^* \geq v_* \geq \tilde{v} \geq u.$$

Now let $\phi(x_1,t)$ be a spatially 1-variable smooth function that touches v^* from above at $(x_0,0) \in B_1' \times \{0\}$ in $\overline{B_1^+} \times [-1,1]$ that satisfies for some open spatial domain $\Omega \subset\subset \mathbb{R}^d$ containing x_0 such that $\Omega \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$ we have $\phi \geq v^*$ in $\Omega \cap \overline{B_1^+}$ and $\phi > v^* + \delta$ on $(\partial\Omega) \cap \overline{B_1^+}$ for some small $\delta > 0$. In fact, by the inequality $v^* \geq u$ we know that

$$\min_{t \in [-1,1]} \phi(\cdot,t) > u + \delta, \text{ on } \left(\partial\Omega \cap \overline{B_1^+}\right) \times [0,\infty).$$

We finish the proof by showing that $\partial_1 \phi(x_0,0) \geq \max f$. We argue by contradiction and assume that $\partial_1 \phi(x_0,0) < \max f$. To that end, we replace ϕ and Ω by

$$\phi(x_1,t) + \mu x_1 + \mu t^2 - \frac{1}{\mu} x_1^2 \quad \text{and} \quad \Omega \cap \{x_1 \leq \mu^2/4\}$$

for some small $\mu > 0$, which ensures that ϕ is a supersolution of the heat equation in $\Omega \times [-t_\mu, t_\mu]$ for some small $t_\mu > 0$ depending on μ and all the previous properties are preserved. Again by Lemma 3.2.3, we can find a sequence of large $T \to \infty$ and constants $C_T = o_T(1)$ such that $\phi_T := \phi(\cdot, t-T) + C_T$ touches u from above at

$$(x_T, s_T) \in B_1' \times (T - t_\mu/2, T + t_\mu/2) \text{ in } \left(\overline{\Omega} \cap \overline{B_1^+}\right) \times [T - t_\mu/2, T + t_\mu/2]$$

with $|x_T - x_0| + |s_T - T| = o_T(1)$. Moreover, we have when $T > 0$ is large,

$$\phi_T(\cdot,T) > u + \delta/2, \quad \text{on } \left(\partial\Omega \cap \overline{B_1^+}\right) \times [0,\infty),$$

and by (3.8.2)

$$\phi_T(\cdot, T) > u, \quad \text{on } \left(\overline{\Omega} \cap \overline{B_1^+}\right) \times \{0\}.$$

For $\eta > 0$ small, we know that

$$p(x) := (\max f)x_1 + \phi_T(0, T)$$

is a stationary solution to (3.1.3), $p(x) \geq \phi_T(x, T)$ for $x \in \overline{\Omega} \cap \{x_1 \leq \eta\} =: \overline{\Omega}_\eta$ with strict inequality when $x_1 > 0$, and $p(x)$ also touches $u(\cdot, T)$ from above at x_T. However, this is a contradiction to the comparison principle 3.5.3 because by the construction $p > u$ on the parabolic boundary $\partial_{\mathrm{p}}^+ (\Omega_\eta \cap B_1^+) \times (0, T + \eta)$ for some sufficiently small $\eta > 0$.

□

3.9 Appendix

3.9.1 Some regularity estimates

Here we present several lemmas that are frequently used.

Lemma 3.9.1. *If a bounded function u is harmonic in B_1^+, and in the sense of viscosity*

$$\partial_1 u^* \leq M \quad \text{and} \quad \partial_1 u_* \geq -M, \ \text{ on } B_1'$$

for some constant $M > 0$, then $u \in C^\alpha_{\mathrm{loc}}(B_1^+ \cup B_1')$, and there is a constant $C > 0$ independent of u such that

$$\|u\|_{C^\alpha(B^+_{1/2})} \leq C \|u\|_{L^\infty(B_1^+)} .$$

Furthermore, if $u^* = u_* = 0$ on $\partial B_1 \cap \{x_1 \geq 0\}$ then $u \in C^\alpha(B_1^+)$ and

$$\|u\|_{C^\alpha(B_1^+)} \leq C \|u\|_{L^\infty(B_1^+)} .$$

Remark 3.9.2. The same result fails if one relaxes to the case that u is merely bounded and u^* is subharmonic and u_* is superharmonic. This is because it is unlikely that $u^* = u_*$ in the measure-theoretic sense in the interior B_1^+.

Lemma 3.9.3. *If a continuous function u is harmonic in B_1^+, and in the sense of viscosity*

$$\partial_1 u = h, \quad \text{on } B_1'$$

for some $h \in C^{\alpha}_{\text{loc}}(B_1')$ and $\alpha > 0$, then $u \in C^{1,\alpha}_{\text{loc}}(B_1^+ \cup B_1')$, and there is a constant $C > 0$ independent of u such that

$$\|u\|_{C^{1,\alpha}(\overline{B^+_{1/2}})} \le C \|u\|_{L^\infty(B_1^+)} .$$

We have a similar parabolic version for the above estimates.

Lemma 3.9.4. *If a bounded function u satisfies $\partial_t u = \Delta u$ in $D_T^+ = B_1^+ \times (0,T]$ for some end time $T > 0$, and in the sense of viscosity*

$$\partial_1 u^* \le M, \ \partial_1 u_* \ge -M, \quad \text{on } D_T'$$

for some constant $M > 0$, then there is some $\alpha > 0$, $u \in C^{\alpha,\alpha/2}_{\text{loc}}(D_T^+ \cup D_T')$, and for any $0 < h < T$ there is a constant $C > 0$ independent of u such that

$$\|u\|_{C^{\alpha,\alpha/2}(B^+_{1/2}\times(h,T))} \le C \|u\|_{L^\infty(B_1^+\times(0,T))} .$$

Lemma 3.9.5. *If a bounded continuous function u satisfies $\partial_t u = \Delta u$ in $B_1^+ \times (0,T]$ for some end time $T > 0$, and in the sense of viscosity*

$$\partial_1 u = h, \quad \text{on } D_T'$$

for some $h \in C^{\alpha}_{\text{loc}}(D_T')$ and $\alpha > 0$, then $u \in C^{1+\alpha,1/2+\alpha/2}_{\text{loc}}(D_T^+ \cup D_T')$, and for any $0 < \tau < T$ there is a constant $C > 0$ independent of u such that

$$\|u\|_{C^{1+\alpha,1/2+\alpha/2}\left(B^+_{1/2}\times(\tau,T]\cup B'_{1/2}\times(\tau,T]\right)} \le C \|u\|_{L^\infty(B_1^+\times(0,T))} .$$

Lemma 3.9.3 and the continuous version of Lemma 3.9.1 can be found in [133]. Here we present a proof for Lemma 3.9.1. The proof of Lemma 3.9.4 is similar to that of Lemma 3.9.1, and the proof of Lemma 3.9.5 can be found in [110].

Proof of Lemma 3.9.1. The proof is essentially similar to the continuous case as in [133], and here we mainly explain the issue of discontinuity near B_1'. It suffices to show that for some $0 < \mu < 1$, we always have for small $r > 0$

$$\operatorname*{osc}_{B^+_{r/2}} u \le \mu \operatorname*{osc}_{B^+_r} u + Mr.$$

To show this inequality, we consider the infimum $\tilde{w}$ of all continuous solutions w to the equation

$$\begin{cases} \Delta w = 0 & \text{in } B_r^+ \\ w \geq u^* & \text{on } \partial B_r \cap \{x_1 \geq 0\} \\ \partial_1 w \leq -M & \text{on } B_r'. \end{cases}$$

Notice that $\tilde{u} := \tilde{w} - h$ with h the classical solution to

$$\begin{cases} \Delta h = 0 & \text{in } B_r^+ \\ h = 0, & \text{on } \partial^+ B_r \\ \partial_1 h = M & \text{on } B_r', \end{cases} \tag{3.9.1}$$

gives the infimum of continuous solutions p to

$$\begin{cases} \Delta p = 0 & \text{in } B_r^+ \\ p \geq u^* & \text{on } \partial^+ B_r \\ \partial_1 p \leq 0 & \text{on } B_r', \end{cases}$$

which, after doing even reflection, by standard Perron's method gives a harmonic function $\tilde{u}$ in B_r such that $\tilde{u} = u^*$ on ∂B_r. Similarly, we can define $\bar{u}$ to be the harmonic function in B_r such that $\bar{u} = u_*$ on ∂B_r.

By L^∞ theory of harmonic functions and the continuity of u on $\partial B_r \cap \{x_1 > 0\}$, we know that

$$\tilde{u} = \bar{u} =: v, \quad \text{on } B_r^+ \cup B_1'.$$

This shows that

$$|u - v| \leq |h|, \quad \text{on } B_r^+,$$

which proves the claim.

In the case that $u^* = u_* = 0$ on $\partial B_1 \cap \{x_1 \geq 0\}$, we observe that by comparison principle, $h \leq u_* \leq u^* \leq -h$ globally on $\overline{B_1^+}$. It then suffices to show that h is in $C^\alpha(B_1^+)$ for some $\alpha > 0$. Observe that $h = \tilde{h} - Mx_1$, where $\tilde{h}$ can be evenly reflected to be a harmonic function on B_1 that satisfies

$$\tilde{h} = M|x_1| \;\; \text{on } \partial B_1.$$

By [141], we know that $\tilde{h}$ is in $C^\alpha(B_1)$ and hence $h \in C^\alpha(B_1^+)$. □

3.9.2 Tangential regularization

In this section, we introduce the tangential regularization procedure used in the parabolic comparison principle. A tangential regularization is an inf- / sup- convolution involving only the tangential x' (and time t if in the parabolic case) variables with an additional harmonic (or caloric) lift. A tangential regularization is preferred in the analysis of nonlinear Neumann problems like (3.1.3), (3.1.7) and (3.3.25), because the commonly used doubling variable method does not apply due to the lack of a uniform obliqueness condition [27, 60], and the also commonly used standard inf/sup-convolutions does not work either due to similar issues.

3.9.3 Tangential regularization for the elliptic case

In the elliptic case for an upper semicontinuous function u on $\overline{B_1^+}$, we extend $u = -\infty$ outside $\overline{B_1^+}$ and define for $\varepsilon > 0$ the following tangential sup-convolution

$$\mathcal{T}^\varepsilon u(x) := \sup\left\{u(y) - \frac{1}{2\varepsilon}|x-y|^2 \ ; \ x_1 = y_1\right\}.$$

For a lower semicontinuous function v on $\overline{B_1^+}$, we define similarly the tangential inf-convolution

$$\mathcal{T}_\varepsilon v(x) := \inf\left\{v(y) + \frac{1}{2\varepsilon}|x-y|^2 \ ; \ x_1 = y_1\right\}.$$

The *elliptic tangential regularization* $\mathcal{H}^\varepsilon u$ and $\mathcal{H}_\varepsilon v$ of u and v are respectively the harmonic lifts of $\mathcal{T}^\varepsilon u$ and $\mathcal{T}_\varepsilon v$, that is, we define $\mathcal{H}^\varepsilon u$ as the infimum of all continuous harmonic functions on $\overline{B_1^+}$ that is above $\mathcal{T}^\varepsilon u$ and $\mathcal{H}_\varepsilon v$ the supremum of all continuous harmonic functions below $\mathcal{T}_\varepsilon v$. The properties of the elliptic tangential regularization can be found in [82].

3.9.4 Tangential regularization for the parabolic case

Let us now focus on the tangential regularization for the parabolic case.

- **Parabolic tangential inf-/sup-convolution**

Definition 3.9.6. Let $U \subset \mathbb{R}^d \times \mathbb{R}$ be a space-time domain, $u \in USC(\overline{U})$ and is finite on $\overline{U}$. Usually we can extend by $u = -\infty$ outside $\overline{U}$ and this extension is still upper semicontinuous. Define for $\varepsilon > 0$ and all $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ the parabolic tangential sup-convolution

$$\mathcal{T}^\varepsilon u(x,t) := \sup\left\{u(y,s) - \frac{1}{2\varepsilon}|x-y|^2 - \frac{1}{2\varepsilon}(t-s)^2 \ ; \ x_1 = y_1, \ (y,s) \in U\right\}.$$

Similarly, for $v \in LSC(\overline{U})$ bounded from below and not identically $+\infty$, we define the parabolic tangential inf-convolution $\mathcal{T}_\varepsilon v = -\mathcal{T}^\varepsilon(-v)$.

Lemma 3.9.7. *Let u be upper semicontinuous and bounded in $\overline{U}$. For every $\varepsilon > 0$, the function $\mathcal{T}^\varepsilon u$ is finite everywhere on $\mathbb{R}^d \times \mathbb{R}$, upper semicontinuous and $\frac{1}{2\varepsilon}$-semi-convex in (x_1, x', t) for any fixed x_1. Moreover, for any $R > 0, 0 < \varepsilon < 1$ we have the following Lipschitz estimate: if $x_1 = y_1$ and $|x|^2 + |y|^2 + |t|^2 + |s|^2 \le R^2$ then*

$$|\mathcal{T}^\varepsilon u(x,t) - \mathcal{T}^\varepsilon u(y,s)| \le C_\varepsilon \left(R + \|u\|_{L^\infty(U)}\right)(|x-y| + |t-s|)\,.$$

Proof. The proof of the pointwise finiteness and semi-convexity in (x_1, x', t) for fixed x_1 is the same as the standard sup-convolutions.

Similar to the standard sup-convolution, by boundedness of u, for sufficiently small $\varepsilon > 0$ there is a $r_\varepsilon > 0$ such that at any $(x_0, t_0) = (x_{0,1}, x_0', t_0) \in \overline{U}$ there will be a $(x_\varepsilon, t_\varepsilon) = (x_{0,1}, x_\varepsilon', t_\varepsilon) \in \overline{U} \cap B_{r_\varepsilon}(x_0, t_0)$ such that

$$\mathcal{T}^\varepsilon u(x_0, t_0) = u(x_\varepsilon, t_\varepsilon) - \frac{1}{2\varepsilon}|x_0' - x_\varepsilon'|^2 - \frac{1}{2\varepsilon}(t_0 - t_\varepsilon)^2.$$

Notice that if $|u| \le M$ then $r_\varepsilon = (100M\varepsilon)^{1/2}$ will suffice because outside the ball $B_{r_\varepsilon}(x_0,t_0)$ we have $|x_0' - x_\varepsilon'|^2 + (t_0 - t_\varepsilon)^2 > 100M\varepsilon$ and hence $\mathcal{T}^\varepsilon u(x_0,t_0) < -M \le u(x_0,t_0)$, which is impossible.

To show the upper semicontinuity we consider any converging sequence of points $(x_n, t_n) = (x_{n,1}, x_n', t_n) \to (x_0, t_0) = (x_{0,1}, x_0', t_0)$ in $\overline{U}$. By the above discussion, there will be a sequence $(x_{n,\varepsilon}, t_{n,\varepsilon}) = (x_{n,1}, x_{n,\varepsilon}', t_{n,\varepsilon}) \in \overline{U} \cap B_{r_\varepsilon}(x_n, t_n)$ such that

$$\mathcal{T}^\varepsilon u(x_n, t_n) = u(x_{n,\varepsilon}, t_{n,\varepsilon}) - \frac{1}{2\varepsilon}|x_n' - x_{n,\varepsilon}'|^2 - \frac{1}{2\varepsilon}(t_n - t_{n,\varepsilon})^2.$$

By compactness of the sequence $(x_{n,\varepsilon}, t_{n,\varepsilon})$ for each fixed ε, we know that in a convergent subsequence n_k with $\lim_{k\to\infty}(x_{n_k,\varepsilon}, t_{n_k,\varepsilon}) = (\tilde{x}_\varepsilon, \tilde{t}_\varepsilon)$

$$\limsup_{k\to\infty} \mathcal{T}^\varepsilon u(x_{n_k}, t_{n_k}) \le u(\tilde{x}_\varepsilon, \tilde{t}_\varepsilon) - \frac{1}{2\varepsilon}|\tilde{x}_\varepsilon' - x_0'|^2 - \frac{1}{2\varepsilon}(\tilde{t}_\varepsilon - t_0)^2 \le \mathcal{T}^\varepsilon u(x_0, t_0),$$

which implies that $\mathcal{T}^\varepsilon u$ is upper semicontinuous at (x_0, t_0).

To show the Lipschitz estimate, we write

$$\mathcal{T}^\varepsilon u(x,t) = u(x_\varepsilon, t_\varepsilon) - \frac{1}{2\varepsilon}|x' - x'_\varepsilon|^2 - \frac{1}{2\varepsilon}(t - t_\varepsilon)^2,$$

and because for (y,s) such that $y_1 = x_1$

$$\mathcal{T}^\varepsilon u(y,s) \geq u(x_\varepsilon, t_\varepsilon) - \frac{1}{2\varepsilon}|y' - x'_\varepsilon|^2 - \frac{1}{2\varepsilon}(s - t_\varepsilon)^2,$$

we have

$$\begin{aligned}\mathcal{T}^\varepsilon u(x,t) - \mathcal{T}^\varepsilon u(y,s) &\leq \frac{1}{2\varepsilon}\left(|y' - x'_\varepsilon|^2 + (s - t_\varepsilon)^2 - |x' - x'_\varepsilon|^2 - (t - t_\varepsilon)^2\right) \\ &\leq C_\varepsilon\left(R + \|u\|_{L^\infty(U)}\right)(|x - y| + |t - s|).\end{aligned}$$

□

Lemma 3.9.8. *Let u be upper semicontinuous and bounded in $\overline{U}$. Let ϕ be a smooth function crossing $\mathcal{T}^\varepsilon u$ from above (strictly) at $(x_0, t_0) = (x_{0,1}, x'_0, t_0) \in \overline{U}$, and let $(x_\varepsilon, t_\varepsilon) = (x_{0,1}, x'_\varepsilon, t_\varepsilon) \in \overline{U}$ be a point close to (x_0, t_0) such that*

$$\mathcal{T}^\varepsilon u(x_0, t_0) = u(x_\varepsilon, t_\varepsilon) - \frac{1}{2\varepsilon}|x'_\varepsilon - x'_0|^2 - \frac{1}{2\varepsilon}(t_\varepsilon - t_0)^2.$$

Then $\phi^\varepsilon(x,t) := \phi(x + x_0 - x_\varepsilon, t + t_0 - t_\varepsilon) - \frac{1}{2\varepsilon}|x'_\varepsilon - x'_0|^2 - \frac{1}{2\varepsilon}(t_\varepsilon - t_0)^2$ crosses u from above (strictly) at $(x_\varepsilon, t_\varepsilon)$ with

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}|x'_\varepsilon - x'_0|^2 + \frac{1}{2\varepsilon}(t_\varepsilon - t_0)^2 = 0.$$

Moreover, we have

$$\nabla\phi^\varepsilon(x_\varepsilon, t_\varepsilon) = \nabla\phi(x_0, t_0) = \frac{1}{\varepsilon}(x'_\varepsilon - x'_0) \quad \text{and} \quad \partial_t\phi^\varepsilon(x_\varepsilon, t_\varepsilon) = \partial_t\phi(x_0, t_0) \leq \frac{1}{\varepsilon}(t_\varepsilon - t_0).$$

Proof. The proof is done by a simple adaptation of the arguments in [51, Proposition 8.6]. □

Corollary 3.9.9. *Let $U = (B_1^+ \cup B'_1) \times [0, \infty)$, and u be a subsolution to* (3.1.3) *on U, then for every $\theta > 0$ there is small $\delta_0 > 0$ so that for all $0 < \delta < \delta_0$, the parabolic tangential regularization $\mathcal{T}^\delta u$ is a subsolution to* (3.1.3) *on*

$$(B^+_{1-\theta} \cup B'_{1-\theta}) \times [\theta, \infty).$$

Proof. By Lemma 3.9.8, any smooth function crossing $\mathcal{T}^\delta u$ from above will also cross u from above with a small additive constant and small-distance translations in time and tangential directions. □

Lemma 3.9.10. *Let u be upper semicontinuous in $\overline{U}$, then the upper half relaxed limit*

$$\limsup_{\varepsilon\to 0}{}^{*}\mathcal{T}^\varepsilon u = \begin{cases} u, & \text{in } \overline{U} \\ -\infty, & \text{elsewhere.} \end{cases}$$

In particular, for any compact subset $K \subset\subset \mathbb{R}^d \times \mathbb{R}$ we have

$$\lim_{\varepsilon\to 0} \max_{(x,t)\in K} \mathcal{T}^\varepsilon u(x,t) = \max_{(x,t)\in K} u(x,t).$$

Proof. This convergence property is a corollary of Proposition 3.7 in [60]. □

- **Caloric lift**

Let $D_T = B_1 \times (0,T)$ be a space-time cylinder and $D_T^+ := B_1^+ \times (0,T]$ the positive interior. For any $g \in C\left(\partial_{\mathrm{p}} D_T^+\right)$ there is a solution $u \in C(\overline{D_T^+}) \cap C^\infty(D_T^+)$ of

$$\begin{cases} \partial_t u = \Delta u & \text{in } D_T^+ \\ u = g & \text{on } \partial_{\mathrm{p}} D_T^+. \end{cases}$$

We show a similar result above a subsolution to $\Delta u \geq \partial_t u$.

Lemma 3.9.11. *Let v be a bounded upper semicontinuous function on $\overline{D_T^+}$ and a subsolution to $\partial_t v \leq \Delta v$ in D_T^+, then there is a unique $u \geq v$ on $\overline{D_T^+}$ such that u solves $\partial_t u = \Delta u$ in the classical sense in D_T^+ and*

$$\limsup_{D_T^+ \ni (y,s)\to(x,t)} u(y,s) = v(x,t)$$

for any $(x,t) \in \partial_{\mathrm{p}} D_T^+$.

Proof. We define

$$u(x) := \inf\left\{p(x)\ ;\ p \in C\left(\overline{D_T^+}\right),\ \partial_t p \geq \Delta p, \quad \text{and} \quad p \geq v\right\}.$$

By standard Perron's method we know that u satisfies the required properties. □

Definition 3.9.12. We call u the caloric lift of v on $\overline{D_T^+}$, and denote

$$\mathcal{H}v := u.$$

For a supersolution w to $\partial_t w \geq \Delta w$ we can define similarly $\mathcal{H}w := -\mathcal{H}(-w)$.

An important property of caloric lift is that it preserves the sub/supersolution conditions of (3.1.3).

Lemma 3.9.13. *Suppose u is a (upper semicontinuous) subsolution to* (3.1.3) *on $\overline{D_T^+}$ in the sense of Definition 3.5.1, then its caloric lift $\mathcal{H}u$ is also a subsolution to* (3.1.3).

Proof. In the interior $\mathcal{H}u$ satisfies the heat equation in the classical sense, so we only have to worry about the boundary condition. By Lemma 3.9.11, we know that $\mathcal{H}u = u$ on the boundary $\partial_{\mathrm{p}} D_T^+$ in the sense of semicontinuity, and because $\mathcal{H}u \geq u$ in the interior, any smooth function crossing $\mathcal{H}u$ from above at the boundary point will also cross u from above, and therefore $\mathcal{H}u$ inherits all the viscosity subsolution conditions of u. □

Let us now discuss some initial regularity of caloric lifts.

Lemma 3.9.14. *Let v be as described in the previous lemma. If $v|_{\partial_{\mathrm{p}} D_T^+}$ is continuous at $(x_0, t_0) \in \partial_{\mathrm{p}} D_T^+$, then the corresponding caloric lift $\mathcal{H}v$ is continuous at (x_0, t_0).*

Proof. Let $\omega(s)$ be a modulus of continuity of $v|_{\partial_{\mathrm{p}} D_T^+}$ at (x_0, t_0) and define, for $(y, s) \in \partial_{\mathrm{p}} D_T^+$,

$$h_\pm(y, s) := v(x_0, t_0) \pm \omega(|y - x_0| + |s - t_0|).$$

By definition $h_+ \geq v \geq h_-$ on $\partial_{\mathrm{p}} D_T^+$. Extend $h_\pm$ to the parabolic interior D_T^+ by solving $\Delta h_\pm = \partial_t h_\pm$. This gives us a continuous upper barrier h_+ and lower barrier h_- of u at (x_0, t_0), making u continuous at (x_0, t_0). □

We use the following interior estimate.

Lemma 3.9.15. *Let u be a bounded solution to the heat equation $\partial_t u = \Delta u + f$ in $B_1 \times (0, T]$ for some $f \in C^{\alpha,\alpha/2}(\overline{B_1} \times [0, T])$, then there exists a constant $C > 0$ depending only on α, $0 < h < T$ and dimension d such that*

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(B_{1/2}\times(h,T])} \leq C\left(\|u\|_{L^\infty(B_1\times(0,T])} + \|f\|_{C^{\alpha,\alpha/2}(B_1\times(0,T])}\right).$$

Proof. See [110]. □

The following lemmas are useful in the proof of Theorem 3.5.3. This first result allows us to show that caloric lifts of Lipschitz (in time) boundary data are Lipschitz (in time).

Lemma 3.9.16. *Let u be a bounded solution to the heat equation $\partial_t u = \Delta u$ in $U \times (t_1, t_2)$ for some relative open domain $U \subset\subset B_1^+ \cup B_1'$ and $u|_{\overline{U'}\times[t_1,t_2]}$ is Lipschitz, then for any relative open subdomain $V \subset\subset U$ and small $r > 0$ there is*

$$\|\partial_t u\|_{L^\infty(V\times(t_1+r,t_2-r))} < \infty.$$

Proof. Let g be a Lipschitz extension of $u|_{\overline{U'}\times[t_1,t_2]}$ to the whole $\mathbb{R}^d \times \mathbb{R}$ with the same Lipschitz constant. We solve for h an auxiliary function satisfying

$$\begin{cases} \partial_t h = \Delta h & \text{in } U^+ \times (t_1, t_2] \\ h = g & \text{on } \partial U \times [t_1, t_2] \\ \Delta h(\cdot, t_1) = 0 & \text{in } U^+. \end{cases}$$

Suppose $L > 0$ is the Lipschitz constant for g, then by comparison principle with the testing function $h(x, t_1) \pm L(t - t_1)$, we know that

$$|h(x,t) - h(x,t_1)| \le L(t - t_1). \tag{3.9.2}$$

On the other hand, by applying the maximum principle on $h(x, t + s) - h(x, t)$ for $s > 0$, combining (3.9.2) and the Lipschitz regularity of g we know that

$$\|\partial_t h\|_{L^\infty(U\times(t_1,t_2))} < \infty.$$

Now the proof follows by considering the decomposition

$$u = (u - h) + h,$$

where $u - h$ is smooth near U' as it satisfies the zero Dirichlet boundary condition on U', while h is Lipschitz in time.

□

This next result shows that a solution of the heat equation is (quantitatively) differentiable at a point where the boundary data is $C^{1,1}$.

Lemma 3.9.17. *Let $U \subset\subset B_1^+ \cup B_1'$ be a relatively open domain. Suppose v is a lower semicontinuous supersolution to the heat equation on $U^+ \times (0,T)$ and $u \in C(U \times (0,T))$ is a subsolution to the heat equation. If v touches u from above at $(x_0,t_0) \in U' \times (0,T)$ and both $u\big|_{U'\times(0,T)}, v\big|_{U'\times(0,T)}$ are $C^{1,1}$ at (x_0,t_0), then there is a sequence of radius $r_k \to 0^+$ and smooth functions ψ_k, ξ_k on on*

$$\left(B_{r_k}^+(x_0) \cup B_{r_k}'(x_0)\right) \times (t_0 - r_k^2, t_0 + r_k^2),$$

such that:

(A) ξ_k touches U from above at (x_0,t_0),

(B) ψ_k touches V from below at (x_0,t_0),

(C) $\partial_t(\xi_k - \psi_k)(x_0,t_0) = 0$,

(D) and
$$\lim_{k\to\infty} |\nabla_x(\xi_k - \psi_k)(x_0,t_0)| \to 0.$$

In particular, if $u = v \in C(U \times (0,T))$ satisfies the heat equation in $U^+ \times (0,T)$ then u is differentiable at (x_0,t_0).

Proof. To define r_k, ξ_k, ψ_k, we first choose a small number $r_1 = r > 0$ such that

$$u \le v \quad \text{in } \overline{B_r^+(x_0)} \times [t_0 - r^2, t_0 + r^2].$$

By the $C^{1,1}$ regularity of $u\big|_{U'\times(0,T)}$ and $v\big|_{U'\times(0,T)}$ at (x_0,t_0), we can find two polynomials on $\partial\mathbb{R}_+^d \times \mathbb{R}$ of the form

$$P(x',t) = u(x_0,t_0) + p\cdot(x' - x_0') + b(t-t_0) - C\left(|x'-x_0'|^2 + (t-t_0)^2\right),$$

and

$$Q(x',t) = u(x_0,t_0) + p\cdot(x' - x_0') + b(t-t_0) + C\left(|x'-x_0'|^2 + (t-t_0)^2\right),$$

where $p \in \partial\mathbb{R}_+^d$, $b \in \mathbb{R}$ and $C > 0$ such that

$$Q \ge v\big|_{U'\times(0,T)} \ge u\big|_{U'\times(0,T)} \ge P$$

and since u touches v from below, we have

- P touches $v\big|_{U'\times(0,T)}$ from below at (x_0,t_0),

- Q touches $u\big|_{U'\times(0,T)}$ from above at (x_0, t_0)

respectively in the domain $\overline{B_r^+(x_0)} \times [t_0 - r^2, t_0 + r^2]$.

We define ξ_1 and ψ_1 to be the caloric lifts (see Definition 3.9.12) in $B_r^+(x_0) \times (t_0 - r^2, t_0 + r^2)$ that share the same boundary data u on

$$\overline{B_r^+(x_0)} \times [t_0 - r^2, t_0 + r^2] \setminus \left(B_r^+(x_0) \cup B_1'(x_0)\right) \times (t_0 - r^2, t_0 + r^2],$$

with

$$\xi_1\big|_{\{x_1=0\}} = Q \quad \text{and} \quad \psi_1\big|_{\{x_1=0\}} = P.$$

By the comparison principle for heat equations, ψ_1 and ξ_1 satisfy the conditions (A) and (B). The condition (C) is satisfied because of boundary regularity of heat equations near (x_0, t_0). Moreover, this construction can be repeated for any shrinking radius $r_k \to 0^+$ with $r_k < r_1 = r$. To complete the construction and check condition (D), we verify the convergence of the gradients of ψ_k, ξ_k. This is achieved by observing that the function

$$H_k(y,s) := \frac{(\psi_k - \xi_k)(r_k y + x_0, r_k^2 s + t_0)}{r_k}$$

satisfies the following heat equation:

$$\begin{cases} \partial_s H_k = \Delta_y H_k & \text{in } B_1^+ \times (-1,1), \\ H_k = 2C\left(r_k|y'|^2 + r_k^3 s^2\right) & \text{on } B_1' \times (-1,1], \\ H_k \equiv 0 & \text{on } \overline{B_1^+} \times [-1,1] \setminus (B_1^+ \cup B_1') \times (-1,1]. \end{cases}$$

By Lemma 3.9.15, we have

$$|\nabla_x(\xi_k - \psi_k)(x_0,t_0)| = |\nabla_y H_k(0,0)| = o_k(1),$$

which completes the construction and condition (D) is checked.

□

Chapter 4

Regularity of a gradient degenerate Neumann problem

This chapter is composed of the work with William Feldman on the regularity theory of a gradient degenerate problem that arises from the homogenized equations in the previous chapter. The work has been published by “Journal de Mathématiques Pures et Appliquées”.

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Regularity theory of a gradient degenerate Neumann problem

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A B S T R A C T

We study the regularity and comparison principle for a gradient degenerate Neumann problem. The problem is a generalization of the Signorini or thin obstacle problem which appears in the study of certain singular anisotropic free boundary problems arising from homogenization. In scaling terms, the problem is critical since the gradient degeneracy and the Neumann PDE operator are of the same order. We show the (optimal) $C^{1,\frac{1}{2}}$ regularity in dimension $d = 2$ and we show the same regularity result in $d \geq 3$ conditional on the assumption that the degenerate values of the solution do not accumulate. We also prove a comparison principle characterizing minimal supersolutions, which we believe will have applications to homogenization and other related scaling limits.

R É S U M É

Nous étudions la régularité et le principe de comparaison pour un problème de Neumann dégénéré en gradient. Le problème est une généralisation du problème de Signorini ou de l'obstacle mince qui apparaît dans l'étude de certains problèmes de frontière libre anisotropes singuliers issus de l'homogénéisation. En termes d'échelle, le problème est critique car la dégénérescence du gradient et l'opérateur de Neumann sont du même ordre. Nous montrons la régularité (optimale) $C^{1,\frac{1}{2}}$ en dimension $d = 2$ et nous montrons le même résultat de régularité pour $d \geq 3$ sous l'hypothèse que les valeurs dégénérées de la solution ne s'accumulent pas. Nous prouvons également un principe de comparaison caractérisant les sursolutions minimales, dont nous pensons qu'il aura des applications à l'homogénéisation et à d'autres limites d'échelle connexes.

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1. Introduction

This paper considers the following critically degenerate Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ \min\{\partial_1 u, |\nabla' u|\} = 0 & \text{on } B_1'. \end{cases} \tag{1.1}$$

Here we have denoted $\partial_i = e_i \cdot \nabla$ for $i = 1, \ldots, d$, $\nabla' = (\partial_2, \ldots, \partial_d)$, $B_1^+ = B_1 \cap \{x_1 > 0\} \subset \mathbb{R}^d$ and $B_1' = B_1 \cap \{x_1 = 0\} \subset \mathbb{R}^{d-1}$. The *contact set* of u, formally defined as

$$\mathcal{C}_u := \{x \in B_1' : \partial_1 u > 0\} \subset \{x \in B_1' : |\nabla' u| = 0\}, \tag{1.2}$$

is of central interest.

This problem appears in the study of certain singularly anisotropic one-phase Bernoulli free boundary problems arising from homogenization (see Section 1.3.2 below). In elliptic PDE terms, the problem (1.1) is an example of a PDE with gradient degeneracy. The non-local PDE operator $\partial_1 u$ and the gradient degeneracy $|\nabla' u|$ are both first-order derivatives making the problem critical.

We will study the regularity and comparison principle for solutions of (1.1). First, we will show that solutions are Lipschitz continuous. Then we prove the optimal $C^{1,1/2}$ regularity of solutions in $d = 2$. In higher dimensions $d \geq 3$, we prove optimal regularity under the condition that u only takes finitely many distinct values on its contact set $\mathcal{C}_u$.

The gradient degenerate Neumann problem (1.1) is closely related to and, in fact, generalizes the well-known Signorini or thin obstacle problem [2,17,21,22,27]

$$\begin{cases} \Delta w = 0 & \text{in } B_1^+ \\ \min\{\partial_1 w, -w\} = 0 & \text{on } B_1'. \end{cases} \tag{1.3}$$

Notice that w in the thin obstacle problem (1.3) is also a viscosity solution to (1.1) with $w \equiv 0$ on $\mathcal{C}_w$ and $w \leq 0$ on the whole flat boundary B_1'. Unlike the thin obstacle problem, the problem (1.1) does not involve any pre-defined obstacle. However, we will show that any viscosity solution u to (1.1) is constant on each component of the "contact set" $\mathcal{C}_u$ defined in (1.2) (see Lemma 2.8 below). Thus our "contact set" $\mathcal{C}_u$ generalizes the role of the contact set in the thin obstacle problem, and our problem falls under the general class of *unconstrained free boundary problems* surveyed in [19].

Although Signorini solutions solve the degenerate Neumann problem (1.1), the problem (1.3) allows additional solutions that do not arise from a Signorini problem. There is, in general, non-uniqueness of solutions to the problem (1.1) even with Dirichlet data posed on the outer boundary $\partial B_1 \cap \{x_1 > 0\}$. Maximal subsolutions of (1.1) just solve the Neumann problem. Minimal supersolutions, on the other hand, generally have nontrivial contact sets $\mathcal{C}_u$. In some cases, the minimal supersolution corresponds to a Signorini problem, but even when $\mathcal{C}_u$ has only finitely many components solutions may bend below *or* above the "obstacle" (see Fig. 1). Our final main result of the paper is a comparison principle (see Theorem 1.6) which characterizes minimal supersolution by one additional non-local viscosity solution property, the *boundary maximum principle*. We expect this comparison principle to allow for regularization arguments, and to have applications in homogenization.

The generalization brings several new challenges in the analysis of regularity. For example, because of the absence of a thin obstacle, it seems unclear that we can obtain semi-convexity/-concavity of a solution as in the thin obstacle case [2,27]. We solve this challenge by proving pointwise differentiability via a different approach that combines the nontangential convergence theories and the Almgren monotonicity formula. There are also possible piling-ups of infinitely many components of $\mathcal{C}_u$ with u having infinitely many different values on them, which might ruin the differentiability of a solution (see Theorem 1.4). However, this challenge seems unattainable in the current context so we will defer this issue to future work.

The problem (1.1) can also be viewed as a critical case of a class of gradient degenerate elliptic problems. In the pioneering work [24], Imbert and Silvestre studied the following type of degenerate elliptic equation

$$|\nabla v|^{\gamma} F(D^2 v) = f$$

with F being uniformly elliptic. This research continued in the case of non-local operators of order $1 < \sigma < 2$ in several works [1,12]

$$|\nabla v|^{\gamma} \Delta^{\sigma/2} v = f. \tag{1.4}$$

In this context our problem (1.1) falls at the critical order $\sigma = 1$ where the gradient degeneracy and the non-local PDE operator are of the same order. Our work is the first to discuss finer properties in this challenging critical case for gradient degenerate PDEs of this type.

1.1. Main results

Matching the optimal $C^{1,1/2}$ regularity of the thin obstacle problem, we will show the following main result on the regularity of (1.1) in dimension 2.

Theorem 1.1. *Suppose that u solves (1.1) and $d = 2$. Then u is in $C^{1,\frac{1}{2}}_{\text{loc}}(B_1^+ \cup B_1')$ and there is a universal $C \geq 1$ so that*

$$\|u\|_{C^{1,\frac{1}{2}}\left(\overline{B^+_{1/2}}\right)} \leq C \|u\|_{L^\infty(B_1^+)}. \tag{1.5}$$

We can also prove similar regularity in dimension $d \geq 3$ under the condition that u takes at most finitely many values on its facets.

Theorem 1.2. *Suppose that u solves (1.1), $d \geq 2$, and $u(\mathcal{C}_u) \subset \mathbb{R}$ is finite. Then u is in $C^{1,\frac{1}{2}}_{\text{loc}}(B_1^+ \cup B_1')$ and*

$$\|u\|_{C^{1,\frac{1}{2}}\left(\overline{B^+_{1/2}}\right)} \leq C \|u\|_{L^\infty(B_1^+)}, \tag{1.6}$$

where C is universal when $d = 2$, and in $d \geq 3$, at most, C depends on d and the minimal gap of the degenerate values as defined below

$$\text{gap}(u) := \min\{|a - b|; a \neq b,\ a, b \in u(\mathcal{C}_u)\}. \tag{1.7}$$

Remark 1.3. The minimal gap is always positive under the assumption $u(\mathcal{C}_u) < \infty$, and it is a useful quantitative parameter of the latter condition. In dimension $d \geq 3$, we can slightly improve the bounding coefficient for a smaller regularity exponent $1/2 > \alpha = \alpha(d) > 0$ by proving the estimate:

$$\|u\|_{C^{1,\alpha}\left(\overline{B^+_{1/2}}\right)} \leq C \|u\|_{L^\infty(B_1^+)},$$

with C depending at most on d and $\#u(\mathcal{C}_u)$. Unlike the positive minimal gap (1.7), the quantity $\#u(\mathcal{C}_u)$ sets no restrictions on the distances between any two distinct degenerate values. See Remark 6.11 for the details.

The proof of the conditional regularity also shows the following result, which is useful to interpret the remaining open issues about the regularity of (1.1). If u were to fail to be differentiable at the origin then u would need to have infinitely many facets in any neighborhood of 0.

Theorem 1.4. *If u solves (1.1) in B_1^+ and u fails to be differentiable at 0 then $\#u(\mathcal{C}_u \cap B_r') = +\infty$ for all $0 < r < 1$.*

Of course, we do not have any example of non-differentiability, so it may be possible to rule this scenario out using other methods. We will further interpret this conditional result below in Section 6.

1.2. *Ideas of the proof*

1.2.1. *Nontangential convergence and almost everywhere differentiability*

In Section 3, we establish the Lipschitz estimate by using a doubling variable method and the Jensen-Ishii lemma [10,25]. To go beyond the Lipschitz regularity, the typical approach in the Signorini problem goes via a semi-concavity/semi-convexity estimate [2,27]. This technique does not seem to work in our setting. Instead, in Section 4, we utilize the classical theory of the non-tangential boundary behavior of bounded harmonic functions [20,23]. Indeed, the gradient ∇u is harmonic and bounded because of the Lipschitz estimate. By applying the non-tangential convergence theory, we show surface measure almost everywhere differentiability (including nontangential directions) of solutions u to (1.1) on B_1'.

1.2.2. *On 2D regularity*

In [3, Section 2], an idea by Hans Lewy was introduced to observe the optimality of the $C^{1,1/2}$ regularity of the thin obstacle problems. In Section 5 we show that this idea can also be applied to the gradient degenerate Neumann problem (1.1) in dimension $d = 2$. Let $\nabla u = (\partial_1 u, \partial_2 u)$ be the bounded gradient and then we can define

$$F = \partial_2 u + i\partial_1 u$$

as a complex analytic function on B_1^+. Its square satisfies

$$G := F^2 = |\partial_2 u|^2 - |\partial_1 u|^2 + 2i\partial_1 u \partial_2 u =: U + iV.$$

By the boundary condition of (1.1), we know that $V \equiv 0$ on the flat boundary B_1', and hence V can be harmonically extended to the whole B_1 via odd extension. By classical complex analysis, this means that G is a complex analytic function in the whole B_1. Now $F = \sqrt{G}$ will admit $C^{1/2}$ regularity across B_1' and hence $u \in C^{1,1/2}$.

1.2.3. *Conditional regularity in $d \geq 3$*

In Sections 6 and 7 we establish the conditional regularity results of Theorem 1.2 and Theorem 1.4.

Our work introduces a distinct method for addressing pointwise differentiability, which avoids relying on the semi-convexity estimate typically used in thin obstacle problems [2,27]. As previously noted, our specific equation (1.1) does not lend itself to semiconvexity-based analysis. Instead, our novel approach integrates the property of non-tangential almost everywhere differentiability with the Almgren monotonicity formula to establish pointwise differentiability. The Almgren monotonicity formula has been extensively applied in the study of thin obstacle problems, see for example [3,18,21] and other references therein. In our case there is an additional error term in the derivative of the Almgren frequency functional which we have, so far, only been able to control using the condition $\#u(\mathcal{C}_u \cap B_1) < \infty$. This is the only place where the condition is used in the proof of pointwise differentiability in dimension $d \geq 3$.

Next, using the pointwise differentiability property, we establish a $C^{1,\alpha}$-type improvement of flatness iteration. In the $C^{1,\alpha}$-iteration, similar to the set-up in [24], it is useful to consider the following tilted boundary condition

$$\min\{\partial_1 u + m_1, |\nabla' u + m'|\} = 0, \tag{1.8}$$

with $m = (m_1, m') \in \mathbb{R}^d$. Unlike the iterations in [24], we show that there is a general constraint on the gradient m: if $\operatorname{osc}_{B_1^+} u \leq 1$ satisfies (1.1) with the boundary condition replaced by (1.8) then the vector $m = (m_1, m') \in \mathbb{R}^d$ will satisfy

$$|\min\{m_1, |m'|\}| \leq K(d),$$

for some positive constant $K(d)$ depending only on the dimension. This dichotomy classifies the allowed gradients "m" in the iterations into two cases $m = (m_1, 0)$ with $m_1 > 0$ or $m = (0, m')$ with $m' \in \mathbb{R}^{d-1}$. We emphasize here that this dichotomy idea and the improvement of flatness procedure essentially does not depend on the finiteness condition $\#u(\mathcal{C}_u \cap B_1) < \infty$, and hence we would obtain a full $C^{1,\alpha}$ estimate as long as we had pointwise differentiability.

1.3. Motivations and literature

1.3.1. Unconstrained free boundary problems and gradient degenerate elliptic equations

The gradient degenerate problem (1.4) has drawn much attention in recent years, and can be in general categorized into the class of *regularity matching* problems, see Section 2.2 in the survey of Figalli and Shahgholian [19] on unconstrained problems. In particular, the homogeneous version of the equation (1.4) can be viewed as a regularity matching problem: for a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} |\nabla u|^\gamma \Delta^{\sigma/2} u = 0 & \text{in } \Omega \\ u = g & \text{outside } \Omega, \end{cases} \tag{1.9}$$

where Ω is a bounded open domain. In this problem u satisfies a non-local elliptic problem outside the free domain $\{|\nabla u| = 0\}$, in the interior of which the gradient vanishes. Multiple regularity results for different choices of γ and σ have been discussed [1,12]. In [12], a $C^{1,\alpha}$ regularity result is obtained for (1.4) for $1 < \sigma < 2$ and σ close to 2. The proof relies on a perturbative method around the case $\sigma = 2$, which is included in the well-known work of Imbert and Silvestre [24]. Recently in [1], under the condition that the exterior datum g admits only one solution to the homogeneous equation (1.9), an optimal $C^{1,\alpha}$ regularity result is obtained for the case $1 < \sigma < 2$ with

$$\alpha(\gamma, \sigma) = \frac{\sigma - 1}{1 + \gamma}. \tag{1.10}$$

As $\sigma \to 2^-$, the estimate remains uniform and coincides with the result when $\sigma = 2$ [24]. The gradient degenerate Neumann problem (1.1) can be categorized into the nonlocal gradient degenerate problem (1.9) in the case that $\sigma = 1$. Indeed, if u is a global solution to (1.1) then we know that [6,30]

$$\partial_1 u = \Delta_{x'}^{1/2} u$$

with $\Delta_{x'}^{1/2}$ the fractional Laplacian on $\mathbb{R}^{d-1}$. Now any global viscosity solution to the equation (1.1) satisfies

$$|\nabla' u| \Delta_{x'}^{1/2} u = 0$$

in the viscosity sense. However, by the optimal regularity exponent as described in (1.10), the $C^{1,\alpha}$ regularity reduces to a Lipschitz one when $\sigma \to 1^+$, which means that (1.1) lies exactly in the critical case of the gradient degenerate problem (1.9).

The original strategy of Imbert and Silvestre [24] relies on the following property – which generalizes to similar nonlinear but local PDEs:

$$\text{all viscosity solutions of } \ |\nabla u|^\gamma \Delta u = 0 \ \text{ in } B_1 \text{ solve } \Delta u = 0 \ \text{ in } B_1.$$

Flat solutions of inhomogeneous problems then inherit regularity from the solutions of the homogeneous problem. However, in the non-local case $1 < \sigma < 2$, we don't have the same property:

$$|\nabla u|^\gamma \Delta^{\sigma/2} u = 0 \quad \text{does not imply} \quad \Delta^{\sigma/2} u = 0,$$

which is due to the nonuniqueness of solutions to the homogeneous problem (1.9). Previous results in the literature either require σ near 2 to inherit regularity from the second order case [12,13], or the most recent result [1] obtain regularity under the assumption that (1.9) has a unique solution to apply a similar improvement of flatness strategy again. In our problem, we also have a similar nonuniqueness issue, but we are specifically interested in general solutions of the homogeneous problem (1.1) in cases of non-uniqueness where the minimal supersolution is nontrivially distinct from the Neumann solution. We also build on several ideas from [24], including the Lipschitz estimate and the formulation of the $C^{1,\alpha}$ iteration, but the source of differentiability is distinct and is more related to the thin obstacle theory. Thus, our techniques combine ideas from the gradient degenerate elliptic PDE theory and the thin obstacle problem.

1.3.2. Singular Bernoulli free boundary problems

Our original motivation to study (1.1) comes from a connection with a singularly anisotropic one-phase Bernoulli problem. Specifically, consider the Bernoulli-type one-phase problem set in the exterior of a compact region K

$$\begin{cases} \Delta u = 0, & \text{in } \{u > 0\} \setminus K, \\ u \equiv 1, & \text{on } K, \\ |\nabla u| = Q(\nabla u), & \text{on } \partial\{u > 0\} \setminus K, \end{cases} \tag{1.11}$$

with the anisotropy Q being a 0-homogeneous function of the form

$$Q(e) = \begin{cases} 1, & e \neq e_1 \\ 2, & e = e_1, \end{cases} \tag{1.12}$$

where $e_1, e_2, \ldots, e_d$ form an orthonormal basis for $\mathbb{R}^d$.

This type of singular anisotropy Q arises from a natural homogenization problem for the classical Bernoulli one-phase problem [5,7,14,15,26]. Specifically, consider the following one-phase problem with laminar oscillatory heterogeneity.

$$\begin{cases} \Delta u_\varepsilon = 0, & \text{in } \{u_\varepsilon > 0\} \setminus K, \\ u_\varepsilon \equiv 1, & \text{on } K, \\ |\nabla u_\varepsilon| = q(x_1/\varepsilon), & \text{on } \partial\{u_\varepsilon > 0\}. \end{cases} \tag{1.13}$$

Here q is a 1-periodic function on $\mathbb{R}$. While the energy minimizing solutions of (1.13) converge to solutions of a classical Bernoulli problem, it is known that the *minimal supersolutions* u_ε instead converge to the minimal supersolution of the anisotropic problem (1.11), see [5,14].

There are some results on the regularity of solutions to (1.11) in the case when K is convex [9,15,16], however little is known without convexity.

The connection between the anisotropic free boundary problem (1.11) and the gradient degenerate Neumann problem (1.1) comes from the formal asymptotic expansion of flat solutions. Such formal asymptotic expansions can be leveraged, rigorously, to obtain regularity of flat solutions in many PDE problems. The general idea was introduced by Savin [29] and first leveraged for free boundary problems in a very influential paper of De Silva [11].

To be more specific: suppose that u solves the homogeneous one-phase problem in B_1

$$\Delta u = 0 \ \text{ in } \ \{u > 0\} \cap B_1, \quad \text{with } \ |\nabla u| = 1 \ \text{ on } \ \partial\{u > 0\} \cap B_1$$

and is ε-flat, i.e.

$$(x_1 - \varepsilon)_+ \leq u(x) \leq (x_1 + \varepsilon)_+ \ \text{ in } \ B_1$$

for some small enough $\varepsilon > 0$. Then one considers the formal asymptotic expansion

$$u(x) = (x_1 + \varepsilon w(x) + o(\varepsilon))_+.$$

Computing the boundary condition

$$1 = |\nabla u|^2 = 1 + 2\varepsilon \partial_1 w + o(\varepsilon) \ \text{ on } \ \partial\{u > 0\} \approx \{x_1 > 0\}$$

one finds that w should solve the Neumann problem

$$\begin{cases} \Delta w = 0 & \text{in } B_1^+ \\ \partial_1 w = 0 & \text{on } B_1'. \end{cases}$$

De Silva's approach [11] shows the rigorous validity of this asymptotic expansion and uses this to establish $C^{1,\alpha}$ regularity of the free boundary of sufficiently flat (universal ε) solutions.

Later in [9], Chang-Lara and Savin studied the regularity of $\partial\{u > 0\}$ when u is constrained in the way that $u = 0$ outside a smooth obstacle domain W_{obs} that contains K. They proved optimal $C^{1,1/2}$ regularity of the free boundary that is near ∂W_{obs} under the assumption that ∂W_{obs} is $C^{1,1}$. The key observation in their paper is that when u is sufficiently flat in $\{u > 0\} \cap B_1(x)$, the free boundary can be well-approximated by the function graph of a solution to the thin obstacle problem (1.3). The derivation from the asymptotic expansion of (1.11) to the equation (1.1) follows a similar logic to [11].

An analogous formal asymptotic expansion of the singular anisotropic Bernoulli problem (1.11) leads to the gradient degenerate Neumann problem (1.1). More specifically if u solves (1.11) in B_1 and is ε-flat

$$(x_1 - \varepsilon)_+ \leq u(x) \leq (x_1 + \varepsilon)_+ \ \text{ in } \ B_1$$

then we can formally expand

$$u(x) = (x_1 + \varepsilon w(x) + o(\varepsilon))_+.$$

If we ignore the higher-order terms, we have, at a free boundary point

$$1 \leq Q(\nabla u)^2 = |\nabla u|^2 = 1 + 2\varepsilon \partial_1 w,$$

which requires that $\partial_1 w \geq 0$. If $|\nabla' w| > 0$ then $\nabla u = e_1 + \varepsilon \nabla w$ is not parallel to e_1 and hence

$$1 = Q(\nabla u)^2 = 1 + 2\varepsilon\partial_1 w.$$

This formally leads to the boundary condition of the limiting problem: $\partial_1 w \geq 0$ and if $|\nabla' w| > 0$ then $\partial_1 w = 0$, which can be simplified as $\min\{\partial_1 w, |\nabla' w|\} = 0$ as illustrated in (1.1).

In Section 9 below we follow the approach of [9,11] to show a rigorous flat asymptotic expansion for directionally monotone solutions of (1.11).

Proposition 1.5. *For all $\eta > 0$ there exists $\varepsilon_0 > 0$ so that if u is a minimal supersolution of* (1.11) *in B_1 and is ε-flat* (9.5) *with slope $p = e_1$ and $\varepsilon \leq \varepsilon_0$ then there is a solution w of* (1.1) *so that*

$$(x_1 + \varepsilon w(x) - \eta\varepsilon)_+ \leq u(x) \leq (x_1 + \varepsilon w(x) + \eta\varepsilon)_+ \quad \text{in} \quad B_{1/2}.$$

Actually, we can show that w is a minimal supersolution of (1.1), see Remark 8.3.

1.4. Non-uniqueness and comparison principle for minimal solutions

As is known in gradient degenerate problems [1], we don't in general have uniqueness for problems of the type (1.9) for $0 < \sigma < 2$. The same phenomenon also occurs when we consider the problem (1.1) with a fixed boundary data on $\partial B_1 \cap \{x_1 \geq 0\}$. The Perron's method minimal supersolution plays an important extremal role. It satisfies an additional viscosity solution property, the *boundary maximum principle* (see Lemma 8.1). Our last main result of the paper is a comparison principle characterizing the minimal supersolution.

Theorem 1.6. *Let v be a super-solution (see Definition 2.2) and u a sub-solution (see Definition 2.1) that satisfies an additional boundary maximum principle as described in Lemma 8.1. If $v \geq u$ on the boundary $\partial B_1 \cap \{x_1 \geq 0\}$, then we have $v \geq u$ on the whole $\overline{B_1^+}$.*

A similar comparison principle for the Bernoulli-type problem can be found in [15, Theorem 5.3]. The usefulness of this sort of theorem is that it gives a "local" viscosity solution characterization of the minimal supersolution. This uniqueness property can be used in the proof of homogenization or other regularization limits, for example as done for related free boundary problems in [15,16].

In general, without the boundary maximum principle, the gradient degeneracy causes the comparison principle to fail. The touching point between a strict subsolution v and supersolution u may occur within the contact set $\mathcal{C}_u$ of the supersolution u, and positivity of $\partial_1 u > 0$ is no contradiction. The boundary maximum principle is enough to rule out this scenario. There are also technical challenges since we must work with general semi-continuous sub and supersolutions and the PDE is on a lower dimensional set. To regularize, we need to use tangential sub-/sup-convolutions with harmonic replacement.

In general, given a continuous boundary condition g on $\partial B_1 \cap \{x_1 \geq 0\}$, we have at least three different methods to generate solutions of (1.1). We can simply solve the Neumann problem, this gives the maximal subsolution. We can solve the thin obstacle problem with obstacle $\max_{\partial' B_1'} g$ from above. And we can find the Perron's method minimal supersolution. The Perron's method minimal supersolution always satisfies the boundary maximum principle, while the Signorini and Neumann solutions may not. In Fig. 1 we show an example where all three of these solutions are distinct.

1.5. Outline

In Section 2 we will discuss the viscosity solutions to the equation (1.1) and define the *contact set* $\mathcal{C}_u$ of a viscosity supersolution u. In Section 3 we establish an interior Lipschitz estimate for all bounded viscosity solutions in any dimensions $d \geq 2$ by applying the doubling variable technique in [24].

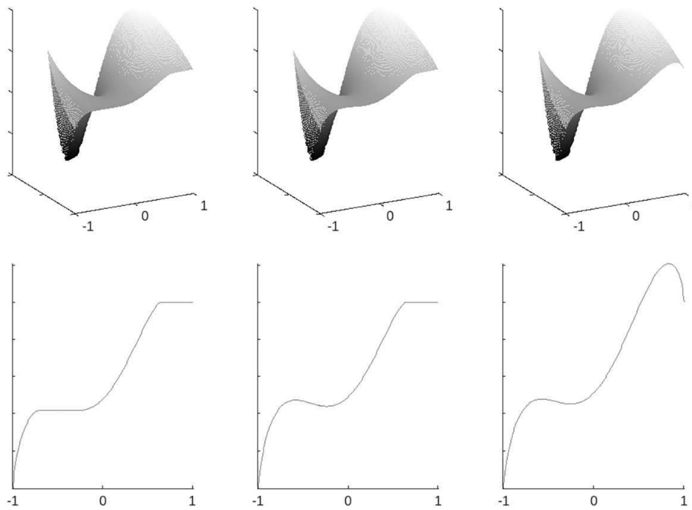

Fig. 1. Different solutions of (1.1) with the same boundary data $g(x_1, x_2) = -43x_1^8 + 19x_1 + 5x_2 - 5$. Top: (Left) the minimal supersolution: $\mathcal{C}_u$ has two components; (Middle) the thin obstacle solution (1.3) with $\max_{\partial' B_1'} g = 0$ as an obstacle from above: it has only one flat component; (Right) the Neumann solution / maximal subsolution. Bottom: the corresponding restrictions to B_1'. Notice that only the minimal supersolution satisfies the boundary maximum principle in this case.

In Section 4 we review some results from the literature on the non-tangential boundary behavior of bounded harmonic functions and show the surface measure almost everywhere differentiability (including non-tangential directions) of a solution u to (1.1) up to the boundary B_1'. In Section 5 we prove Theorem 1.1 by applying the almost everywhere differentiability up to B_1' and the complex analytic arguments.

In Section 6 we prove the Almgren monotonicity formula under the additional condition $\#u(\mathcal{C}_u) < \infty$. In the same section, we establish the improvement of flatness and hence the $C^{1,\alpha}$ regularity by using the monotonicity formula. In Section 7 we finish the proof of Theorem 1.2 by using the Almgren monotonicity again. In Section 9 we show the flat asymptotic expansion of (1.11) gives rise to the problem (1.1).

1.6. Acknowledgments

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2. Preliminaries

2.1. Notations

- $d = n + 1 \geq 2$ are dimensions.
- $(x_1, x') = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ are the coordinate functions. $e_1, \cdots, e_2$ form an orthonormal basis for $\mathbb{R}^d$. ∂_i, $i = 1, \cdots, d$ are the partial derivatives with respect to the directions e_i. $\nabla' = (\partial_2, \cdots, \partial_d)$ is the tangential gradient.
- $B_r(x)$ is the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. $B_r = B_r(0)$.
- $\partial\Omega$ is the boundary of an open domain $\Omega \subset \mathbb{R}^d$.
- $\Omega^+ = \Omega \cap \{x_1 > 0\}$.
- $\Omega' = \Omega \cap \{x_1 = 0\}$. $\partial'\Omega'$ is the relative boundary of Ω' in $\{x_1 = 0\}$. $B_r' = B_r'(0)$.
- $\overline{\Omega}$ is the closure of Ω.
- The notation $A \sqcup B$ denotes disjoint union of sets A and B.

2.2. Viscosity solutions

Let us discuss the definition of viscosity solutions to the equation (1.1).

Definition 2.1. A function $u \in \mathrm{USC}\left(\overline{B_1^+}\right)$ is a *subsolution* of (1.1) if u is subharmonic in B_1^+ and whenever φ smooth touches u from above at $x_0 \in B_1'$ with $\Delta\varphi(x_0) < 0$

$$\partial_1\varphi(x_0) \geq 0.$$

Definition 2.2. A function $u \in \mathrm{LSC}\left(\overline{B_1^+}\right)$ is a *supersolution* of (1.1) if u is superharmonic in B_1^+ and whenever φ smooth touches u from below at $x_0 \in B_1'$ with $\Delta\varphi(x_0) > 0$ then

$$\min\{\partial_1\varphi(x_0), |\nabla'\varphi|(x_0)\} \leq 0.$$

In other words

$$\text{if } |\nabla'\varphi|(x_0) > 0 \ \text{ then } \ \partial_1\varphi(x_0) \leq 0.$$

A continuous function is called a *viscosity solution* if it is both sub- and supersolutions.

Remark 2.3. There is no comparison principle and no uniqueness for the solutions as defined above. However, in Section 8, we will discuss the comparison principle for a supersolution and a *strong* subsolution, i.e., a subsolution that satisfies the boundary maximum principle. This comparison principle characterizes the minimal supersolutions to the problem (1.1).

We provide some special example solutions.

Example 2.4. Any solution to the Signorini problem

$$\begin{cases} \Delta w = 0, & \text{in } B_1^+ \\ w \leq c, & \text{on } B_1' \\ \partial_1 w = 0, & \text{on } \{w < c\} \cap B_1' \\ \partial_1 w \geq 0, & \text{on } B_1', \end{cases} \tag{2.1}$$

where $c \geq \sup_{\partial' B_1'} g$ is some constant. A simple example solution to this equation for $c = 0$ is $w(x, y) = -\mathrm{Re}\left((x + iy)^{3/2}\right)$;

Example 2.5. The sign-reversed Signorini problem

$$\begin{cases} \Delta w^- = 0, & \text{in } B_1^+ \\ w^- \geq \tilde{c}, & \text{on } B_1' \\ \partial_1 w^- = 0, & \text{on } \{w^- > \tilde{c}\} \cap B_1' \\ \partial_1 w^- \geq 0, & \text{on } B_1', \end{cases} \tag{2.2}$$

where $\tilde{c} \leq \inf_{\partial' B_1'} g$ is some constant. An example solution is $w^-(x, y) = \mathrm{Re}\left((x + iy)^{5/2}\right)$.

Lemma 2.6. *Let u_k be a family of continuous viscosity solutions to (1.1) which converge uniformly in on compact subsets of B_1^+ to a limit u_∞, then u_∞ is also a viscosity solution.*

We omit the proof since it follows the standard argument from the viscosity solution theory.

2.3. Contact, non-contact set, and the thin free boundary

Let us now study the behavior of a supersolution u on the flat boundary B_1'. First, we give a formal definition of the contact set. This is named in analogy to the thin obstacle problem, but there is, technically speaking, no obstacle to be contacted.

Definition 2.7. Let u be a supersolution to (1.1), then the *contact set* $\mathcal{C}_u$ is defined by:

$$\mathcal{C}_u := \{x \in B_1' \ : \ \exists \varphi \in C^\infty \text{ touching } u \text{ from below in } \overline{B_1^+} \text{ at } x \text{ with } \partial_1 \varphi(x) > 0\}.$$

Our first result says that $\mathcal{C}_u$ is open.

Lemma 2.8. *Let u be a supersolution, then $\mathcal{C}_u$ is relatively open in B_1' and u is constant on each component of $\mathcal{C}_u$.*

We will return to the proof in a moment. First, we give some additional definitions, also named in analogy to the thin obstacle problem.

Definition 2.9. Define the *non-contact set* $\mathcal{N}_u$ to be the relative interior of $B_1' \setminus \mathcal{C}_u$ and the *free boundary* $\Gamma_u := B_1' \setminus (\mathcal{C}_u \cup \mathcal{N}_u)$.

Given these definitions we have

$$B_1' = \mathcal{C}_u \sqcup \mathcal{N}_u \sqcup \Gamma_u. \tag{2.3}$$

Also note that, from Lemma 2.8, the free boundary Γ_u is relatively closed in B_1' and also

$$\Gamma_u = \partial' \mathcal{C}_u \ \text{ and } \ \Gamma_u = \partial' \mathcal{N}_u.$$

Remark 2.10. We can extend the definitions of contact/non-contact/free-boundary sets to a larger class of problems. Suppose u is a viscosity solution to (1.1) with the boundary condition replaced by the following tilted version

$$\min\{\partial_1 u + p_1, |\nabla' u + p'|\} = 0, \text{ on } B_1'$$

for some $p = (p_1, p') \in \mathbb{R}^d$, then that is equivalent to say that $u + p \cdot x$ is a viscosity solution to the original equation (1.1), and hence we may define

$$\mathcal{C}_u := \mathcal{C}_{u+p\cdot x}, \mathcal{N}_u := \mathcal{N}_{u+p\cdot x}, \text{ and } \Gamma_u := \Gamma_{u+p\cdot x}$$

correspondingly.

Proof of Lemma 2.8. Let $x_0 \in \mathcal{C}_u$, and then we may assume, by translation and rescaling, that $x_0 = 0$ and $u(0) = 0$, and u satisfies the following one-sided flatness condition:

$$u(x) \geq \beta x_1 - \varepsilon, \, x \in \overline{B_1^+} := \overline{B}_1 \cap \{x_1 \geq 0\},$$

where $\beta > 0$ is the inward normal slope of the touching test function in the definition of $\mathcal{C}_u$, and $\varepsilon > 0$ can be made arbitrarily small (at the cost of rescaling to a smaller radius depending on $\beta > 0$ and the C^1 modulus of the touching test function).

It then suffices to show that u must be identically equal to $u(0) = 0$ in a small neighborhood of $0 \in B_1'$. To that end, let us consider the following family of harmonic parabola barriers

$$v_{t,s}(x_1, x') := -\delta\left(|x' - s|^2 - |s|^2\right) + \frac{\beta}{2}(x_1)_+ + (d-1)\delta x_1^2 - \varepsilon + \varepsilon t,$$

where $\delta > 0$ is a small number. It is not hard to observe that when $\varepsilon \ll \delta < \min\{1, \beta\}/(100d)$ and $s \in \{0\} \times \mathbb{R}^{d-1}$, $|s| < 1/4$, we have for $x \in \partial B_1 \cap \{x_1 \geq 0\}$

$$\begin{aligned} \beta x_1 - \varepsilon - v_{1,s}(x) &= \frac{\beta}{2}(x_1)_+ - \varepsilon + \delta\left(|x' - s|^2 - |s|^2\right) - (d-1)\delta x_1^2 \\ &= \frac{\beta}{2}(x_1)_+ + \delta|x'|^2 - 2\delta x' \cdot s - (d-1)\delta x_1^2 - \varepsilon \\ &\geq \frac{\beta}{2}(x_1)_+ - d\delta x_1^2 + \delta - 2\delta x' \cdot s - \varepsilon \\ &\geq \left(\frac{\beta}{2} - d\delta\right)(x_1)_+ + \delta - 2\delta|s| - \varepsilon \\ &> 0. \end{aligned} \tag{2.4}$$

Now we have

$$v_{0,s}(x) \leq u(x),\ |s| < 1/4 \text{ and } x \in \overline{B_1^+}.$$

We consider for each $|s| < 1/4$ the largest $t^* \geq 0$ such that

$$v_{t^*,s}(x) \leq u(x) \text{ for all } x \in \overline{B_1^+}.$$

Note that $t^* \leq 1$ because $v_{1,s}(0) = 0 = u(0)$.

Because $v_{t^*,s}$ are harmonic on B_1^+, the touching point cannot be in the interior. Also because of (2.4), the touching point cannot occur at $\partial B_1 \cap \{x_1 \geq 0\}$ either.

Let $\tilde{x} \in B_1'$ be a touching point. We claim that $\tilde{x} = s$. Indeed, we first observe that

$$\partial_1 v_{t^*,s}(\tilde{x}) = \frac{\beta}{2} > 0.$$

If $\tilde{x} \neq s$ then $\nabla' v_{t^*,s}(\tilde{x}) \neq 0$, which contradicts the super-solution condition of u at $\tilde{x}$.

Now, let $h(s) = t^*(s)$ be defined for each specific $|s| < 1/4$. Because by definition h is $C^{1,1}$ on the lower side and hence h is Lipschitz, and because all the lower touching parabolas have the touching points at the peaks, which means that the gradient of h must be 0 almost everywhere and hence equals some constant. □

3. Lipschitz regularity

In the course of proving the Lipschitz regularity of solutions to (1.1) it is convenient to consider a slightly more general class of boundary conditions that arise from renormalizations of the form $u(x) \to u(x) - p' \cdot x$. Let $p' \cdot e_1 = 0$ be a fixed vector orthogonal to e_1 and consider the variant of (1.1)

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ \min\{\partial_1 u, |\nabla' u + p'|\} = 0 & \text{on } B_1'. \end{cases} \tag{3.1}$$

We will prove a Lipschitz estimate on this general class of equations independent of the vector p'.

Lemma 3.1. *Let $p' \cdot e_1 = 0$. There is a constant $C(d) \geq 1$, independent of p', such that, if u is a continuous viscosity solution of (3.1), then*

$$\|\nabla u\|_{L^\infty\left(B^+_{1/2}\right)} \leq C \|u\|_{L^\infty(B^+_1)}. \tag{3.2}$$

In particular, u is locally Lipschitz continuous in $B^+_1 \cup B'_1$.

The idea of the proof is from [24, Lemma 4]. Basically, this is a version of the Bernstein method for proving Lipschitz regularity in nonlinear elliptic and parabolic equations which uses doubling of variables when differentiating in the PDE is not possible due to insufficient regularity and lack of a good smoothing procedure. The origin of the idea goes back to [25].

Proof. By homogeneity of the equation, we can assume that $\|u\|_{L^\infty(B^+_1)} \leq 1/2$. It suffices to show that we can find $L_1 > 0$ and $L_2 > 0$ such that, for all $x_0 \in B^+_{1/2}$ (and hence $x_0 \cdot e_1 > 0$),

$$M = \sup_{x,y \in \overline{B^+_1}} u(x) - u(y) - L_1\omega(|x-y|) - L_2|x-x_0|^2 - L_2|y-x_0|^2 \leq 0, \tag{3.3}$$

where $\omega(x) = s - \frac{2}{3}s^{3/2}$ if $s \leq 1$ and $\omega(s) = \omega(1)$ if $s \geq 1$. If one proves such an inequality then the Lipschitz constant will be bounded from above by all $L > L_1 + L_2$. Indeed, by boundedness and continuity of u in $\overline{B^+_1}$, it suffices to consider the case when $|x-y| < 1$ and $x, y \in B^+_1$. In this case, we choose $x_0 = y$ and obtain

$$u(x) - u(y) \leq L_1\omega(|x-y|) + L_2|x-y|^2 \leq (L_1 + L_2)|x-y|.$$

Assume towards a contradiction that $M > 0$. Note that, since u is continuous the maximum in (3.3) is achieved. Suppose that $(x, y) \in \overline{B^+_1} \times \overline{B^+_1}$ is a pair that achieves the maximum. Then $x \neq y$ since, otherwise, $M \leq 0$ contradicting the assumption. Note that this is where we use the fact that u is a continuous viscosity solution, for semi-continuous viscosity solutions $u^*(x) - u_*(x)$ can be strictly positive allowing the maximum to occur when $x = y$.

Then we obtain, by the assumption $M > 0$,

$$L_1\omega(|x-y|) + L_2|x-x_0|^2 + L_2|y-x_0|^2 < u(x) - u(y) \leq |u(x)| + |u(y)| \leq 1.$$

By choosing $L_2 = (4/r)^2$ for some fixed small number $1 \gg r > 0$ we obtain that $|x - x_0| \leq r/3$ and $|y - x_0| \leq r/3$. Now we may assume that both $x \neq y$ are contained in $\overline{B}_r(x_0) \cap \overline{B^+_1} \subset\subset B^+_{2/3} \cup B'_{2/3}$.

We now apply the Jensen-Ishii Lemma, see [10, Theorem 3.2], to construct a limiting sub-jet (q_x, X) of u at x and super-jet (q_y, Y) of u at y, where

$$q_x = q + 2L_2(x - x_0) \quad \text{and} \quad q_y = q - 2L_2(y - x_0), \tag{3.4}$$

with $q = L_1\omega'(|x-y|)\frac{x-y}{|x-y|}$ and for all small $\eta > 0$ (dependent of the distance $\text{dist}(x, y)$)

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \preccurlyeq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2L_2 + \eta)Id, \tag{3.5}$$

with

$$Z = L_1 \left[\left(\frac{1}{|x-y|} - \frac{1}{|x-y|^{1/2}} \right) Id + \left(\frac{1}{2|x-y|^{1/2}} - \frac{1}{|x-y|} \right) \frac{(x-y) \otimes (x-y)}{|x-y|^2} \right]$$
$$=: L_1 \left[\left(\frac{1}{|x-y|} - \frac{1}{|x-y|^{1/2}} \right) Id + \left(\frac{1}{2|x-y|^{1/2}} - \frac{1}{|x-y|} \right) j \otimes j \right]. \tag{3.6}$$

Notice that we have the identity

$$j \cdot Zj = -\frac{L_1}{2} \frac{1}{|x-y|^{1/2}}.$$

For notational convenience, we will merely discuss the limit case in the rest of the proof, although it will be more accurate to discuss everything before taking a limit. Let us first discuss the case that both $x, y \in B_1^+$. By harmonicity of u in B_1^+ we know that

$$\mathrm{tr}(X - Y) \geq 0.$$

On the other hand, if we apply any vector of the form $(v, v)^T$ to (3.5) we obtain

$$(X - Y)v \cdot v \leq (4L_2 + 2\eta)|v|^2, \tag{3.7}$$

and if we apply $(j, -j)^T$ then we have

$$(X - Y)j \cdot j \leq 4L_2 + 2\eta - 2L_1|x-y|^{-1/2} \leq 4L_2 + 2\eta - L_1,$$

when $r \geq |x - y| > 0$ is chosen small. Suppose $\{j, \tilde{e}_2, \cdots, \tilde{e}_d\}$ is an orthonormal basis for $\mathbb{R}^d$ then we obtain

$$\mathrm{tr}(X - Y) = (X - Y)j \cdot j + \sum_{i=2}^{d} (X - Y)\tilde{e}_i \cdot \tilde{e}_i \leq d(4L_2 + 2\eta) - L_1 < 0$$

if one chooses L_1 large enough.

We also claim that $x \notin B_1'$ or otherwise because we assumed $x_0 \cdot e_1 > 0$

$$q_x \cdot e_1 = L_1 \omega'(|x-y|) \frac{-y_1}{|x-y|} - 2L_2 x_0 \cdot e_1 < 0,$$

contradicting the Neumann subsolution condition of u at $x \in B_1'$.

It then suffices to consider the case that $x \in B_1^+$ and $y \in B_1'$. In this case, we apply the supersolution condition and because

$$q_y \cdot e_1 = L_1 \omega'(|x-y|) \frac{x_1}{|x-y|} + 2L_2(x_0 \cdot e_1) > 0,$$

we know that $q_y = (q_y \cdot e_1)e_1 - p'$ with $q_y \cdot e_1 > 0$ according to the supersolution condition 2.2. Now we arrive at this last case that $y = (0, y') \in \mathcal{C}_u$ according to Lemma 2.8 and because u restricted to B_1' is linear on connected components of $\mathcal{C}_u$ we have the following inequality

$$e \cdot Ye \leq 0, \text{ for all } e \perp e_1.$$

In particular, by combining this inequality with (3.7) we obtain

$$e \cdot Xe \leq (4L_2 + 2\eta)|v|^2, \text{ for all } e \perp e_1. \tag{3.8}$$

Let $j = \frac{x-y}{|x-y|} =: (j_1, j')$ as before with $j' = \beta\tilde{e} \in B_1'$ for some $1 > \beta \geq 0$, then if we apply $(j+\tilde{e}, \tilde{e})^T$ to (3.5) we will obtain

$$(j+\tilde{e}) \cdot X(j+\tilde{e}) \leq \tilde{e} \cdot Y\tilde{e} + j \cdot Zj + 10L_2 + 10\eta \leq 10L_2 + 10\eta - L_1,$$

and we may also apply $(j-\tilde{e}, -\tilde{e})^T$ to obtain, similarly,

$$(j-\tilde{e}) \cdot X(j-\tilde{e}) \leq 10L_2 + 10\eta - L_1.$$

For the case $\beta > 0$ we can take

$$\left\{ \frac{j+\tilde{e}}{\sqrt{|j_1|^2 + |1+\beta|^2}}, \frac{j-\tilde{e}}{\sqrt{|j_1|^2 + |1-\beta|^2}}, \overline{e}_3, \cdots, \overline{e}_d \right\}$$

as an orthonormal basis of $\mathbb{R}^d$ and combine all the above estimates to obtain

$$\begin{aligned}
\operatorname{tr}(X) &= \frac{1}{|j_1|^2 + |1+\beta|^2}(j+\tilde{e}) \cdot X(j+\tilde{e}) \\
&\quad + \frac{1}{j_1^2 + |1-\beta|^2}(j-\tilde{e}) \cdot X(j-\tilde{e}) + \sum_{i=3}^{d} \overline{e}_i \cdot X\overline{e}_i \\
&\leq \frac{1}{2-2\beta}\left(10L_2 + 10\eta - L_1\right) + (d-1)\left(10L_2 + 10\eta\right) - L_1 \\
&< 0,
\end{aligned}$$

where on $\overline{e}_i \cdot X\overline{e}_i$ we have used the bound (3.8). In the case $\beta = 0$ we may choose $\tilde{e} = e_2$ and $\overline{e}_i = e_i$ for each $i = 3, \cdots, d$ and then obtain a similar result. The contradiction of the inequality to the harmonicity of u leads to the proof of the lemma. □

4. Nontangential convergence

According to the estimate in the prior section, we know that the gradient ∇u of a viscosity solution u to the equation (1.1) is bounded and harmonic on B_r^+ for all $r < 1$. We will apply classical harmonic analysis results on the boundary behavior and Poisson integral formulae for bounded harmonic functions in Lipschitz domains.

The following result can be found in the paper of Hunt and Wheeden [23, Page 311], the exact statement in dimension $d = 2$ can also be found in the book of Garnett and Marshall [20, Corollary 2.5].

Theorem 4.1. *Suppose h is a bounded harmonic function in a Lipschitz and starlike domain Ω then there is a bounded function f on $\partial\Omega$ such that h converges to f nontangentially almost everywhere and h can be recovered from the Poisson integral of f on $\partial\Omega$.*

In our paper we will focus on the case that $\Omega = B_1^+$, which satisfies the conditions as described in Theorem 4.1

Now let u be a viscosity solution of (1.1). By Theorem 4.1 there is a full measure set $E = E_u \subset B_1'$ so that the nontangential limit of ∇u exists at each $y \in E$. Furthermore, since, again by Lemma 3.1, $u\big|_{B_1'}$ is a Lipschitz continuous function on B_1', it is differentiable in the tangential variables almost everywhere on B_1'. Thus we may also, without loss, assume that $u\big|_{B_1'}$ is differentiable in the tangential directions at all

$y \in E$. We notice that at this stage we don't know whether the nontangential limit of $\nabla' u$ coincides with the tangential gradient of $u\big|_{B_1'}$.

Combining the above information, there is a bounded $\mathbb{R}^d$-valued function P on B_1' that satisfies

$$\nabla u(x) \to P(y),\ \forall x \to y \in E \text{ nontangentially,} \tag{4.1}$$

for a full-measure subset $E \subset B_1'$. In particular, we denote

$$\sigma := e_1 \cdot P \text{ and } \tau := |P'| = \sqrt{|P|^2 - \sigma^2}. \tag{4.2}$$

In the following, we would like to show that u is differentiable in E and in particular, $P'(y) = \nabla' u\big|_{B_1'}(y)$ for all $y \in E$.

Lemma 4.2. *For any bounded viscosity solution u to the problem (1.1), there is a full measure subset $E_u \subset B_1'$, on which u is differentiable and the gradient $\nabla u(y) = P(y)$ for all $y \in E_u$ with P as defined in (4.1). The corresponding σ and τ as defined in (4.2) satisfies*

$$\min\{\sigma(y), \tau(y)\} = 0,\ \forall y \in E_u.$$

Moreover, u satisfies the Neumann boundary condition $\partial_1 u = \sigma$ on B_1' in the distributional weak sense:

$$\int_{B_1^+} \nabla u \cdot \nabla \phi + \int_{B_1'} \sigma \phi = 0, \quad \textit{for all } \phi \in C^\infty_{\text{loc}}(B_1^+ \sqcup B_1').$$

Proof. Let $r > 0$ be small, and we would like to consider the following families of functions with $|x| \leq 1, x_1 \geq 0$

$$u_r(x) = \frac{u(y + rx) - u(y)}{r}.$$

By the Lipschitz estimate in Lemma 3.1, we know that the above family of functions has convergent subsequences. Let $r_k \to 0$ be a subsequence such that u_{r_k} converges uniformly to some other Lipschitz function u_∞ in $\overline{B_1^+}$. By classical viscosity solution theory, we know that u_∞ also has to be a viscosity solution to (1.1).

On the other hand, by the nontangential convergence of ∇u to the boundary, there is a full-measure subset E_u of B_1' such that the nontangential limit $P(y)$ as defined in (4.1) exists for $y \in E_u$, and for $x_1 > 0$

$$\begin{aligned} u_\infty(x) &= \lim_{k \to \infty} \frac{u(y + r_k x) - u(y)}{r_k} \\ &= \lim_{k \to \infty} \frac{\int_0^{r_k} \nabla u(y + tx) \cdot x dt}{r_k} \\ &= P(y) \cdot x. \end{aligned}$$

This equality is also true for $x_1 = 0$ because u_∞ is Lipschitz continuous up to B_1'. Since $P(y)$ is uniquely determined, we know that the above convergence of u_{r_k} holds for any convergent subsequences of $r \to 0^+$, and hence we obtain differentiability at $y \in E_u$. In particular, we have $\nabla' u\big|_{B_1'}(y) = P'(y)$ for all $y \in E$. Now the lemma is proved by observing that any viscosity solution to (1.1) that takes the form $P(y) \cdot x$ will satisfy

$$\min\{P_1(y), |P'|(y)\} = \min\{\sigma(y), \tau(y)\} = 0.$$

To show that u satisfies the Neumann boundary condition in the distributional weak sense, we first observe that, by interior regularity of harmonic functions, for any $\phi \in C_0^\infty(B_1^+ \sqcup B_1')$

$$\int\limits_{B_1^+ \cap \{x_1 > 1/k\}} \nabla u \cdot \nabla \phi + \int\limits_{B_1^+ \cap \{x_1 = 1/k\}} \partial_1 u \phi = 0.$$

By applying the nontangential convergence of $\partial_1 u(1/k, x') \to \sigma(x')$ as $k \to \infty$ and the Lipschitz estimate 3.1, we know that after sending $k \to \infty$,

$$\int\limits_{B_1^+} \nabla u \cdot \nabla \phi + \int\limits_{B_1'} \sigma \phi = 0. \quad \square$$

Corollary 4.3. *For an arbitrary $p' \in \mathbb{R}^d$ such that $p' \cdot e_1 = 0$, a viscosity solution $w_{p'}$ of (3.1) also satisfies*

$$\min\{\partial_1 w_{p'}(x), |\nabla' w_{p'}(x) + p'|\} = 0, \text{ for almost all } x \in B_1',$$

in the sense of nontangential convergence. In particular, there exists a constant $L = L\left(d, \|w_{p'}\|_{L^\infty(B_1^+)}\right) > 0$ such that if $|p'| > L$ then $w_{p'}$ satisfies the zero Neumann boundary condition on $B_{1/2}'$ in the classical sense.

Proof. The nontangential convergence can be derived similarly to Lemma 4.2. On the other hand, by the Lipschitz estimate, Lemma 3.1, the function $w_{p'}$ is uniformly Lipschitz with the Lipschitz constant $L = L(d, \|w_{p'}\|_{L^\infty}) > 0$ independent of the choice of p'. If one choose $|p'| > L$, then we have $|\nabla' w_{p'}(x) + p'| > 0$ almost everywhere on $B_{1/2}'$, which implies that $\partial_1 w_{p'} = 0$ almost everywhere on $B_{1/2}'$. By the second part of Lemma 4.2, it implies that $w_{p'}$ satisfies the zero Neumann boundary condition on $B_{1/2}'$ in the distributional weak sense. By classical regularity theory for Neumann problems, this implies that $w_{p'}$ satisfies the Neumann boundary condition in the classical sense on $B_{1/2}'$. $\square$

Corollary 4.4. *There is a full-measure set $E_u \subset B_1'$ such that $\sigma(y) = 0$ for all $y \in E_u \cap (\mathcal{N}_u \sqcup \Gamma_u)$.*

Proof. We pick E_u to be the set of differentiability of u and $y \in E_u \cap (\mathcal{N}_u \sqcup \Gamma_u)$. Let us now consider the blow-up function $v(x) = v_r(x) = \frac{u(rx+y)-u(y)}{r}$ satisfies the following $\varepsilon = \varepsilon_r$-flatness

$$\begin{aligned} v_r(x) &= \frac{u(rx+y) - u(y)}{r} \\ &= p(y) \cdot x + O(\varepsilon_r) \\ &= \sigma(y)(x_1)_+ + \nabla' u(y) \cdot x + O(\varepsilon_r), \end{aligned}$$

where $x \in \overline{B_1^+}$, $r > 0$ and as $r \to 0^+$, the flatness $\varepsilon_r \to 0$. By Lemma 4.2 we know that $\sigma(y) \geq 0$ and if $\sigma(y) > 0$ then $\tau = |\nabla' u(y)| = 0$, and hence we may without loss write

$$v_r(x) = \sigma(y)(x_1)_+ + O(\varepsilon_r).$$

We argue similarly to Lemma 2.8 by contradiction: if $\sigma(y) > 0$, then we construct a function of the form

$$\sigma(y)x_1/2 - \delta(x_2)^2 + 2\delta(x_1)^2 - \nu.$$

Choose $\delta > 0$ and ν properly, so that the function is below $\sigma(y)x_1 - C\varepsilon_r$ on $\partial B_1 \cap \{x_1 \geq 0\}$ for small $r > 0$, and it will touch v_r only on B_1'. If the touching point is 0, then because 0 is in $\mathcal{N}_v \sqcup \Gamma_v$, $\sigma(y) \leq 0$. If the touching point is not 0 then the touching point has to be in $\mathcal{N}_v \sqcup \Gamma_v$ too by the supersolution condition of v, which still implies that $\sigma(y) \leq 0$. □

5. Optimal regularity in dimension $d = 2$

In this section, we present the proof of the optimal $C^{1,1/2}$ regularity of any viscosity solutions u to the equation (1.1) in dimension $d = 2$.

The idea starts with the classic use of complex variables, originally due to Hans Lewy, and with well-known application in thin obstacle problems, see [2, Section 2]. Consider the complex analytic function

$$F = \partial_2 u + i\partial_1 u,$$

and its square

$$G := F^2 = |\partial_2 u|^2 - |\partial_1 u|^2 + 2i\partial_1 u\partial_2 u =: U + iV.$$

We focus first on the imaginary part

$$V = 2\partial_1 u\partial_2 u.$$

Since it is the imaginary part of an analytic function V is harmonic, it is also bounded in B_1^+ due to Lemma 3.1. Furthermore, by Lemma 4.2, we have that, for almost every $y \in B_1'$,

$$V(x) \to 0 \text{ as } x \in B_1^+ \to y. \tag{5.1}$$

In other words, $V = 0$ on B_1' in the sense of nontangential convergence. By applying Theorem 4.1 then V satisfies zero Dirichlet boundary condition in the classical sense on B_1' and so V can be odd extended to a harmonic function in the whole disc B_1, which we still denote by V.

Then, by classical complex analysis, V admits a unique harmonic conjugate in the entire B_1. It must agree with U in the upper half ball and so we denote it as U, a harmonic extension of U to B_1. Notice that the odd symmetry of V implies that U is even symmetric with respect to $x_1 \mapsto -x_1$.

Thus we proved the following lemma.

Lemma 5.1. *The function F is analytic in B_1^+ and its square $G = F^2$ has a unique analytic continuation to the whole disc B_1.*

With this lemma, we are now able to prove Theorem 1.1.

Proof of Theorem 1.1. According to Lemma 5.1 we know that U has a harmonic extension to the whole disc B_1. Also we have the formula $U = |\partial_1 u|^2 - |\partial_2 u|^2$ and so, given that the supports of $|\partial_1 u|$ and $|\partial_2 u|$ are disjoint on B_1', we claim that

$$\sigma = |\partial_1 u| = \sqrt{U_-} \text{ a.e. on } B_1', \tag{5.2}$$

where $U = U_+ - U_-$ is the standard decomposition into positive and negative parts. Since U is harmonic in the entire B_1 and therefore U_- is locally Lipschitz in B_1, the identity (5.2) implies that $\sigma \in C^{1/2}_{\text{loc}}(B_1')$.

To prove the claim we first observe that by the nontangential limits of $\partial_j u$, Theorem 4.1 and Lemma 4.2, for almost every $y \in B_1'$

$$G(x) = |\partial_2 u|^2 - |\partial_1 u|^2 + 2i\partial_1 u \partial_2 u \to \tau(y)^2 - \sigma(y)^2 \quad \text{as} \quad x \in B_1^+ \to y \text{ non-tangentially.}$$

On the other hand, we know that G is defined and holomorphic in B_1 so the non-tangential limits must agree with the value of the function

$$U(y) = G(y) = \tau(y)^2 - \sigma(y)^2 \text{ for a.e. } y \in B_1'.$$

Then, using that $\sigma(y)\tau(y) = 0$ almost everywhere on B_1',

$$U_-(y) = \sigma(y)^2 \quad \text{and} \quad U_+(y) = \tau(y)^2 \quad \text{on} \quad B_1'.$$

This justifies the claim (5.2).

Then, since u solves, in the distributional weak sense,

$$-\Delta u = 0 \quad \text{in} \quad B_1^+ \quad \text{with} \quad \partial_1 u = \sigma = \sqrt{U_-} \quad \text{on} \quad B_1',$$

by standard $C^{1,\alpha}$ estimates for the Neumann problem with a $C^{0,\alpha}$ boundary condition we obtain that $u \in C^{1,\frac{1}{2}}_{\text{loc}}(B_1^+ \cup B_1')$. □

6. Conditional regularity in dimension $d \geq 3$

In this section, we prove $C^{1,\alpha}$ regularity of a solution u to the problem (1.1) under the following additional condition. Instead of assuming that $u(\mathcal{C}_u)$ is finite, as stated in Theorem 1.2, we make an equivalent (see Remark 6.3) assumption with the relevant parameter more clearly quantified:

$u(\mathcal{C}_u)$ is a finite set so that for any connected components I and J of $\mathcal{C}_u$ with $u(I) \neq u(J)$ the separation condition $\operatorname{dist}(I, J) \geq \delta$ holds. $(\mathbf{A}_\delta)$

In terms of this separation hypothesis, we aim to prove the $C^{1,\alpha}$ regularity result.

Theorem 6.1. *Let u be a viscosity solution to (1.1) that satisfies condition ($\mathbf{A}_\delta$), then there is a small $\alpha = \alpha(d) \in (0, 1)$ such that $u \in C^{1,\alpha}_{\text{loc}}(B_1^+ \sqcup B_1')$ and there is a constant $C = C(d)\delta^{-\alpha} > 0$ such that*

$$\|u\|_{C^{1,\alpha}\left(\overline{B^+_{1/2}}\right)} \leq C \|u\|_{L^\infty(B_1^+)}.$$

Remark 6.2. The controlling constant "$C = C(d)\delta^{-\alpha}$" can actually be replaced by "$C(d, N)$" with $N = \#u(\mathcal{C}_u \cap B_1)$ (see Remark 6.11). We retain the current exposition for the convenience of the proof.

The proof will make use of the well-known Almgren frequency formula, which has seen frequent use in the thin obstacle problem [3,18,21,28]. The main reason for our conditioning on hypothesis ($\mathbf{A}_\delta$) is to guarantee the monotonicity of the frequency function. It will be made clear in the computations in Section 6.1 that the possible occurrence of infinitely many connected components of $\mathcal{C}_u$ piling up on a single point seems to ruin the monotonicity property.

Remark 6.3. Even though the condition ($\mathbf{A}_\delta$) is somewhat artificial because we cannot verify it in many interesting cases, it is indeed satisfied in the case of the classical Signorini problem and it demonstrates the central difficulty of our problem (1.1). The Signorini problem corresponds to the case that $u(\mathcal{C}_u) = \{0\}$ is a singleton and *also* $u \leq 0$ on B_1'. The singleton case also includes the cases of the sign-reversed Signorini problem as introduced in Example 2.5. However, we are not able to make any general guarantee on when a particular boundary condition may admit a solution to this sign-reversed Signorini problem.

Remark 6.4. As mentioned above the hypothesis that $u(\mathcal{C}_u)$ is finite, and hypothesis ($\mathbf{A}_\delta$) are in fact equivalent, the latter just quantifying a useful parameter. Let u be a viscosity solution with $\|u\|_{L^\infty(B_1^+)} = 1$ so that u is Lipschitz with Lipschitz constant at most $L = L(d) > 0$. Suppose additionally that $u(\mathcal{C}_u)$ finite. Then call $\delta = L^{-1} \min\{|u(z) - u(w)| : \ z, w \in \mathcal{C}_u \text{ and } u(z) \neq u(w)\}$, which is positive due to the set $u(\mathcal{C}_u)$ being finite. Let I, J be a pair of components of $\mathcal{C}_u$ such that $u(I) \neq u(J)$. By Lipschitz continuity of u, we have

$$\operatorname{dist}(I, J) \geq \frac{|u(x) - u(y)|}{L} \geq \delta,$$

where $x \in I, y \in J$.

6.1. *Almgren monotonicity formula*

In this section, we will study the monotonicity of the Almgren frequency function for the gradient degenerate Neumann problem (1.1). Due to the above remarks it suffices to consider the case that

$$u(\mathcal{C}_u \cap B_1) = \{0\}. \tag{6.1}$$

When we prove Theorem 6.1 below we will just make an initial re-scaling to a ball of radius δ to achieve this hypothesis. The initial scaling determines the dependence on δ in the theorem.

The following computations, if not particularly mentioned, are obtained after a mollification procedure and an appropriate use of Lemma 4.2. Let u be the even extension of a viscosity solution to (1.1) such that $u(0) = 0$. Consider the frequency functional

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} = \frac{r D(r)}{H(r)}. \tag{6.2}$$

Differentiating the denominator gives

$$H'(r) = \frac{d-1}{r} H(r) + 2 \int\limits_{\partial B_r} u \partial_\nu u, \tag{6.3}$$

where ∂_ν is the unit outer normal derivative on ∂B_r. Now we aim to integrate by parts in the second term. Recall that σ, as defined in (4.2), is the nontangential limit of $\partial_1 u$ on B_1' from B_1^+. We can justify, using the distributional weak formulation as discussed in Lemma 4.2, that the distributional Laplacian of u is given by

$$\Delta u = 2\sigma d\mathcal{H}^{d-1}|_{B_1'} \ \text{ in } \ B_1.$$

Using this identity we find

$$\begin{aligned} H'(r) &= \frac{d-1}{r} H(r) + 2 \int\limits_{B_r} |\nabla u|^2 + 4 \int\limits_{B_r'} u\sigma \\ &= \frac{d-1}{r} H(r) + 2D(r) + 4c(r). \end{aligned} \tag{6.4}$$

Remark 6.5. This final term $c(r) := \int_{B_r'} u\sigma$ is a major difficulty that we are currently only able to deal with via conditioning on the hypothesis ($\mathbf{A}_\delta$), which has allowed us to reduce to the case $u(\mathcal{C}_u \cap B_1) = \{0\}$. In this case, we have $u \equiv 0$ on $\mathcal{C}_u \cap B_1$, and on the other hand, we observe by Corollary 4.4 that $\sigma = 0$ a.e. on $(\mathcal{N}_u \sqcup \Gamma_u) \cap B_1$, which shows that $\sigma u = 0$ a.e. on the whole B_1' and hence $c(r) = 0$, $0 < r < 1$.

On the other hand, we also have after a mollification procedure, Rellich's formula

$$\begin{aligned}\int_{\partial B_r} |\nabla u|^2 &= \frac{d-2}{r}\int_{B_r} |\nabla u|^2 + 2\int_{\partial B_r} (\partial_\nu u)^2 - \frac{2}{r}\int_{B_r}(x\cdot\nabla u)\Delta u \\ &= \frac{d-2}{r}\int_{B_r} |\nabla u|^2 + 2\int_{\partial B_r} (\partial_\nu u)^2 ,\end{aligned} \tag{6.5}$$

where $\Delta u = 2\sigma\, d\mathcal{H}^{d-1}\big|_{\mathcal{C}_u}$ and by Lemma 4.2, $(x\cdot\nabla u)\sigma = (x'\cdot\nabla' u)\sigma = 0$ a.e. on B_1'. The mollification argument for obtaining (6.5) is valid because of the Lipschitz estimate in Lemma 3.1. Collecting these computations we have proved the monotonicity formula for u

$$\begin{aligned}\frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \\ &= 2\left(\frac{\int_{\partial B_r} (\partial_\nu u)^2}{\int_{\partial B_r} u\partial_\nu u} - \frac{\int_{\partial B_r} u\partial_\nu u}{\int_{\partial B_r} u^2}\right) \\ &\geq 0.\end{aligned}$$

Notice that if $N(r) = \kappa$ for $0 < r < 1$ then $N'(r) = 0$ and by the above Cauchy-Schwartz inequality we know that there is $g(r)$ for each $0 < r < 1$ such that

$$\partial_\nu u = g(r)u.$$

To determine g we observe on the other hand,

$$r\frac{d}{dr}\log H(r) = d - 1 + 2N(r) = d - 1 + 2\kappa,$$

which implies that $H(r) = H(1)r^{2\kappa+d-1}$ and by (6.3)

$$g(r)\int_{\partial B_r} u^2 = \int_{\partial B_r} u\partial_\nu u = \frac{1}{2}\left(H' - \frac{d-1}{r}H\right) = \frac{\kappa}{r}H(r),$$

and thus $g(r) \equiv \kappa/r$. This implies that u is a κ-homogeneous function. We summarize the above computations in a theorem.

Theorem 6.6 *(Almgren Monotonicity Formula). Let $d \geq 2$ and u be a viscosity solution to (1.1) in B_1^+, evenly extended to B_1, which has $u(\mathcal{C}_u \cap B_1) = \{0\}$, $u(0) = 0$ and $0 \in \Gamma_u$. Then the quantity*

$$N(r) = N(r, u) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is monotone increasing in $0 < r < 1$. Moreover, if $N(r) \equiv \kappa$ for all $0 < r < 1$ then u is a κ-homogeneous function in B_1.

6.2. Pointwise differentiability

Let u be a viscosity solution evenly extended to the whole ball B_1 that satisfies ($\mathbf{A}_\delta$). We would like to consider the following blow-up sequence at a fixed point $0 \in \Gamma_u \subset B_1'$, $tx \in B_1$ and we also assume $u(0) = 0$

$$u_t(x) = \frac{u(tx)}{t} \quad \text{for} \quad 0 < t < 1.$$

Notice that $\|u_t\|_{C^{0,1}(\overline{B}_1)} \le 2\|u\|_{C^{0,1}(\overline{B}_{1/2})}$ as $t \to 0^+$, and when t is sufficiently small, by condition ($\mathbf{A}_\delta$), $u_t(\mathcal{C}_{u_t}) = \{0\}$. In particular, we have after passage to a subsequence $t_k \to 0$, there is an $u_0 \in C^{0,1}(\overline{B}_1)$ such that $u_{t_k} \to u_0$ uniformly in $\overline{B}_1$.

Lemma 6.7 *(Blow-up limit at the free boundary). Suppose u is a viscosity solution to (1.1) and satisfies the condition ($\mathbf{A}_\delta$). If $0 \in \Gamma_u$ and $u(0) = 0$ then*

$$\frac{u(tx)}{t} \to 0 \quad \textit{as } t \to 0 \textit{ uniformly on } \overline{B_1^+}.$$

Remark 6.8. The differentiability of a solution w to (1.1) up to B_1' can be obtained partially by using interior regularity of Neumann or Dirichlet problems near $\mathcal{N}_w$ and $\mathcal{C}_w$ respectively. This lemma completes the proof of pointwise differentiability by establishing the differentiability on Γ_w.

Proof. We prove by contradiction and assume there is a blow-up limit $u_0 \neq 0$ and $u_{t_k} \to u_0$ uniformly on $\overline{B_1^+}$ as $k \to \infty$. By the argument in (6.4), we know that $H_{u_0}(r) > 0$ for all $0 < r < 1$ and hence for each r we can find $k_0(r) > 0$ such that $H_{u_{t_k}}(r) > \frac{1}{2}H_{u_0}(r) > 0$ for all $k > k_0(r)$.

To prove the lemma, we notice that by the Almgren monotonicity formula (see Theorem 6.6),

$$N(r, u_{t_k}) = N(rt_k, u) \to N(0^+, u) = \kappa \ge 0,$$

as $k \to \infty$, and on the other hand, we know that

$$v_k = \frac{u_{t_k}}{\left(\int_{\partial B_r} u_{t_k}^2\right)^{1/2}} = \frac{u_{t_k}}{H_{u_{t_k}}^{1/2}(r)}$$

satisfies for sufficiently large k and a constant $C > 0$ independent of k

$$\|v_k\|_{L^2(\partial B_r)} = 1, \ \|v_k\|_{L^\infty(B_r)} \le C, \ \text{ and } \int_{B_r} |\nabla v_k|^2 \le N(rt_k, u) \le N(1, u) \le C. \tag{6.6}$$

By interior estimates of harmonic functions, we have local uniform convergence of ∇v_k to $\nabla u_0 / \|u_0\|_{L^2(\partial B_r)}$ in $B_r \setminus B_r'$, and because of boundedness of their L^∞ norms by the Lipschitz estimate in Lemma 3.1, we have the strong convergence of v_k in $H^1(B_r)$. This shows that

$$N(r, u_{t_k}) = \frac{r\int_{B_r} |\nabla u_{t_k}|^2}{\int_{\partial B_r} u_{t_k}^2} \to N(r, u_0), \ \text{ as } k \to \infty.$$

From this we obtain that $N(r, u_0) = \kappa$ for all $r > 0$, and therefore u_0 is κ-homogeneous on $\mathbb{R}^d$ according to Theorem 6.6. If $u_0 \neq 0$ then it can only be a 1-homogeneous function by Lipschitz regularity.

Thus u_0 has the form

$$u_0(r, \theta) = rh(\theta), \ r \ge 0, \ \theta \in \partial B_1 \cap \{x_1 \ge 0\}.$$

Since u_0 is also a viscosity solution to (1.1), the function h must satisfy

$$\begin{cases} \Delta_\theta h(\theta) + (d-1)h(\theta) = 0, & \theta \in \partial B_1 \cap \{x_1 > 0\}, \\ \min\left\{-\partial_{\vec{n}} h(\theta), \sqrt{|\nabla_\tau h(\theta)|^2 + h^2(\theta)}\right\} = 0, & \theta \in \partial' B_1', \end{cases} \tag{6.7}$$

where Δ_θ is the Laplace-Beltrami operator on $\partial B_1 \cap \{x_1 > 0\}$, $\partial_{\vec{n}}$ is the outer normal derivative of $\partial B_1 \cap \{x_1 \geq 0\}$ on the boundary $\partial' B_1'$ and ∇_τ the tangential gradient on $\partial' B_1'$.

We first claim that

$$\int_{\partial' B_1'} h = 0.$$

This can be obtained by the following Green's formula with the linear function $\ell(x) = p \cdot x$, which all satisfy $\Delta_\theta \ell + (d-1)\ell = 0$ on the sphere,

$$\begin{aligned} 0 &= \int_{\partial B_1 \cap \{x_1>0\}} \ell(\Delta_\theta h + (d-1)h) - h(\Delta_\theta \ell + (d-1)\ell) \\ &= \int_{\partial' B_1'} \ell \partial_{\vec{n}} h - h \partial_{\vec{n}} \ell \\ &= \int_{\partial' B_1'} \ell \partial_{\vec{n}} h + h p_1 \end{aligned}$$

The claim is proved by taking $p = e_1$.

On the other hand, we claim that we can apply h and do integration by parts to obtain

$$\begin{aligned} \int_{\partial B_1 \cap \{x_1>0\}} |\nabla_\theta h|^2 &= (d-1) \int_{\partial B_1 \cap \{x_1>0\}} h^2 + \int_{\partial' B_1'} h \partial_{\vec{n}} h \\ &= (d-1) \int_{\partial B_1 \cap \{x_1>0\}} h^2. \end{aligned} \tag{6.8}$$

For the above equality, we used that

$$\int_{\partial' B_1'} h \partial_{\vec{n}} h = 0. \tag{6.9}$$

This is because, by Lemma 4.2, $\min\{\partial_1 u_0(x), |\nabla' u_0|(x)\} = 0$ for almost every $x \in B_1'$ in the nontangential convergence sense, which by homogeneity implies

$$\min\left\{-\partial_{\vec{n}} h(\theta), \sqrt{|\nabla_\tau h(\theta)|^2 + h^2(\theta)}\right\} = 0,$$

for almost all $\theta \in \partial' B_1'$ also in the sense of nontangential convergence. By a similar proof of the second part of Lemma 4.2, we can justify the validity of integration by parts and the fact that $h \partial_{\vec{n}} h = 0$ almost everywhere on $\partial' B_1'$.

Now we can subtract off a linear function by considering $\tilde{h} = h - \gamma x_1$ so that $\tilde{h}$ also satisfies (6.8) and $\int_{\partial B_1 \cap \{x_1>0\}} \tilde{h} = 0$. For this mean zero condition just choose

$$\gamma = \frac{\int_{\partial B_1 \cap \{x_1>0\}} h}{\int_{\partial B_1 \cap \{x_1>0\}} (x_1)_+}.$$

Notice that $\gamma \leq 0$ because

$$(d-1)\int_{\partial B_1\cap\{x_1>0\}} h = \int_{\partial B_1\cap\{x_1>0\}} -\Delta_\theta h = \int_{\partial' B_1'} \partial_{\vec{n}} h \le 0.$$

As for (6.8), because $\Delta_\theta \tilde{h} + (d-1)\tilde{h} = 0$ in $\partial B_1 \cap \{x_1 > 0\}$ it suffices to check (6.9) for $\tilde{h}$. This is proved by

$$\begin{aligned}\int_{\partial' B_1'} \tilde{h}\partial_{\vec{n}}\tilde{h} &= \int_{\partial' B_1'} h\partial_{\vec{n}}\tilde{h}\\ &= \int_{\partial' B_1'} h\partial_{\vec{n}}h - \gamma\int_{\partial' B_1'} h\\ &= 0.\end{aligned}$$

If $\tilde{h} \equiv 0$ then $u_0 \equiv \gamma|x_1|$ and because u_0 is also a solution to (1.1) and by prior discussions $\gamma \le 0$, we obtain $\gamma = 0$.

If $\tilde{h} \not\equiv 0$, then

$$\frac{\int_{\partial B_1\cap\{x_1>0\}} |\nabla_\theta \tilde{h}|^2}{\int_{\partial B_1\cap\{x_1>0\}} \tilde{h}^2} = d-1. \tag{6.10}$$

On the other hand, we know that the second Neumann eigenvalue

$$\lambda := \lambda_{N,2}(\partial B_1 \cap \{x_1 > 0\}) = \inf_{\substack{0\neq g\in H^1(\partial B_1\cap\{x_1>0\})\\ \int_{\partial B_1\cap\{x_1>0\}} g=0}} \frac{\int_{\partial B_1\cap\{x_1>0\}} |\nabla_\theta g|^2}{\int_{\partial B_1\cap\{x_1>0\}} g^2} > 0$$

is equal to $d-1$ and the minimizing functions g must be restrictions of linear functions of the form $p'\cdot x$ with $p' \in \mathbb{R}^d$ and $p'\cdot e_1 = 0$ [32, Chapter 3]. By (6.10), we know that $\tilde{h}$ must be equal to a Neumann second eigenfunction and so

$$u_0(x) = \gamma|x_1| + p'\cdot x$$

is a smooth solution to (1.1) for some p'. Because, again, u_0 is a viscosity solution to (1.1) and $\gamma \le 0$, we know that $\gamma = 0$ and $u_0(x) = p'\cdot x$ for some $p'\cdot e_1 = 0$.

To show that $p' = 0$, we argue by contradiction and assume that there is a sequence $s_j \to 0^+$ such that $f_j = u_{s_j}$ converges locally uniformly to $p'\cdot x$ for some $p' \neq 0$ as $j \to \infty$. Now, we obtain a sequence $\varepsilon_j \to 0^+$ such that

$$\overline{f}_j = \frac{f_j - p'\cdot x}{\varepsilon_j}$$

is uniformly bounded on B_1^+ and satisfies boundary condition

$$\min\{\partial_1 \overline{f}_j, |\nabla' \overline{f}_j + p'/\varepsilon_j|\} = 0, \text{ on } B_1'$$

in the viscosity sense. According to Corollary 4.3, we know that when j is large $\overline{f}_j$ must satisfy the zero Neumann boundary condition on $B'_{1/2}$ and hence $u_{s_j} = f_j$ must also satisfy this condition, which contradicts the assumption that $0 \in \Gamma_{u_t}$ for all $t > 0$. □

6.3. Improvement of flatness

Combining Lemma 6.7, the interior regularity of zero Neumann and Dirichlet problems, we obtain pointwise differentiability (including nontangential directions) of u on B_1'. To obtain the improvement of flatness results, we plan to first consider the following three different cases respectively

(I) The first case deals with the small gradients, and it essentially corresponds to the original boundary condition

$$\min\{\partial_1 u, |\nabla' u|\} = 0.$$

(II) The second case deals with large tangential gradients, and it essentially corresponds to the tangentially modified boundary condition

$$\min\{\partial_1 u, |\nabla' u + q'|\} = 0,$$

with $q' \cdot e_1 = 0$ and $|q'|$ large.

(III) The last case deals with large inner normal derivatives, and it essentially corresponds to the normal modified boundary condition

$$\min\{\partial_1 u + q_1, |\nabla' u|\} = 0,$$

with $q_1 > 0$ large.

In fact, it turns out that these three cases are enough to derive the full improvement of flatness results. This can be obtained by Lemma 6.14 that shows the dichotomy of gradients: a solution u, with $\operatorname{osc} u \leq 1$, to

$$\min\{\partial_1 u + q_1, |\nabla' u + q'|\} = 0$$

will require $|\min\{q_1, |q'|\}|$ to be bounded by a constant independent of u. Now, with a bounded cost, we can modify u so that it is contained in one of the three categories we introduced above.

Lemma 6.9 *(Improvement of Flatness I). Let u be a viscosity solution to (1.1) in B_1 with either $u(\mathcal{C}_u \cap B_1) = \{0\}$ or $\mathcal{C}_u \cap B_1 = \emptyset$, and $\operatorname{osc}_{B_1^+} u \leq T_1$ for some fixed $T_1 > 0$. There is a $1/2 > \mu = \mu(d, T_1) > 0$ such that for each u as described there is $1/2 > \nu = \nu(u) \geq \mu$ such that*

$$\inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{B_\nu^+} \{u - p \cdot x\} \leq \frac{1}{2}\nu. \tag{6.11}$$

Remark 6.10. Notice that here T_1 is absorbed in $\nu(u)$ and $\mu(d, T_1)$, instead of writing νT_1 and μT_1. We will use similar notations in the following improvement of flatness lemmas, where T_1, T_2 and T_3 are constants to be determined.

Remark 6.11. This lemma can actually be improved to the case that $\# u(\mathcal{C}_u \cap B_1) \leq N$ for some fixed positive integer N with a slight modification to the proof. In this way we can replace the constant "$C = C(d)\delta^{-\alpha}$" in Theorem 6.1 by "$C(d, N)$". That is, the controlling constant can be made independent of the minimal gap in $u(\mathcal{C}_u \cap B_1)$, but solely depending on the dimension and the number of elements in it.

Proof. According to Lemma 6.7, for blow-ups on Γ_u, and the interior regularity of the solution on $\mathcal{C}_u \sqcup \mathcal{N}_u$, we have the following convergence

$$\inf_{p\in\mathbb{R}^d}\frac{\operatorname{osc}_{B_r^+}\{u-p\cdot x\}}{r}\to 0,\ \text{ as } r\to 0^+.$$

Thus for each u there is a $\nu'=\nu'(u)\in(0,1)$ such that for every $0<r<\nu'$, the following inequality holds

$$\inf_{p\in\mathbb{R}^d}\operatorname*{osc}_{B_r^+}\{u-p\cdot x\}\le\frac{1}{4}r. \tag{6.12}$$

We now define

$$\eta(u)=\max\{0<s\le 1\,;\,(6.12)\text{ holds for } r=s\}>0.$$

It then suffices to show a uniform positive lower bound for η's since we will obtain (6.11) immediately by taking $\nu=\eta/2$. We argue by contradiction and assume that there exists a sequence of functions u_j that satisfy the conditions described in the statement while

$$\eta(u_j)\to 0 \text{ as } j\to\infty. \tag{6.13}$$

Because $\operatorname{osc}_{B_1^+}(u_j)\le T_1$, we know by the Lipschitz estimate, Lemma 3.1, that (up to a subsequence) u_j converges locally uniformly to some u_∞ in $B_1^+\sqcup B_1'$. By classical viscosity solution theories, u_∞ also satisfies the conditions as described in the statements. Indeed, it suffices to check that either $u_\infty\left(\mathcal{C}_{u_\infty}\cap B_1\right)=\{0\}$ or $\mathcal{C}_{u_\infty}\cap B_1=\emptyset$. Suppose there is a number $s\neq 0$ such that $s\in u_\infty\left(\mathcal{C}_{u_\infty}\cap B_1\right)$, then we can find a relatively open component $I_s\subset\mathcal{C}_{u_\infty}\cap B_1$ such that $u_\infty(I_s)=s$. By the local uniform convergence of u_j to u_∞ on $B_1^+\sqcup B_1'$ we know that for some small $\delta>0$, u_j are uniformly close to s on $I_s\cap B_{1-\delta}$ for large j. By the assumption $u_j(\mathcal{C}_{u_j}\cap B_1)=\{0\}$, we know that u_j satisfy the zero Neumann boundary condition on $I_s\cap B_{1-\delta}$, which implies that u_∞ also satisfies the zero Neumann boundary condition on $I_s\cap B_{1-\delta}$, contradicting the assumption that $I_s\subset\mathcal{C}_{u_\infty}$. A similar proof can show that, in general, the number of degenerate values of the blow-up limit u_∞ does not exceed the number of the degenerate values of u_j's. This addresses the issues as discussed in Remark 6.11.

On the other hand, by the pointwise differentiability of u_∞, we know that for some small $\tilde\eta>0$ there must be some $\tilde p\in\mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_{\tilde\eta}^+}\{u_\infty-\tilde p\cdot x\}\le\frac{1}{8}\tilde\eta,$$

which contradicts the assumption (6.13). □

Lemma 6.12 *(Improvement of Flatness II). Let u be a viscosity solution to* (1.1) *with the boundary condition replaced by*

$$\min\{\partial_1 u,|\nabla' u+q'|\}=0,$$

for some $q'\in\{0\}\times\mathbb{R}^{d-1}$, and $\operatorname{osc}_{B_1^+}u\le T_2$ for some fixed $T_2>0$. There exists $J=J(d,T_2)>0$ such that if $|q'|>J$, then there exists $\iota=\iota(d,T_2)>0$ with $1/2>\iota$ satisfying

$$\inf_{p\in\mathbb{R}^d}\operatorname*{osc}_{B_\iota^+}\{u-p\cdot x\}\le\frac{1}{2}\iota.$$

Proof. According to Lemma 3.1 and Corollary 4.3, we know that the Lipschitz constant $L=L\big(d,\|u\|_{L^\infty(B_1^+)}\big)>0$ of u in $B_{1/2}^+$ is independent of the choice of q'. If one chooses $J=2L\,(d,T_2)>L$, then u satisfies the zero Neumann boundary condition on $B_{1/2}'$ and $\operatorname{osc}_{B_{1/2}^+}u\le T_2$, and then the improvement of flatness comes naturally from the smoothness of Neumann solutions. □

Lemma 6.13 *(Improvement of Flatness III). Let u be a viscosity solution to (1.1) with the boundary condition replaced by*

$$\min\{\partial_1 u + q_1, |\nabla' u|\} = 0,$$

for some $q_1 \in \mathbb{R}$, and $\operatorname{osc}_{B_1^+} u \le T_3$ for some fixed $T_3 > 0$. There exists $I = I(d, T_3) > 0$ such that if $q_1 > I$, then there exists $\gamma = \gamma(d, T_3) > 0$ with $1/2 > \gamma$ satisfying

$$\inf_{p\in\mathbb{R}^d} \operatorname*{osc}_{B_\gamma^+}\{u - p\cdot x\} \le \frac{1}{2}\gamma. \tag{6.14}$$

Proof. Let us study the family of functions

$$w = x_1 + \frac{u}{q_1},$$

where $q_1 > I$ to be chosen. Let $\varepsilon = T_3/I$, we know that w are bounded solutions to (1.1) that are uniformly flat in the sense that

$$x_1 - \varepsilon \le w \le x_1 + \varepsilon.$$

By a similar argument to Lemma 2.8, we may choose $I = 400\max\{1, T_3\}d$ so that $\varepsilon \le 1/(400d)$. We then obtain $B'_{1/4} \subset \mathcal{C}_w$. This also implies that $u \equiv C$ for some constant C on $B'_{1/4}$. By an odd reflection, $u - C$ can be extended to a harmonic function in $B_{1/4}$ with $\operatorname{osc}_{B_{1/4}}\{u - C\} = \operatorname{osc}_{B_{1/4}}\{u\} \le T_3$. By applying the interior regularity of harmonic function we can determine the constant $0 < \gamma < 1/4$ that satisfies (6.14). □

Lemma 6.14 *(Dichotomy of Gradients). Suppose $\operatorname*{osc}_{B_1^+}(u) \le 1$ solves*

$$\begin{cases} \Delta u = 0, & \text{in } B_1^+ \\ \min\{\partial_1 u + m_1, |\nabla' u + m'|\} = 0 & \text{on } B_1' \end{cases} \tag{6.15}$$

in the viscosity sense for some $m = (m_1, m') \in \mathbb{R}^d$. Then there is a constant $K = K(d) > 0$ such that

$$|\min\{m_1, |m'|\}| \le K.$$

Proof. Suppose that there exist a sequence u_j satisfying $\operatorname{osc}_{B_1^+}(u_j) \le 1$ and (6.15), but $|\min\{m_{j,1}, |m_j'|\}| \to \infty$ as $j \to \infty$.

We can assume that either (i) $\min\{m_{j,1}, |m_j'|\} = m_{j,1}$ for all j or (ii) $\min\{m_{j,1}, |m_j'|\} = |m_j'|$ for all j. We consider the two cases separately.

First suppose that $\min\{m_{j,1}, |m_j'|\} = m_{j,1}$. We claim that

$$\liminf_{j\to\infty} |m_j'|/|m_{j,1}| > 0.$$

It suffices to consider the case that $m_{j,1} < 0$. To show this we consider

$$w_j^1 = \frac{u_j}{|m_{j,1}|} + \frac{m_{j,1}}{|m_{j,1}|}x_1,$$

which by assumption on u_j is a bounded sequence of functions. Observe that w_j^1 satisfy the boundary condition

$$\min\{\partial_1 w_j^1, |\nabla' w_j^1 + m_j'/|m_{j,1}||\} = 0.$$

If a subsequence $|m_j'|/|m_{j,1}| \to 0^+$, then by the compactness of w_j^1 (this is because of the Lipschitz estimate, Lemma 3.1), after passage to a subsequence, w_j^1 will converge locally uniformly to a viscosity solution to (1.1). On the other hand, this sequence converges uniformly to $-x_1$, which is not a viscosity solution to (1.1) and shows the claim.

We further claim that it's impossible that $|m_{j,1}| \to \infty$. We divide into cases depending on whether $m_j'/|m_{j,1}|$ stays bounded or not. If $m_j'/|m_{j,1}|$ stays bounded then, after passage to a subsequence, we can suppose that $m_j'/|m_{j,1}| \to m_\infty' \in \{0\} \times \mathbb{R}^{d-1}$ with $|m_\infty'| > 0$ (by the first claim). This implies that the limit function $w_\infty^1 = \pm x_1$ is a viscosity solution of $\min\{\partial_1 w_\infty^1, |\nabla' w_\infty^1 + m_\infty'|\} = 0$ on B_1', which is not true. If $m_j'/|m_{j,1}| \to \infty$ then by Corollary 4.3, for sufficiently large j, w_j^1 satisfies zero Neumann boundary condition on $B_{1/2}'$, which also contradicts the form of the limit function $w_\infty^1 = \pm x_1$ because of compactness of zero Neumann solutions with bounded oscillation on $B_{1/2}^+$.

In the case $\min\{m_{j,1}, |m_j'|\} = |m_j'|$ we define

$$w_j^2 = \frac{u_j}{|m_j'|} + \frac{m_j'}{|m_j'|} \cdot x + \frac{m_{j,1}}{|m_j'|} x_1.$$

This implies that w_j^2 satisfies the original boundary condition (1.1). In this case, we observe that when j is large, w_j^2 (after passage to a subsequence) has one-sided flatness in the way that

$$w_j^2(x) \geq m_\infty' \cdot x + x_1 - o_j(1), \text{ for all } x \in \overline{B_1^+}$$

with $\lim_{j\to\infty} \dfrac{m_j'}{|m_j'|} = m_\infty' \in \{0\} \times \mathbb{R}^{d-1}$ in a proper subsequence, $|m_\infty'| = 1$, and $|w_j^2(0)| \leq 1/|m_j'| = o_j(1)$. This contradicts the supersolution condition of w_j^2 for large j according to a similar argument to the proof of Lemma 4.4. □

6.4. $C^{1,\alpha}$-iteration

Before proving the theorem, we write a lemma that summarizes the improvement of flatness results in the previous subsection. Define the set of allowed gradients

$$\mathcal{T} = \{(q_1, q') \in \mathbb{R}^d\, ; \, \min\{q_1, |q'|\} = 0\},$$

and for $R > 0$ we define the fattening

$$\mathcal{T}_R = \{(q_1, q') \in \mathbb{R}^d\, ; \, |\min\{q_1, |q'|\}| \leq R\}.$$

Lemma 6.15. *Suppose that u is harmonic in B_1^+ with*

$$\min\{\partial_1 u + q_1, |\nabla' u + q'|\} = 0 \quad \text{on} \quad B_1',$$

for some $q = (q_1, q') \in \mathcal{T}_R$ and $\operatorname{osc}_{B_1^+}(u) \leq 1$. If $v(\mathcal{C}_v \cap B_1)$ has at most one element with $v := q \cdot x + u$ (which is a viscosity solution to (1.1); see Remark 2.10), then there is a $1/2 > \nu = \nu(u) > 0$ and $\kappa = \kappa(d, R) > 0$ such that $\nu \geq \kappa$ and

$$\inf_{p \in \mathbb{R}^d} \operatorname{osc}_{B_\nu^+}(u - p \cdot x) \leq \frac{1}{2}\nu.$$

Proof. We define $\bar{q} \in \mathcal{T}$ as follows

$$\bar{q} := \begin{cases} q', & \text{if } \min\{q_1, |q'|\} = q_1 \\ q_1 e_1, & \text{if } \min\{q_1, |q'|\} = |q'|, \end{cases} \quad \text{and} \quad \tilde{q} := q - \bar{q}.$$

Notice that $|\tilde{q}| \le R$ because $q \in \mathcal{T}_R$ and

$$\tilde{q} := \begin{cases} q_1 e_1, & \text{if } \min\{q_1, |q'|\} = q_1 \\ q', & \text{if } \min\{q_1, |q'|\} = |q'|. \end{cases}$$

Now the function $w := \tilde{q} \cdot x + u$ satisfies (1.1) with the boundary condition replaced by

$$\min\{\partial_1 w + \overline{q}_1, |\nabla' w + \overline{q}'|\} = 0,$$

where $\overline{q} = (\overline{q}_1, \overline{q}') \in \mathcal{T}$, and $\operatorname{osc}_{B_1^+}(w) \le 1 + 2R$.

If $|\overline{q}| \ge 2\max\{I(d, 1+2R), J(d, 1+2R)\} =: \Lambda(d, R)$, then the improvement of flatness follows from Lemma 6.12 and 6.13, where $J(d, T_2)$ and $I(d, T_3)$ are defined. Here we choose $T_2 = T_3 = 1 + 2R$.

If $|\overline{q}| \le \Lambda(d, R)$, then we know that $v = u + q \cdot x$ satisfies the original (1.1) with $\operatorname{osc}_{B_1^+}(v) \le 1 + 2R + 2\Lambda(d, R)$. Since $v(\mathcal{C}_v)$ has at most one element, we can apply Lemma 6.9 to obtain the improvement of flatness with $T_1 = 1 + 2R + 2\Lambda(d, R)$.

We may define $\kappa(d, R) = \min\{\mu(d, T_1), \iota(d, T_2), \gamma(d, T_3)\} > 0$. □

Proof of Theorem 6.1. First, we re-scale to reduce to the case that $u(\mathcal{C}_u \cap B_1) = \{0\}$. If u satisfies condition ($\mathbf{A}_\delta$), then we can consider $w(x) := \frac{u(\delta x + x_0) - u(x_0)}{\delta}$ and observe that $w(\mathcal{C}_w \cap B_1)$ has at most one element. Furthermore w will be bounded independent of δ due to the Lipschitz estimate (Lemma 3.1) and

$$[u]_{C^{1,\alpha}(B_{\delta/2}(x_0))} \le \delta^{-\alpha} [w]_{C^{1,\alpha}(B_{1/2})}.$$

To prove that u is $C^{1,\alpha}_{\text{loc}}(B_1^+ \sqcup B_1')$ it suffices to show that u is $C^{1,\alpha}$ at 0 in the sense that there is $C = C(d) > 0$ and $p \in \mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_r^+}(u - p \cdot x) \le C r^{1+\alpha}, \; r \in (0, 1). \tag{6.16}$$

Indeed, by classical arguments this implies $C^{1,\alpha}_{\text{loc}}$ of u when restricted to B_1', which then implies the $C^{1,\alpha}_{\text{loc}}$ regularity of u in the whole $B_1^+ \sqcup B_1'$ by classical estimates for Dirichlet problems.

To show (6.16), it suffices to find a sequence (q_k, r_k) such that $q_k \in \mathbb{R}^d$, and

$$\operatorname*{osc}_{B_{r_k}^+}(u - q_k \cdot x) \le r_k^{1+\alpha} \quad \text{for all } k \in \mathbb{N} \tag{6.17}$$

where $r_k \to 0$ as $k \to \infty$ and $\frac{1}{2} \ge \frac{r_{k+1}}{r_k} \ge \kappa(d) > 0$ for all k and some $\kappa(d) > 0$. If this is done then the constant C in (6.16) will take the form $\kappa^{-(1+\alpha)}$.

We start with $u_0 = u$ such that $\operatorname{osc}_{B_1^+}(u_0) \le 1$ and $q_0 = 0$. We fix $R = K(d)$ as in Lemma 6.14, $T_1 = 1 + 2K(d) + 2\Lambda(d, K(d))$ and $T_2 = T_3 = 1 + 2K(d)$ as in the proof of Lemma 6.15. By applying Lemma 6.15 to u_0, we obtain $1/2 > \nu_1 \ge \kappa(d, K(d)) =: \kappa(d) > 0$ and $p_1 \in \mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_{\nu_1}^+}(u - p_1 \cdot x) \le \frac{1}{2}\nu_1.$$

We now choose $\alpha > 0$ small so that $\kappa^\alpha > 1/2$. Suppose for $k \geq 1$ we have already constructed $q_k \in \mathbb{R}^d$ (notice that we already have $q_1 = p_1$ and $r_1 = \nu_1$) such that (6.17) holds true. We then consider for $x \in B_1^+ \sqcup B_1'$

$$u_k(x) = r_k^{-1-\alpha}\left(u(r_k x) - q_k \cdot (r_k x)\right).$$

Notice that $\operatorname{osc}_{B_1^+}(u_k) \leq 1$ and u_k satisfies (1.1) with boundary condition replaced by

$$\min\{\partial_1 u_k + r_k^{-\alpha} q_{k,1}, |\nabla' u_k + r_k^{-\alpha} q_k'|\} = 0.$$

By Lemma 6.14 we obtain that $r_k^{-\alpha} q_k \in \mathcal{T}_K$, and then we may apply Lemma 6.15 with $R = K(d)$ to u_k and obtain $1/2 > \nu_{k+1} \geq \kappa$, $p_{k+1} \in \mathbb{R}^d$ such that

$$\operatorname*{osc}_{B_{\nu_{k+1}}}(u_k - p_{k+1} \cdot x) \leq \frac{1}{2}\nu_{k+1}.$$

Setting $r_{k+1} = r_k \nu_{k+1}$ and $q_{k+1} = q_k + r_k^\alpha p_{k+1}$, we will obtain

$$\operatorname*{osc}_{B^+_{r_{k+1}}}(u - q_{k+1} \cdot x) \leq r_k^{1+\alpha}\frac{1}{2}\nu_{k+1} \leq r_{k+1}^{1+\alpha}. \quad \square$$

7. Conditional optimal regularity in $d \geq 3$

In this section we discuss the optimal $C^{1,1/2}$ regularity of a viscosity solution u to (1.1) satisfying the condition $\#u(\mathcal{C}_u \cap B_1) < +\infty$. The proof uses the Almgren monotonicity formula Theorem 6.6 again in a similar way to results for the thin obstacle problem. Let us start with a more detailed version of Theorem 1.2.

Theorem 7.1. *Let u be a viscosity solution to (1.1) that satisfies condition ($\mathbf{A}_\delta$), then there is a constant $C(d,\delta) = C(d)\delta^{-1/2} > 0$ such that*

$$\|u\|_{C^{1,1/2}(\overline{B^+_{1/2}})} \leq C\,\|u\|_{L^\infty(B_1^+)}.$$

To obtain optimal regularity we would like to consider functions of the form

$$w_t(x) = \frac{u(tx)}{\left(\frac{1}{t^{d-1}}\int_{\partial B_t} u^2\right)^{1/2}},$$

where u is evenly extended to the whole ball B_1, $tx \in B_1$. We would like to consider the blow-up limit of w_t at the base point $0 \in \Gamma_u$ and $u(0) = 0$. Notice that w_t is controlled in the sense that

$$\|w_t\|_{L^2(\partial B_1)} = 1. \tag{7.1}$$

On the other hand, by the Almgren monotonicity, we have

$$\int_{B_1} |\nabla w_t|^2 = N(1, w_t) = N(t, u) \leq N(T, u),\ 0 < t \leq T.$$

Unlike (6.6), we don't immediately have a uniform L^∞ bound for w_t, and thus these L^2 estimates are not enough for working with the blow-up limits, and we need an additional L^2 to L^∞ estimate to proceed.

As discussed in Remark 6.4 it suffices to consider the case that $u(\mathcal{C}_u \cap B_1) = \{0\}$, since if u satisfies ($\mathbf{A}_\delta$) then it satisfies $\#u(\mathcal{C}_u \cap B_\delta) = 1$ in all balls of radius δ.

7.1. An L^2 to L^∞ estimate

When $u(\mathcal{C}_u \cap B_1) = \{0\}$ then u also solves the following *no-sign Signorini problem*

$$\begin{cases} \Delta w = 0, & \text{in } B_1^+ \\ \min\{\partial_1 w, |w|\} = 0, & \text{on } B_1' \\ w = g, & \text{on } \partial B_1 \cap \{x_1 \geq 0\}, \end{cases} \tag{7.2}$$

in the viscosity sense. Note that solutions to this problem may only be $C^{1/2}$ regular, for example, $w(x) = \mathrm{Re}(x_2 + i|x_1|)^{1/2}$ solves, but we are just using this as a convenient setting to prove the $L^2 \to L^\infty$ estimate.

Remark 7.2. Given a viscosity solution w to (7.2), we will obtain a partition

$$B_1' = \mathcal{C}_w \sqcup \Gamma_w \sqcup \mathcal{N}_w,$$

where $\mathcal{N}_w = \{|w| > 0\} \cap B_1'$ is open and $\Gamma_w = \partial' \mathcal{N}_w$ is called the *free boundary* of w. Notice that the definitions of these sets are essentially different from those for solutions to (1.1) in Remark 2.10. An example that shows this difference is $w(x_1, x_2) = \mathrm{Re}(x_2 + i|x_1|)^{1/2}$. This example is a solution to (7.2) but not (1.1). Notice that for any $N > 0$ there exists a smooth function ϕ_N touching w from below at 0, while $\partial_1 \phi_N(0) > N > 0$, which means that $0 \in \mathcal{C}_w$ if in the sense of Definition 2.7, but it is in fact contained in Γ_w by the definitions of $\mathcal{N}_w$ and Γ_w as described above. However, the definitions will *coincide* if a solution solves both (1.1) and (7.2).

Lemma 7.3. *Let w be a continuous viscosity solution to the equation (7.2), then there is a constant $C = C(d) > 0$ such that*

$$\|w\|_{L^\infty(B_{1/2}^+)} \leq C \|g\|_{L^2(\partial B_1 \cap \{x_1 \geq 0\})}. \tag{7.3}$$

Remark 7.4. Using the same proof we know that for some constant $C > 0$ and all $r > 0$

$$\|w\|_{L^\infty(B_{r/2}^+)} \leq C r^{-d/2} \|w\|_{L^2(B_r^+)},$$

where C is independent of r.

Proof. Let $g_+ = \max\{g, 0\}$, $g_- = \min\{g, 0\}$, and denote v_+, v_- respectively the Neumann solution with boundary data g_+, g_-. By classical theories $v_\pm$ are smooth in $B_1^+ \sqcup B_1'$ up to the flat boundary. We claim that any continuous viscosity solution w to (7.2) has to satisfy

$$v_- \leq w \leq v_+, \text{ in } \overline{B_1^+}.$$

Once we prove the claim the estimate (7.3) will follow from the classical theories for Neumann solutions. The upper bound $w \leq v_+$ can be immediately obtained by observing that w is also a Neumann subsolution and $g \leq g_+$ on $\partial B_1 \cap \{x_1 \geq 0\}$. To obtain the lower bound we consider the following maximization problem for small $\beta > 0$

$$\max_{x \in \overline{B_1^+}} v_-(x) - w(x) + \beta(x_1)_+ - \beta.$$

By comparison principle of harmonic functions and $g_- \leq g$ on $\partial B_1 \cap \{x_1 \geq 0\}$, the maximum point x_* must occur in B_1' if the maximum value is positive. Let us first consider the case that $g_- \neq 0$. In this case $v_- < 0$ in $B_1^+ \sqcup B_1'$ by strong maximum principle, and hence

$$w(x_*) < v_-(x_*) - \beta < 0.$$

Moreover $v_- + \delta(x_1)_+ + C$ touches w from below at x_* for some constant C, which contradicts the supersolution condition of w at x_*.

In the case $g_- = 0$ we would like to show that $w \geq 0$. Let us similarly consider the following maximization problem

$$\max_{x \in \overline{B_1^+}} \beta(x_1)_+ - \beta - w(x).$$

Also by maximum principle the maximum point x_* can only occur on B_1' if the maximum value is positive. This shows that

$$w(x_*) < -\beta < 0,$$

and then $\beta(x_1)_+ + C$ touches w from below at x_*, which also contradicts the supersolution condition of w. □

7.2. Blow-up profiles

In this section, we discuss the possible blow-up profiles of the function sequence w_t as discussed after Theorem 7.1.

According to (7.1), we know that the blow-up sequence w_t have bounded L^2-norm on the boundary portion $\partial B_1 \cap \{x_1 \geq 0\}$. By applying Lemma 7.3, we obtain boundedness of w_t in $L^\infty(B_{1/2})$ (when extended to the whole ball by even reflection). Now using the $C^{1,\alpha}$ estimate, Theorem 6.1, the sequence of functions w_t is bounded in $C^{1,\alpha}\left(\overline{B_{1/4}}\right)$, which shows the following lemma.

Lemma 7.5. *The sequence of functions w_t is compact in both $H^1(B_{1/4})$ and $C^1\left(\overline{B_{1/4}}\right)$.*

By applying this lemma, we can find $t_j \to 0^+$ such that $w_{t_j} \to w_0$ in both $H^1(B_{1/4})$ and $C^1\left(\overline{B_{1/4}}\right)$ as $j \to \infty$. On the other hand, we have for $0 < r < 1/4$

$$N(r, w_0) = \lim_{j\to\infty} N(r, w_{t_j}) = \lim_{j\to\infty} N(rt_j, u) = N(0^+, u) =: \kappa.$$

Applying Almgren's monotonicity formula, Theorem 6.6, we obtain the following characterization of all the blow-up limits.

Proposition 7.6. *Let u be a viscosity solution to (1.1) that satisfies the condition ($\mathbf{A}_\delta$), then the blow-up limit w_0 as defined above is a nonzero global solution to (7.2), and is homogeneous of degree $\kappa = N(0^+, u) > 1$.*

Now, we would like to classify all the κ-homogeneous solutions w_κ to (7.2) in dimension $d = 2$. It can be checked (see Appendix A) that after reflection and normalization, any nonzero homogeneous viscosity solution w_κ of degree $\kappa \geq 0$ has to take one of the forms for $k \in \mathbb{Z}_+$ in Table 1.

Table 1
Classification of κ-homogeneous solutions to (7.2) in dimension $d = 2$. See the proof in Appendix A.

κ	$w_\kappa(x_1, x_2)$	$\mathcal{C}_{w_\kappa} \cap B_1$	$\Gamma_{w_\kappa} \cap B_1$	$\mathcal{N}_{w_\kappa} \cap B_1$
1	$\lvert x_1\rvert$	B_1'	$\emptyset$	$\emptyset$
$2k+1$	$\mathrm{Im}\,((x_2 + i\lvert x_1\rvert)^\kappa)$	$B_1' \setminus \{0\}$	$\{0\}$	$\emptyset$
$\frac{2k-1}{2}$	$\mathrm{Im}\,((x_2 + i\lvert x_1\rvert)^\kappa)$	$\{0\} \times (0,1)$	$\{0\}$	$\{0\} \times (-1,0)$
k	$\pm\mathrm{Re}\,((x_2 + i\lvert x_1\rvert)^\kappa)$	$\emptyset$	$\emptyset$	B_1'

Remark 7.7. In the case $\kappa = \frac{4k-1}{2}$ we have

$$\mathrm{Im}\,((x_2 + i|x_1|)^\kappa) = -\mathrm{Re}\,((-x_2 + i|x_1|)^\kappa)$$

correspond to the *nontrivial* homogeneous solutions to the Signorini problem in Example 2.4. In the case $\kappa = \frac{4k-3}{2}$ we have

$$\mathrm{Im}\,((x_2 + i|x_1|)^\kappa) = \mathrm{Re}\,((-x_2 + i|x_1|)^\kappa)$$

correspond to the *nontrivial* homogeneous solutions to the sign-reversed Signorini problem in Example 2.5.

In particular, if $\kappa > 1$ then we know that $\kappa \geq 3/2$. In higher dimensions, we can also obtain this property by using the ACF monotonicity formula, see [21,28].

Theorem 7.8. *Let w be a homogeneous viscosity solution to* (7.2) *of degree $2 > \kappa > 1$. Then $\kappa = 3/2$, and*

$$w(x) = \mathrm{Im}(x_2 + i|x_1|)^{3/2} = -\mathrm{Re}(-x_2 + i|x_1|)^{3/2},$$

after a possible rotation in $\mathbb{R}^{d-1}$ and normalization.

Sketch of Proof. We outline the idea of the proof here. See [21,28] for detailed proofs. Given w, extended evenly to the whole ball B_1, we would like to consider the following two functions with $e \in \{0\} \times \mathbb{R}^{d-1}$

$$v_+ = \max\{\partial_e w, 0\},\ v_- = \max\{\partial_{-e} w, 0\} = \max\{-\partial_e w, 0\}.$$

By $C^{1,\alpha}$ regularity, Theorem 6.1, we can discuss everything in classical setting. Therefore, by the boundary condition (1.1), we know that $v_\pm$ are harmonic wherever they are positive, which shows that both of them are subharmonic. On the other hand, we have $v_- \cdot v_+ = 0$. By the ACF monotonicity formula,

$$\phi_e(r) = \frac{1}{r^4} \int\limits_{B_r} \frac{|\nabla v_+|^2}{|x|^{d-2}} \int\limits_{B_r} \frac{|\nabla v_-|^2}{|x|^{d-2}} = r^{4(\kappa-2)} \phi_e(1)$$

is monotone in $r > 0$. When $1 < \kappa < 2$ then the monotonicity implies that $\phi_e(1) = 0$, which means that one of $v_\pm$ is identically zero and hence $\partial_e w$ is either nonnegative or nonpositive on the entire $\mathbb{R}^d$. We denote

$$S_+ = \{e \in \mathbb{S}^{d-2} := \partial' B_1' : \partial_e u \geq 0\},$$

and

$$S_- = \{e \in \mathbb{S}^{d-2} := \partial' B_1' : \partial_e u \leq 0\}.$$

Notice that $S_+ = -S_- \neq \emptyset$, $\mathbb{S}^{d-2} = S_+ \cup S_-$ and both are closed subsets. Since whenever $d > 2$, $\mathbb{S}^{d-2}$ is connected and hence $S_+ \cap S_- \neq \emptyset$. One can choose $e^{(1)} \in S_+ \cap S_-$ to reduce to the orthogonal subspace of

$\{0\} \times \mathbb{R}^{d-1}$ with respect to $e^{(1)}$. Proceed with the same procedure we can obtain an orthogonal sequence $e^{(1)}, \cdots, e^{(d-2)}$ such that we can reduce to the subspace spanned by $e^* \in \{0\} \times \mathbb{R}^{d-1}$ with $e^* \cdot e^{(j)} = 0$ for all $j = 1, \cdots, d-2$, which is equivalently solving the problem in the case that $d = 2$. In this case we know that the only homogeneous solution with $1 < \kappa < 2$ is of the form

$$w(x) = \mathrm{Im}(x_2 + i|x_1|)^{3/2} = -\mathrm{Re}(-x_2 + i|x_1|)^{3/2}. \quad \square$$

7.3. Optimal regularity in dimension $d \geq 3$ under condition ($\mathbf{A}_\delta$)

To prove Theorem 7.1, we follow the framework in [22] and start with the estimation near $0 \in \Gamma_u$.

Lemma 7.9. *Let u be a viscosity solution to (1.1) that satisfies condition ($\mathbf{A}_\delta$) with $\mathrm{osc}_{B_1^+} u \leq 1$, $u(0) = 0$ and $0 \in \Gamma_u$. Then $|\nabla u(0)| = 0$ and there is $r_0 = C(d)\delta > 0$ such that*

$$|v(x)| \leq \overline{C}|x|^{3/2}, \ |x| \leq r_0,$$

where $\overline{C} > 0$ is universal.

Proof. For the proof recall the definition of the frequency function $N(u, r) = \frac{rD(r)}{H(r)}$ in (6.2) with $D(r) = \int_{B_r} |\nabla u|^2$ and $H(r) = \int_{\partial B_r} u^2$. By Theorem 7.8 we know that for $0 < r < r_0$

$$\frac{3}{2} \leq N(0^+, u) \leq N(r, u).$$

This implies that

$$r\frac{d}{dr}\log H(r) \geq d + 2 \ \text{ for } \ 0 < r < r_0,$$

and then

$$H(r) \leq Cr^{d+2}.$$

After integrating with respect to r we know that

$$\int\limits_{B_r} u^2 \leq Cr^{d+3}.$$

The proof is now completed by applying Remark 7.4. $\square$

Now we can prove the optimal regularity estimate for all the viscosity solutions to (1.1).

Proof of Theorem 7.1. We follow the reflection arguments in Theorem 6.7 in [22]. Similar to the proof of Theorem 6.1, it suffices to consider the case that $u(\mathcal{C}_u)$ has at most one element. For each $x_0 \in B_{1/2}^+ \sqcup B_{1/2}'$ we define $d(x_0) = \mathrm{dist}(x_0, \Gamma_u)$ with Γ_u the free boundary of u. We claim that either

$$B(x_0, d(x_0)) \cap B_1' \subset \mathcal{C}_u, \text{ or } B(x_0, d(x_0)) \cap B_1' \subset \mathcal{N}_u. \tag{7.4}$$

Indeed, by the partition $B_1' = \mathcal{C}_u \sqcup \mathcal{N}_u \sqcup \Gamma_u$ we know that $A := B(x_0, d(x_0)) \cap B_1'$ can be partitioned into

$$A = (A \cap \mathcal{C}_u) \cup (A \cap \mathcal{N}_u) \cup (A \cap \Gamma_u).$$

We assume without loss that $A \neq \emptyset$. By definition of $d(x_0)$, we know that $A \cap \Gamma_u = \emptyset$ and so if neither of 7.4 is satisfied then A will be a disconnected set, which contradicts of the fact that it is an open subball of B_1'. If $A_{x_0} \subset \mathcal{C}_u$ then we extend u to the whole $B(x_0, d(x_0))$ by odd extension, and if $A_{x_0} \subset \mathcal{N}_u$ then we extend u to the whole $B(x_0, d(x_0))$ by even extension. By Schwarz reflection, these are smooth extensions.

To show $C^{1,1/2}$ it suffices to show that for $|\xi - \eta| \le \frac{1}{32}$, $\xi, \eta \in B_{1/2}^+ \sqcup B_{1/2}'$,

$$|\nabla u(\xi) - \nabla u(\eta)| \le C|\xi - \eta|^{1/2},$$

for some constant $C > 0$. In the following $C > 0$ means a universal constant that may change from line to line. If $d(\xi) \ge \frac{1}{16}$ (or symmetrically $d(\eta) \ge \frac{1}{16}$), then we can use the smoothness of u in $B(\xi, \frac{1}{32})$ in either the case of odd or even extension as described above. If $d(\eta) \le d(\xi) \le \frac{1}{16}$ and $|\xi - \eta| \ge d(\xi)/2$, then by the gradient estimate we have

$$\begin{aligned} |\nabla u(\xi)| &\le \frac{C}{d(\xi)} \sup_{B(\xi, d(\xi))} |u| \\ &\le \frac{C}{d(\xi)} \sup_{B(\xi_0, 2d(\xi))} |u| \\ &\le C d(\xi)^{1/2} \\ &\le C|\xi - \eta|^{1/2}, \end{aligned}$$

where $\xi_0 \in \Gamma_u$ is a point that $|\xi - \xi_0| = d(\xi)$. Similarly $|\nabla u(\eta)| \le C|\xi - \eta|^{1/2}$. In the last case that $d(\eta) \le d(\xi) \le \frac{1}{16}$ and $|\xi - \eta| < d(\xi)/2$, we use the interior estimate for second order derivatives of harmonic functions and obtain

$$\begin{aligned} |\nabla u(\xi) - \nabla u(\eta)| &\le \frac{C|\xi - \eta|}{d(\xi)^2} \sup_{B(\xi, d(\xi))} |u| \\ &\le \frac{C|\xi - \eta|}{d(\xi)^2} \sup_{B(\xi_0, 2d(\xi))} |u| \\ &\le C|\xi - \eta| d(\xi)^{-1/2} \\ &\le C|\xi - \eta|^{1/2}. \quad \square \end{aligned}$$

8. Minimal supersolution and comparison principle

In this section, we study the minimal supersolutions to (1.1) by using Perron's method and prove the characterizing comparison principle. Given a fixed continuous boundary data g on the boundary portion $\partial B_1 \cap \{x_1 \ge 0\}$, a minimal supersolution with respect to the boundary data g is defined as

$$v_g(x) := \inf\{v(x)\,;\, v \text{ is a supersolution to (1.1) and } v \ge g \text{ on } \partial B_1 \cap \{x_1 \ge 0\}\}.$$

Unlike the general viscosity solutions to (1.1), a minimal supersolution will satisfy an additional *strong subsolution condition*.

Definition 8.1 *(Strong subsolution).* An upper semicontinuous function u is called a strong subsolution to (1.1) if it is a subsolution and there are no C^1 up-to-boundary function of the form $\varphi(x_1, x') \equiv \psi(x_1)$ that touches u from above in $\Omega_h \cap \overline{B_1^+}$ at some $x_0 \in B_1'$ and $\varphi > u$ in $\overline{\Omega_h \setminus \Omega} \cap \overline{B_1^+}$ where Ω is an arbitrary open domain of $\mathbb{R}^d$ containing x_0 and $\Omega_h = \bigcup_{y \in \Omega} B_h(y)$ for some small $h > 0$ so that $\overline{\Omega_h} \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$.

We will show in the following subsections that this strong subsolution condition is equivalent to the boundary maximum principle, and is indeed a necessary condition for a minimal supersolution.

Remark 8.2. In Section 9, we will discuss how the flat asymptotic expansion of the minimal supersolutions to (1.11) gives rise to the strong subsolution property of the asymptotic limit. This, by the comparison principle, Theorem 1.6 (which we will prove in this section) will then lead to the equivalence of the three notions of solutions: the minimal supersolutions, the solutions that satisfy the strong subsolution condition (or the asymptotic expansion limit arising in (1.11)) and the solutions that satisfy the boundary maximum principle.

To prove the comparison principle (see Theorem 1.6) for supersolutions and strong subsolutions we face several difficulties due to the degeneracy of the problem (1.1). The degenerate Neumann boundary condition in (1.1) is incompatible with the classical doubling variable arguments. We cannot use the classical sub-/sup-convolution either, because otherwise the sub-/super-solutions are not preserved under the mollification procedure. We overcome this issue by introducing the "tangential" sub-/sup-convolution technique along with a harmonic lift (see Section 8.2 and 8.3).

Remark 8.3. Combining the discussions in Section 8 and 9, we know that the following three functions are equal to each other if they share the same boundary data on the boundary portion $\partial B_1 \cap \{x_1 \geq 0\}$

1. the asymptotic expansion of the Singular Bernoulli problem as discussed in Section 9, which satisfies an additional strong subsolution condition as defined in Definition 8.1;
2. the viscosity solution to (1.1) that satisfies an additional boundary maximum principle as described in Lemma 8.1;
3. the minimal supersolution to (1.1) as discussed in Section 8.

8.1. Boundary maximum principle

Let us now show that the strong subsolution condition is equivalent to the boundary maximum principle. To that end, let us recall the concept of sub/sup-convolutions.

Definition 8.4. Let $U \subset \mathbb{R}^d$ be a domain and $u, v : U \to \mathbb{R}$. The sup-convolution of $u \in \mathrm{USC}\left(\overline{U}\right)$ is defined for $\varepsilon > 0$

$$u^\varepsilon(x) = \sup_{y \in \overline{U}} \left\{ u(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\},\ x \in \mathbb{R}^n.$$

The inf-convolution of $v \in \mathrm{LSC}\left(\overline{U}\right)$ is defined as

$$v_\varepsilon(x) = \inf_{y \in \overline{U}} \left\{ v(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\},\ x \in \mathbb{R}^n.$$

For the convenience of discussing the limit of inf/sup-convolutions, let us also introduce the half-relaxed limits of Barles [4].

Definition 8.5. Let u^k be a family of functions that is bounded from above, then the upper half relaxed limit of u^k is defined as

$$\limsup_{k\to\infty}{}^* u^k(z) := \lim_{k\to\infty} \sup_{n>k,\ |z-x|\leq 1/k} u^n(x).$$

If v_k is a family of functions that is bounded from below, then the lower half relaxed limit is defined as

$$\liminf_{k\to\infty}{}_* v_k(z) := \lim_{k\to\infty} \inf_{n>k,\, |z-x|\le 1/k} v_n(x)$$

Remark 8.6. Let us recall some of the basic properties of sub/sup-convolutions (see [10,31] and the references therein).

1. It is known that v_ε and u^ε are, respectively, semiconcave and semiconvex, specifically $v_\varepsilon(x) - \frac{1}{2\varepsilon}|x|^2$ is concave and $u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2$ is convex. In particular, both u^ε and v_ε are also Lipschitz continuous.
2. If u is subharmonic in a domain Ω then u^ε is also subharmonic in a slightly smaller domain $\Omega_\varepsilon \subset\subset \Omega$.
3. The lower and upper half relaxed limits defined above are always, respectively, lower and upper semi-continuous.
4. When u is upper semi-continuous, then we have the following half-relaxed convergence

   $$u = \limsup_{\varepsilon\to 0^+}{}^* u^\varepsilon.$$

   A similar convergence holds for lower semi-continuous v:

   $$v = \liminf_{\varepsilon\to 0^+}{}_* v_\varepsilon.$$

5. If $u = \limsup^* u^k$ on a compact set $K \subset \mathbb{R}^d$, then we have

   $$\limsup_{k\to\infty} \max_K \left(u^k\right) \le \max_K(u).$$

   A similar statement holds true for $v = \liminf_* v_k$, then

   $$\liminf_{k\to\infty} \min_K (v_k) \ge \min_K(u).$$

6. Let u be an upper semi-continuous function and call its sup-convolutions u^ε, then because $u^\varepsilon \ge u$, on a compact set $K \subset \mathbb{R}^d$ we have

   $$\lim_{\varepsilon\to 0} \max_K (u^\varepsilon) = \max_K(u).$$

   A similar result also holds for lower ones.

Next, we show that strong subsolutions satisfy a *boundary maximum principle.*

Lemma 8.7 *(Boundary Maximum Principle). Let u be a strong subsolution as defined in Definition 8.1, then we have for any subdomain $\Omega \subset\subset B_1'$*

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial'\Omega} u(x), \tag{8.1}$$

where $\partial'\Omega$ is defined as the relative boundary of B_1' in $\{x_1 = 0\}$.

Remark 8.8. We notice that if a subsolution u satisfies (8.1), then it will also satisfy the strong subsolution condition.

Proof. We extend u to $\overline{B_1}$ evenly and consider it as an upper semi-continuous function defined on the whole $\mathbb{R}^d$ by setting it to be $-\infty$ outside $\overline{B_1}$. It can be observed that u is subharmonic in B_1. Let u^ε be a family of sup-convolutions of u, then according to Remark 8.6 we know that u^ε are Lipschitz, subharmonic in $B_{1-\gamma(\varepsilon)}$, and u^ε converges in $\overline{B_1^+}$ to u as $\varepsilon \to 0$ in the sense

$$u = \limsup_{\varepsilon\to 0^+}{}^* u^\varepsilon.$$

Suppose there is a subdomain $\Omega \subset\subset B_1'$ and positive numbers $\delta, \rho > 0$ such that

$$\max_{\overline{\Omega}} u \geq \max_{\overline{\Omega_\rho\setminus\Omega}} u + \delta,$$

with $\Omega_\rho := \bigcup_{x\in\Omega} B_\rho'(x) \subset\subset B_1'$. According to Remark 8.6 (5) and (6), for sufficiently small $\varepsilon > 0$ we have

$$u^\varepsilon(y) \leq \max_{\overline{\Omega}} u - 2\delta/3 \quad \text{for} \quad y \in \overline{\Omega_\rho \setminus \Omega}. \tag{8.2}$$

On the other hand, we pick for convenience a subsequence $\varepsilon_j \to 0^+$ as $j \to \infty$, and using Remark 8.6 (6) again on the set $K = \overline{\Omega}$, we obtain a sequence $u_j := u^{\varepsilon_j}$ such that

$$\lim_{j\to\infty} \max_{\overline{\Omega}} u_j = \max_{\overline{\Omega}} u, \tag{8.3}$$

and the following property is satisfied by combining (8.2) and (8.3)

$$\max_{\overline{\Omega_\rho\setminus\Omega}} u_j \leq \max_{\overline{\Omega}} u_j - \delta/2, \tag{8.4}$$

for sufficiently large j.

To make a contradiction, we assume $L_j > 0$ to be the Lipschitz constants of u_j (we may, without loss, assume that $L_j \to \infty$ as $j \to \infty$), then we construct

$$w_j(x) = \max_{\overline{\Omega}} u_j + 100L_j(x_1)_+,$$

that touches u_j from above in $\Omega_\rho \times [0, r]$ at some $x_j \in \Omega$ with $r > 0$ small so that $\Omega_\rho \times [0, r] \subset\subset B_1^+ \sqcup B_1'$. We claim that $w_j - u_j \geq \delta/2 > 0$ on the fattened boundary $\overline{\Omega_\rho \setminus \Omega} \times [0, r] \bigcup \overline{\Omega_\rho} \times [r(1-\rho), r]$. Indeed, on the set $\overline{\Omega_\rho} \times [\delta/L_j, r]$, we always have

$$w_j - u_j \geq 98\delta > \delta/2.$$

For $z \in \overline{\Omega_\rho \setminus \Omega} \times [0, \delta/L_j]$, we have by (8.4), for sufficiently large j

$$\begin{aligned} w_j(z) &= \max_{\overline{\Omega}} u_j + 100L_j(x_1)_+ \\ &\geq \max_{\overline{\Omega_\rho\setminus\Omega}} u_j + \delta/2 + 100L_j(x_1)_+ \\ &\geq u_j(z) + \delta/2. \end{aligned}$$

This completes the claim.

Now there is a sequence $c_j \to 0$ (which is because $w_j(0) = \max_{\overline{\Omega}} u_j \to \max_{\overline{\Omega}} u$) such that $w_j + c_j$ touches u from above in $\Omega_\rho \times [0, r]$. We claim that the touching must be at some point in Ω. This is because w_j

are harmonic the touching must occur on the boundary of $\Omega_\rho \times (0, r)$, in which case we can reduce to the boundary portion Ω due to the strict ordering on the fattened boundary $\overline{\Omega_\rho \setminus \Omega} \times [0, r] \bigcup \overline{\Omega_\rho} \times [r(1-\rho), r]$.

Indeed we can choose j sufficiently large so that $|c_j| \ll \delta/4$, and then we have by the previous claim the strict inequality

$$w_j + c_j > u_j + c_j + \delta/2 > u + \delta/4 \ \text{ in } \ \overline{\Omega_\rho \setminus \Omega} \times [0, r] \bigcup \overline{\Omega_\rho} \times [r(1-\rho), r],$$

which contradicts the strong subsolution property. □

8.2. Tangential sub/sup-convolution

For the comparison principle proof, we will use a procedure based on inf/sup-convolutions in the tangential variables and harmonic replacement. This is natural for the nonlinear Neumann problem, which could also be viewed as a nonlinear fractional order PDE problem on the lower dimensional B_1'.

Let us now define the tangential inf/sup-convolutions.

Definition 8.9. Suppose $u, -v \in \mathrm{USC}(\overline{B_1^+})$, then the *tangential sup-convolution* of u is defined as

$$\mathcal{T}^\varepsilon u(x) := \sup_{x+h \in \overline{B_1^+}(0);\, h \in \mathbb{R}^{d-1}} \left\{ u(x+h) - \frac{1}{2\varepsilon}|h|^2 \right\},$$

where $x = (x_1, x')$. The tangential inf-convolution of v is defined as $\mathcal{T}_\varepsilon v := -\mathcal{T}^\varepsilon(-v)$.

Remark 8.10. Let $u \in \mathrm{USC}(\overline{B_1^+})$, then we may naturally extend $u(x_1, x') = -\infty$ for $x \notin \overline{B_1^+}(0)$ and then $u \in \mathrm{USC}([0,1] \times \mathbb{R}^{d-1})$. The advantage of this extension is that we may extend the tangential sup-convolution formula to

$$\mathcal{T}^\varepsilon u(x) = \sup_{h \in \mathbb{R}^{d-1}} \left\{ u(x+h) - \frac{1}{2\varepsilon}|h|^2 \right\} \quad \text{for} \quad x \in [0,1] \times \mathbb{R}^{d-1}.$$

Similar extension can be done to v.

Now we will briefly establish several properties of the tangential inf/sup-convolutions that follow from or have very similar proofs to the properties of standard inf/sup convolutions which were collected in Remark 8.6.

Lemma 8.11. *For any $\varepsilon > 0$ small and fixed $x_1 \in [0, 1]$, the tangential sup-convolution $\mathcal{T}^\varepsilon u(x_1, x')$ is semi-convex in $x' \in \mathbb{R}^{d-1}$. For every ε, $\mathcal{T}^\varepsilon u$ is upper semi-continuous in $[0,1] \times \mathbb{R}^{d-1}$.*

Proof. The first statement is an immediate consequence of the same result for the classical sup-convolution recalled in Remark 8.6. To show the second statement, we consider a sequence of points $z_n \to z \in \overline{B_1^+}$. For each z_n there corresponds an $h_n \in B_2'(0)$ such that

$$\mathcal{T}^\varepsilon u(z_n) = u(z_n + h_n) - \frac{1}{2\varepsilon}|h_n|^2.$$

By compactness of h_n we may obtain after passage to a subsequence

$$h_n \to h_\infty,$$

which implies that

$$\begin{aligned}\limsup_{n\to\infty}\mathcal{T}^\varepsilon u(z_n) &= \limsup_{n\to\infty} u(z_n+h_n) - \frac{1}{2\varepsilon}|h_n|^2\\ &\le u(z+h_\infty) - \frac{1}{2\varepsilon}|h_\infty|^2\\ &\le \mathcal{T}^\varepsilon u(z). \quad \square\end{aligned}$$

Lemma 8.12. *Let u be an upper semi-continuous function, u^ε the classical sup-convolution, then*

$$u^\varepsilon \ge \mathcal{T}^\varepsilon u \ge u, \tag{8.5}$$

and in particular,

$$\limsup_{\varepsilon\to 0^+}{}^* \mathcal{T}^\varepsilon u = u.$$

Proof. The inequality (8.5) can be obtained by the definition directly. The half-relaxed limit can be obtained by applying the inequality (8.5) and Remark 8.6 (4). $\square$

Lemma 8.13. *Let $u \in \mathrm{USC}(\overline{B_1^+})$ and $x_0 \in \overline{B_1^+}$. If a smooth function ϕ touches $\mathcal{T}^\varepsilon u$ (strictly) from above at x_0, and*

$$\mathcal{T}^\varepsilon u(x_0) = u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_0 - x_\varepsilon|^2 \quad \textit{for some} \quad x_\varepsilon \in x_0 + \mathbb{R}^{d-1}.$$

Then $\psi(x) = \phi(x + x_0 - x_\varepsilon) + \frac{1}{2\varepsilon}|x_\varepsilon - x_0|^2$ will touch u (strictly) from above at $x_\varepsilon \in \overline{B_1^+}$ that is close to x_0, and $\nabla'\psi(x_\varepsilon) = \nabla'\phi(x_0) = \frac{1}{\varepsilon}(x_\varepsilon - x_0) \in \mathbb{R}^{d-1}$.

The proof is omitted since it is similar to the standard sup-/inf-convolution, and the details of the proof can also be in [8, Proposition 8.6].

Applying this lemma it is standard to check that the inf-convolution and sup-convolutions preserve viscosity super and subsolution properties respectively.

Corollary 8.14. *If u is a supersolution (subsolution) to 2.2, then $\mathcal{T}_\varepsilon u$ ($\mathcal{T}^\varepsilon u$) is still a supersolution (subsolution) 2.2 (or 2.1) in $\overline{B_{1-\gamma}^+}$ for some small $\gamma = \gamma(\varepsilon) > 0$. Moreover, if u is a strong subsolution, then so is $\mathcal{T}^\varepsilon u$ in $\overline{B_{1-\gamma}^+}$. The constant $\gamma \to 0$ as $\varepsilon \to 0^+$.*

8.3. Harmonic lift

In this section, we study the harmonic lift of a given bounded subharmonic function v on $\overline{B_1^+}$. The results are standard but we want to carefully enumerate the properties of the harmonic lift since v will only be upper-semicontinuous.

Using Perron's method, we can define for $z \in \overline{B_1^+}$

$$w(z) := \inf\{u(z)\,;\ u \in C\left(\overline{B_1^+}\right), \text{superharmonic and } u \ge v \text{ in } B_1^+\}. \tag{8.6}$$

Note that $\max_{\overline{B_1^+}} v$ is one such superharmonic function so the infimum is well-defined. By the standard arguments in Perron's method for the Laplacian, we have $w = w^* = w_*$ is continuous and harmonic in the interior B_1^+.

Lemma 8.15. *Let w be as defined in (8.6), then $w^* = v$ on ∂B_1^+.*

Proof. Indeed, since $v = v^*$, we have by the equivalent form of upper envelope

$$v(z) = v^*(z) = \inf\{h(z)\,;\, h \in C\left(\overline{B_1^+}\right),\, h \geq v \text{ on } \overline{B_1^+}\},\; z \in \overline{B_1^+}$$

there will be a sequence of continuous functions $h_{n,z} \in C\left(\overline{B_1^+}\right)$ for $z \in \partial B_1^+$, such that $h_{n,z}(z) \to v^*(z) = v(z)$ as $n \to \infty$. Now we construct $w_{n,z}$ such that

$$\begin{cases} \Delta w_{n,z} = 0, & \text{in } B_1^+ \\ w_{n,z} = h_{n,z}, & \text{on } \partial B_1^+, \end{cases}$$

and the $w_{n,z}$ are continuous up to ∂B_1^+ because the boundary data is continuous and the domain is outer regular. By definition of w, we always have

$$w \leq w_{n,z} \text{ in } \overline{B_1^+}.$$

This shows that we have

$$w^*(z) \leq w_{n,z}(z),\, \forall n,$$

which shows the inequality $w^* \leq v$. The other side can be obtained directly from the definition of w. □

Definition 8.16. Let $v \in \mathrm{USC}\left(\overline{B_1^+}\right)$ be a bounded subharmonic function, then we defined its harmonic lift to be, with w from (8.6), $w^* \in \mathrm{USC}\left(\overline{B_1^+}\right) \cap C^\infty\left(B_1^+\right)$, which is harmonic in B_1^+ and $w^* = v$ on ∂B_1^+. The lower semicontinuous harmonic lift of a bounded superharmonic function in B_1^+ is defined similarly.

Suppose u is a bounded subsolution to in the sense of Definition 2.1 then u is also a subharmonic function. We denote the harmonic lift of u as $\hat{u}$. For a supersolution v, in the sense of Definition 2.2, we may do a similar procedure and obtain $\hat{v}$.

Next, we show that the harmonic lifts of sub/ supersolutions are still sub/supersolutions.

Lemma 8.17. *Suppose u, v are respectively bounded sub-/supersolutions to (1.1). Then $\hat{u}, \hat{v}$ will still be sub-/supersolutions. Moreover, if u satisfies the boundary maximum principle then $\hat{u}$ will as well.*

Proof. The interior PDE follows from the properties of harmonic lift established above. Because of the inequalities $\hat{u} \geq u$ and $\hat{u} = u$ on the boundary ∂B_1^+, if a test function φ touches $\hat{u}$ from above on B_1' then it also touches u from above at the same point. The viscosity subsolution property of $\hat{u}$ follows immediately from this observation. The supersolution property is similar.

As for the boundary maximum principle, this property depends only on the values on B_1', and $\hat{u} = u$ on B_1'. □

8.4. Comparison principle

Let us now prove the comparison principle in dimension $d \geq 2$.

Proof of Theorem 1.6. It suffices to consider bounded u, v since we can replace, for some big $N > 0$, u by $\max\{u, -N\}$ and v by $\min\{v, N\}$. Note that u is upper semi-continuous and v is lower semi-continuous the compact set $\overline{B_1^+}$, so there is some $N > 0$ large so that $u \leq N$ and $v \geq -N$. This combined with $v \geq u$ on the boundary ensures that $\min\{v, N\} \geq \max\{u, -N\}$ on the boundary for N large enough.

We first claim that for all $s > 0$ there is a $\delta = \delta(u, v, s) > 0$ such that

$$v + s \geq u \ \text{ on } \bigcup_{x \in \partial B_1 \cap \{x_1 \geq 0\}} B_\delta(x) \cap \overline{B_1^+} =: D_\delta.$$

Indeed, otherwise there will be a $s_0 > 0$ and a sequence $x_j \to x \in \partial B_1 \cap \{x_1 \geq 0\}$ such that $v(x_j) + s_0 \leq u(x_j)$ for all j, then we have

$$u(x) - v(x) \geq \limsup_{j \to \infty} u(x_j) - v(x_j) \geq s_0 > 0,$$

which contradicts the assumption.

For $\varepsilon > 0$, we now study in the smaller domain $U_\varepsilon = B^+_{1-\gamma(\varepsilon)}$, with both ε and $\gamma = \gamma(\varepsilon) > 0$ sufficiently small. Fix $s_0 > 0$ small and in the following we always assume $\gamma(\varepsilon) < \delta(u, v, s_0) =: \delta_0$. On U_ε we define

$$\hat{u}^\varepsilon := \widehat{\mathcal{T}^\varepsilon u} \ \text{ and } \ \hat{v}_\varepsilon := \widehat{\mathcal{T}_\varepsilon v}.$$

By Lemma 8.17 these are, respectively, an upper semicontinuous strong subsolution of (1.1) satisfying boundary maximum principle, and a lower semicontinuous supersolution of (1.1) in U_ε. We write $\hat{u}$ and $\hat{v}$, dropping the ε-dependence when it is not important. Notice that because $\mathcal{T}^\varepsilon u$ and $\mathcal{T}_\varepsilon v$ are continuous when restricted to B_1', $\hat{u}$ and $\hat{v}$ are also continuous up to B_1' by standard boundary barrier arguments for harmonic functions.

Now we make a perturbation to a strict supersolution. Let $\eta \gg \varepsilon > 0$ be a fixed small number, we further consider the following modified functions

$$\overline{u} = \hat{u}^\varepsilon = \hat{u} \ \text{ and } \ \overline{v} = \hat{v}_\varepsilon - \eta(x_1)_+ + 10\eta.$$

It then suffices to show that

$$\max_{\overline{U_\varepsilon}} (\overline{u} - \overline{v}) = \max_{\partial U_\varepsilon \setminus U_\varepsilon'} (\overline{u} - \overline{v}). \tag{8.7}$$

Indeed, if we have established this equality, then on one hand we have

$$\begin{aligned} \max_{\overline{U_\varepsilon}} (\overline{u} - \overline{v}) &\geq \max_{\overline{U_\varepsilon}} (\mathcal{T}^\varepsilon u - \mathcal{T}_\varepsilon v) - 100\eta \\ &\geq \max_{\overline{U_\varepsilon}} (u - v) - 100\eta, \end{aligned}$$

and on the other hand, we have by Lemma 8.15, 8.11 and Remark 8.6 (5)

$$\begin{aligned} \max_{\partial U_\varepsilon \setminus U_\varepsilon'} (\overline{u} - \overline{v}) &= \max_{\partial U_\varepsilon \setminus U_\varepsilon'} (\mathcal{T}^\varepsilon u - \mathcal{T}_\varepsilon v + \eta(x_1)_+ - 10\eta) \\ &\leq \max_{D_{\delta_0}} \mathcal{T}^\varepsilon (u - v) + 100\eta \\ &= \max_{D_{\delta_0}} (u - v) + o_\varepsilon(1) + 100\eta \\ &\leq s_0 + o_\varepsilon(1) + 100\eta. \end{aligned}$$

Sending $\varepsilon, s_0, \eta \to 0^+$ completes the proof.

Let $\hat{x}$ be a point in $\overline{U}_\varepsilon$ where the maximum in (8.7) is achieved. We need to show that $\hat{x}$ cannot be in $U_\varepsilon \sqcup U_\varepsilon'$. By strong maximum principle for harmonic functions $\hat{x} \notin U_\varepsilon$.

We now show that $\hat{x} \notin U'_\varepsilon$. According to Remark 2.10 we partition the flat boundary portion B'_1 into

$$B'_{1-\gamma(\varepsilon)} = \mathcal{N}_{\overline{v}} \sqcup \mathcal{C}_{\overline{v}} \sqcup \Gamma_{\overline{v}},$$

with $\mathcal{N}_{\overline{v}} = \mathcal{N}_{\hat{v}}, \mathcal{C}_{\overline{v}} = \mathcal{C}_{\hat{v}}$ and $\Gamma_{\overline{v}} = \Gamma_{\hat{v}}$. For convenience, we will write them as $\mathcal{N}, \mathcal{C}$ and Γ respectively.

We may without loss assume that $\hat{x} \in \mathcal{N} \sqcup \Gamma$, because if $\hat{x} \in \mathcal{C}$ then $\hat{u} - \hat{v} = \hat{u} - K$ in a component of $\mathcal{C}$ for some constant K, and by boundary maximum principle of $\hat{u}$ there will be another point $\hat{x}^* \in \Gamma = \partial'\mathcal{C}$ such that $\overline{u}(\hat{x}^*) = \overline{u}(\hat{x})$. We can then replace $\hat{x}$ by this new $\hat{x}^*$.

Next, we observe that, by the definition of tangential sub/sup-convolutions, at the maximum point $\hat{x}$ there exist two quadratic functions P_1, P_2 on $\mathbb{R}^{d-1}$ and a constant c such that $P_1 \geq \overline{v}\big|_{B'_1} + c \geq \overline{u}\big|_{B'_1} = \hat{u}\big|_{B'_1} \geq P_2$ and $P_1(\hat{x}) = P_2(\hat{x}) =: l$. In particular, for some $q \in \mathbb{R}^{d-1}$ we can write

$$P_1(x') = l + q \cdot (x' - \hat{x}) + \frac{1}{2\varepsilon}|x' - \hat{x}|^2, \text{ and } P_2(x') = l + q \cdot (x' - \hat{x}) - \frac{1}{2\varepsilon}|x' - \hat{x}|^2. \tag{8.8}$$

For convenience, we assume $c = 0$, since it will not affect the proof. We claim that for any $h > 0$ small there exists a smooth function w that touches $\overline{v}$ from below at $\hat{x}$ and

$$\frac{\partial w}{\partial x_1}(\hat{x}) \geq -h.$$

This claim will immediately imply that $\hat{x} \notin \mathcal{N} \sqcup \Gamma$ and lead to a contradiction because otherwise $w^\eta = w + \eta(x_1)_+ - 10\eta$ will touch $\hat{v}$ from below at $\hat{x}$ and $\partial_1 w^\eta(\hat{x}) \geq \eta - h > 0$, violating the supersolution condition.

To prove the existence of such w, we let $\varepsilon \gg \tau > 0$ be a small number and consider in the half ball $B^+_\tau(\hat{x})$ the following functions $w_{\tau,i}$ with $i = 1, 2$:

$$\begin{cases} \Delta w_{\tau,i}(x) = 0, & x \in B^+_\tau(\hat{x}), \\ w_{\tau,i}(x) = \hat{u}(x), & x \in \partial B^+_\tau(\hat{x}) \cap \{x_1 \geq 0\}, \\ w_{\tau,i}(0, x') = P_i(x'), & x = (0, x') \in B^+_\tau(\hat{x}) \cap \{x_1 = 0\}. \end{cases} \tag{8.9}$$

The boundary data is potentially discontinuous on $\partial B^+_\tau(\hat{x}) \cap \{x_1 \geq 0\}$, so we are solving (8.9) by Perron's method as in Section 8.3. However, the boundary data on B'_τ is polynomial so $w_{\tau,i}$ are smooth in a neighborhood of $\hat{x}$. Now $w_{\tau,1}$ touches $\hat{u}$ from above and $w_{\tau,2}$ touches $\hat{u}$ from below at $\hat{x}$, because of the ordering $(w_{\tau,1})_* \geq \hat{u} \geq w^*_{\tau,2}$ on the boundary ∂B^+_τ. According to the subsolution condition of $\hat{u}$,

$$\frac{\partial w_{\tau,1}}{\partial x_1}(\hat{x}) \geq 0.$$

Notice that, on the other hand, $w_{\tau,2}$ touches $\overline{v}$ from below at $\hat{x}$. We then just need to show that $\partial_1 w_{\tau,2}(\hat{x})$ is sufficiently close to $\partial_1 w_{\tau,1}(\hat{x})$ when τ is chosen small. To that end, we need to control $\tilde{w}_\tau = w_{\tau,1} - w_{\tau,2}$, which will satisfy the following equation,

$$\begin{cases} \Delta \tilde{w}_\tau(x) = 0, & x \in B^+_\tau(\hat{x}), \\ \tilde{w}_\tau(x) = 0, & x \in \partial B^+_\tau(\hat{x}) \cap \{x_1 \geq 0\}, \\ \tilde{w}_\tau(0, x') = P_1(x') - P_2(x') =: P(x'), & x = (0, x') \in B^+_\tau(\hat{x}) \cap \{x_1 = 0\}, \end{cases} \tag{8.10}$$

where by (8.8) $P(x') = \frac{1}{\varepsilon}|x' - \hat{x}|^2$. By the transformation $w_\tau(z) := \frac{\varepsilon}{\tau^2}\tilde{w}_\tau(\tau z + \hat{x})$, we observe that w_τ satisfies

$$\begin{cases} \Delta w_\tau(z) = 0, & z \in B_1^+(0), \\ w_\tau(z) = 0, & z \in \partial B_1^+(0) \cap \{z_1 \geq 0\}, \\ w_\tau(0, z') = |z'|^2, & z = (0, z') \in B_1^+(0) \cap \{z_1 = 0\}, \end{cases} \tag{8.11}$$

which has a universal bound C on its Lipschitz constant in $B_{1/2}$ and so

$$|\partial_1 \tilde{w}_\tau(0)| = \frac{\tau}{\varepsilon} |\partial_1 w_\tau(0)| \leq C\varepsilon^{-1}\tau,$$

which can be made arbitrarily small by choosing τ small enough depending on ε. Note that $\varepsilon > 0$ is a fixed positive number in this argument. □

8.5. Minimal supersolutions are exactly the solutions satisfying boundary maximum principle

In this section, we show that the minimal supersolution to Equation (1.1) will satisfy the strong subsolution condition, and hence the boundary maximum principle.

To construct a minimal supersolution let us first write (1.1) in the following form

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ \min\{\partial_1 u, |\nabla' u|\} = 0 & \text{on } B_1' \\ u = g & \text{on } \partial B_1 \cap \{x_1 \geq 0\}, \end{cases} \tag{8.12}$$

with g an arbitrary continuous function.

Definition 8.18. An upper semicontinuous function u is called a viscosity subsolution to (8.12) if it is a subsolution to (1.1) and $u \leq g$ on $\partial B_1 \cap \{x_1 \geq 0\}$. Similarly, a lower semicontinuous function v is called a viscosity supersolution to (8.12) if it is a supersolution to (1.1) and $v \geq g$ on $\partial B_1 \cap \{x_1 \geq 0\}$.

One can easily check that

$$w_{\text{sub}}(x) := -\|g\|_\infty,$$

is a subsolution to (8.12) that satisfies boundary maximum principle on B_1'. Similarly, we know that

$$w_{\text{sup}} = \|g\|_\infty$$

is a supersolution to (8.12).

Define the Perron's method minimal supersolution

$$v_{\min}(x) := \inf\{v(x)\,;\, v \text{ is a supersolution to (8.12) }\}. \tag{8.13}$$

Proposition 8.19. *The function $v_{\min}$ is the unique viscosity solution to (8.12) that satisfies the strong subsolution condition, Definition 8.1, and equivalently the boundary maximum principle.*

Remark 8.20. By applying comparison principle, Theorem 1.6, it can be observed immediately that all viscosity solutions to (8.12) are bounded from above by the solution $v_{\rm N}$ to the mixed boundary problem

$$\begin{cases} \Delta v_{\rm N} = 0, & \text{in } B_1^+ \\ v_{\rm N} = g, & \text{on } \partial B_1 \cap \{x_1 \geq 0\} \\ \partial_1 v_{\rm N} = 0, & \text{on } B_1', \end{cases} \tag{8.14}$$

and from below by the minimal supersolution $v_{\min}$ to (8.12). In Example 2.4, the solution $w(x,y) = -\mathrm{Re}\left((x+iy)^{3/2}\right)$ to the Signorini problem coincides with the minimal supersolution to (1.1) because w satisfies the boundary maximum principle. In Example 2.5, the solution $w^-(x,y) = \mathrm{Re}\left((x+iy)^{5/2}\right)$ to the sign-reversed Signorini problem is also the minimal supersolution because it also satisfies boundary maximum principle.

Proof. By the discussion above Proposition 8.19, we have established that $w_{\text{sub}} \leq v := v_{\min} \leq w_{\text{sup}}$ is a well-defined bounded function. By applying a slight modification of the classical Perron's method, we know that the lower envelope v_* is a supersolution, and $(v_*)^*$ is a subsolution. Moreover, $(v_*)^* = v_*$ in B_1^+ is harmonic, and we also have $(v_*)^* = v_* = g$ on $\partial B_1 \cap \{x_1 \geq 0\}$.

It then suffices to show that $(v_*)^*$ also satisfies the boundary maximum principle. Indeed, with the boundary maximum principle we can apply the comparison principle 1.6, which implies that $(v_*)^* \leq v_*$, and then $v = (v_*)^* = v_*$ on the whole $\overline{B_1^+}$, which implies the continuity of v up to boundary and that v is a viscosity solution to (8.12).

We would like to show using the strong subsolution condition, which is equivalent to the boundary maximum principle according to Remark 8.8. Suppose for a bounded supersolution $\tilde{v}$ to (8.12), the upper envelope $\tilde{v}^*$ does not satisfy the strong subsolution condition, then there is a smooth up to boundary function $\psi(x_1, x') \equiv \phi(x_1)$ (we may without loss assume that $\phi''(x_1) \leq 0$) that touches $\tilde{v}^*$ from above at $z \in B_1'$, and there is a domain Ω containing z with $\Omega_h = \bigcup_{x\in\Omega} B_h(x) \cap \overline{B_1^+} \subset\subset B_1^+ \sqcup B_1'$ for some small $h > 0$, so that

$$\psi \geq \tilde{v}^* + s, \quad \text{on } \overline{\Omega_h \setminus \Omega},$$

for some small constant $s > 0$. To reach a contradiction, we consider the function

$$v_c = \begin{cases} \min\{\tilde{v}, \psi - s/2\}, & \text{in } \overline{\Omega_h} \\ v, & \text{elsewhere.} \end{cases}$$

The proof will complete if we observe that v_c is a supersolution to (8.12) but there is a point $z_0 \in \Omega$ such that $v_c(z_0) < \tilde{v}(z_0)$. Indeed, $v_c = v$ outside Ω and $\psi - s/2$ is a supersolution, and hence v_c is a supersolution to (8.12). On the other hand, by definition, there is a sequence $z_j \in B_1^+ \sqcup B_1'$ such that $z_j \to z$ and $\tilde{v}(z_j) \to \tilde{v}^*(z)$ as $j \to \infty$. One can check that $v_c(z_j) < \tilde{v}(z_j)$ for a sufficiently large j. □

9. Flat asymptotic expansion of a singular Bernoulli problem

In this section, we study the following discontinuous anisotropic model

$$\begin{cases} \Delta u = 0, & \text{in } \{u > 0\} \setminus \overline{W}, \\ u \equiv 1, & \text{on } \overline{W}, \\ |\nabla u|^2 = Q^2(\nabla u), & \text{on } \partial\{u > 0\}, \end{cases} \tag{9.1}$$

with the anisotropy Q being a 0-homogeneous function of the form

$$Q(e) = \begin{cases} 1, & e \neq e_1 \\ 2, & e = e_1, \end{cases} \tag{9.2}$$

where $e_1, e_2, \ldots, e_d$ form an orthonormal basis for $\mathbb{R}^d$. More rigorously, we will assume u to be a minimal supersolution to (9.1). For the definitions and discussion of such solutions, see [14,15].

We already know that when W is a convex domain, then the free domain $\{u > 0\}$ is also convex and the regularity problem of the free boundary can be easily reduced to the case discussed in [9], and the optimal regularity is exactly $C^{1,1/2}$.

In the case that W is non-convex, we would like to divide the free boundary $\partial\{u > 0\}$ into two disjoint parts:

$$\partial\{u > 0\} =: \{|\nabla u| > 1\} \sqcup R =: \Lambda \sqcup R, \tag{9.3}$$

where "$|\nabla u| > 1$" is defined in the sense of viscosity. We would like to show the following facts:

1. Λ is relatively open in $\partial\{u > 0\}$;
2. Λ is composed of open subsets of orthogonal hyperplanes of e_1.

Based on the second statement we would like to show that the free boundary will be $C^{1,\alpha}$ in a neighborhood of $\Lambda \cup \partial'\Lambda$. Following the idea of [9,11], we would like to study the asymptotic expansion of the solution u near $\Lambda \cup \partial'\Lambda$.

Let us begin with a more precise definition of Λ.

Definition 9.1. Λ is composed of all the points $x \in \partial\{u > 0\}$ such that there is a smooth function ϕ touching u from below at x satisfying $\nabla\phi(x)$ parallel to e_1 and $|\nabla\phi(x)| > 1$.

According to this definition, we know that any point $x \in \Lambda$ is an inner regular point of $\partial\{u > 0\}$. For the convenience of the arguments, we may without loss assume that $x = 0$ and the solution u satisfies a half-flatness condition in a unit ball $B_1 = B_1(0)$ for small $\varepsilon > 0$

$$u(x) \geq \nabla\phi(0) \cdot x - \varepsilon =: \alpha x_1 - \varepsilon, \tag{9.4}$$

for some $\alpha > 1$ and we can also assume $B_2(2e_1) \cap B_1 \subset \{u > 0\} \cap B_1$.

Let us now prove that $\partial\{u > 0\}$ is flat near $x = 0$.

Lemma 9.2. *With the above assumptions, we show that there is a small number $\delta > 0$ such that $B_\delta \cap \{u > 0\} = B_\delta \cap \{x_1 > 0\}$.*

Proof. Observe first that by (9.4) we know that for $\overline{x} = \frac{1}{5}e_1$

$$u\left(\overline{x} + t\right) - \frac{1}{5} \geq \frac{1}{5}(\alpha - 1) - \varepsilon \gg \varepsilon,$$

for $t = (0, t_2, \cdots, t_d)^T \in \{0\} \times \mathbb{R}^{d-1}$, $|t| < 1/20$. By Harnack inequality, we may without loss assume that

$$u(x) \gg \varepsilon, \forall x \in B_{1/10}\left(\overline{x}\right).$$

Let w be a positive function such that it is strictly subharmonic in the annulus $A = B_{3/4}(\overline{x}) \setminus \overline{B_{1/20}(\overline{x})}$ and is 1 on the inner boundary and 0 on the outer boundary. We extend w to be constantly 1 in the inner disc. It is enough to take

$$w = c(|x - \overline{x}|^{-\gamma} - (3/4)^{-\gamma}), x \in A$$

with c chosen so that the conditions above are satisfied.

Now we follow De Silva's argument and compare

$$u(x) \geq x_1 - \varepsilon + C_0\varepsilon(w(x) - \tau) =: v_\tau(x),\ x \in B_{1/2}?$$

This is true indeed for $\tau = 1$, and we want to show that this is also true for $\tau = 0$. Let $1 \geq \tau^* \geq 0$ be the smallest number such that the above inequality holds then by strict subharmonicity and strict inequality inside $B_{1/20}(\overline{x})$ the touching point can only be at the boundary. But because of strict inequality $|\nabla v_{\tau^*}| > 1$ at the touching point, the touching point can only be the origin, which is naturally excluded by choosing $C_0 < 1/2$. This shows that $\tau^* \leq 0$ and hence the inequality also holds for $\tau = 0$.

The same argument shows that $v_0(x + h(0)e_1)$ touches u from below at the origin for some unique $h(0) \approx \varepsilon$. Let $\|h\|_{L^\infty(-\delta,\delta)} \leq C\varepsilon$ be the function such that for each $|t| \ll 1/20$ we have that $v_0(x-t+h(t)e_1)$ touches u from below exactly at the boundary. Similar as before, all the touching points will be of the form $(h(t) + c)e_1 - t$ for some constant c and each fixed t.

Now the function $h(t)$ is semi-convex and Lipschitz. Moreover because of the existence of upper touching functions having gradient zero at each $|t| < \delta$ we conclude that $\nabla h(t) = 0$ in the viscosity sense and hence $h(t) \equiv h(0)$ is a constant. $\square$

9.1. A Harnack inequality assuming both flatness and the free boundary being a function graph

We are interested in the regularity of the free boundary near the relative boundary $\partial'\Lambda$ of Λ in $\partial\{u > 0\}$. Let $0 \in \partial'\Lambda$ and we consider a solution u to (9.1) that is restricted to the unit ball $B_1(0)$. Unlike in the case of Chang-Lara and Savin [9], we don't necessarily have the inner or outer regularity of the free boundary at 0, and so we don't immediately obtain the differentiability of u near the origin. However, because of the definition of Λ and Lemma 9.2, we do know that if u admit differentiability at 0 then $|\nabla u|(0) = 1$ and hence after a rescaling and a small translation, the following *flatness* condition will be satisfied:

$$(x \cdot p)_+ \leq u(x) \leq (x \cdot p + \varepsilon)_+,\ x \in B_1(0),\ p \in \partial B_1(0). \tag{9.5}$$

For the convenience of analysis, we also assume that the free boundary is the *function graph* of a continuous function f on the set $\{x \cdot p = 0\}$.

The main difficulty of our problem is that we don't a priori know the position of the facets Λ, but we do know the following dichotomy: either $\partial\{u > 0\} \cap B_{1/(400d)} = \Lambda_u \cap B_{1/(400d)}$ (i.e. the whole free boundary in $B_{1/(400d)}$ is completely flat), or the following Harnack inequality is satisfied.

Lemma 9.3. *Let $p(x) := x \cdot p$ and $l = \frac{1}{100d}$. There exist constants $\overline{\varepsilon} = \overline{\varepsilon}(d) > 0$ and $l/2 > \mu = \mu(d) > 0$ such that if u satisfies the anisotropic boundary condition $|\nabla u| = Q(\nabla u)$ with Q defined in (9.2) on the free boundary $\partial\{u > 0\} \cap B_1$ that is a function graph, u is harmonic in its positive set satisfying the ε-flatness condition (9.5) with $0 < \varepsilon \leq \overline{\varepsilon}$, then if at $\overline{x} = \frac{1}{5}p$*

$$u(\overline{x}) \geq p(\overline{x}) + \frac{\varepsilon}{2}, \tag{9.6}$$

either

i. the free boundary satisfies $\partial\{u > 0\} \cap B_{l/4} = \Lambda_u \cap B_{l/4}$, and so is completely flat in $B_{l/4}$;
ii. or there is a point $x^* \in \partial\{u > 0\} \cap B_{l/4}$ not contained in $\Lambda_u \cap B_{l/4}$ and

$$u \geq (p(x) + c\varepsilon)_+, \quad \text{in } \overline{B}_\mu, \tag{9.7}$$

for some $0 < c = c(d) < 1$.

If $u(\overline{x}) \leq p(\overline{x}) + \frac{\varepsilon}{2}$, *then similarly*

$$u \leq (p(x) + (1-c)\varepsilon)_+, \ \ in\ \overline{B}_\mu. \tag{9.8}$$

Remark 9.4. This Harnack inequality doesn't assume $0 \in \partial\{u > 0\} \cap B_1$. It also shows that there is no degeneracy in the flatness parameter $\bar{\varepsilon}$ as $p \approx e_1$.

Proof. We focus on the first case that $u(\overline{x}) \geq p(\overline{x}) + \frac{\varepsilon}{2} = 1/5 + \frac{\varepsilon}{2}$, because the second case exactly falls in the case of De Silva [11]. In this case, by classical Harnack inequality, we obtain for some universal constant $m > 0$

$$u(x) - p(x) \geq m\varepsilon, \forall x \in B_{1/10}\left(\overline{x}\right). \tag{9.9}$$

Let w be a positive function such that it is strictly subharmonic in the annulus $A = B_{l+1/5}(\overline{x}) \setminus \overline{B}_{1/20}(\overline{x})$ for some $l > 0$ to be determined, and is 1 on the inner boundary and 0 on the outer boundary. We extend w to be constantly 1 in the inner disc. It is enough to take

$$w = \tilde{c}(|x-\overline{x}|^{-\gamma} - (l+1/5)^{-\gamma}), x \in A$$

with $\tilde{c}$ chosen so that the conditions above are satisfied. Indeed, w is strictly subharmonic in A when $\gamma > d-2$. We will choose $\gamma = d-1$ in the following proof.

Now we follow De Silva's argument and compare for $z \in B_{l/2}(0)$ and $\tau \geq 0$

$$u(x) \geq p(x) + m\varepsilon(w(x-z) - \tau) =: v_{\tau,z}(x),\ x \in B_{l/2}? \tag{DS}$$

Notice that we always have the above inequality for $\tau \geq 1$ because of the ε-flatness assumption (9.5). Even for $\tau = 0$, by (9.9) we still have

$$u(x) \geq v_{0,z}(x),\ z \in B_{l/2}(0), \ \text{ and } x \notin \overline{A+z}. \tag{9.10}$$

On the other hand, we claim that the portion of the barrier free boundary $\partial\{v_{\tau,z} > 0\} \setminus \{p(x) = 0\}$ is contained in $B_{r(l)}(z')$ with $r(l) = \sqrt{3l/5 + 3l^2/4}$ and $z' = z - p(z)p$. Indeed, the barrier free boundary portion $\partial\{v_{\tau,z} > 0\} \setminus \{p(x) = 0\}$, no matter what $z \in B_{l/2}(0)$ and $\tau \geq 0$, is contained in $A + z \cap \{p(x) < 0\} \subset B_{r(l)}(z')$.

Let us now choose a proper $l = l(d) > 0$ so that for all $0 < \varepsilon \leq \overline{\varepsilon}$, z, τ, the barrier free boundary portion $\partial\{v_{\tau,z} > 0\} \setminus \{p(x) = 0\}$ is the graph of a convex function on $\{p(x) = 0\}$. According to implicit function theorem, for $\varepsilon > 0$ small, the boundary $\partial\{v_{\tau,z} > 0\}$ can be denoted by a function $\xi_1 = g(\xi')$, with (ξ_1, ξ') a new Euclidean coordinate system of $\mathbb{R}^d$ such that $\partial_{\xi_1} = p \cdot \nabla$. It then suffices to show that, for a constant $l = l(d)$, g is always a convex function. Indeed, we can observe that if we write $v_{\tau,z}(\xi_1, \xi') = \xi_1 - \varepsilon\phi(\xi)$, with $\phi(\xi) = -m(w(\xi - z) - \tau)$, we obtain

$$(1 - \varepsilon\partial_{\xi_1}\phi)\nabla_{\xi'} g = \varepsilon \nabla_{\xi'}\phi, \tag{9.11}$$

and hence

$$\left(\nabla_{\xi'}\right)^2 g = \varepsilon \left(\nabla_{\xi'}\right)^2 \phi + O(\varepsilon^2). \tag{9.12}$$

But we know that

$$\frac{1}{m\tilde{c}}\left(\nabla_{\xi'}\right)^2 \phi = \frac{\gamma}{|\xi - z - \overline{x}|^{\gamma+2}} \mathrm{Id}_{(d-1)\times(d-1)} - \gamma(\gamma+2)\frac{(\xi' - z')\otimes(\xi' - z')}{|\xi - z - \overline{x}|^{\gamma+4}}, \tag{9.13}$$

with the smallest eigenvalue

$$\gamma - \gamma(\gamma+2)\frac{|\xi' - z'|^2}{|\xi' - z'|^2 + |\xi_1 - z_1 - 1/5|^2} > 0,$$

will require

$$|\xi' - z'|^2 < \frac{1}{\gamma+1}|\xi_1 - z_1 - 1/5|^2 = \frac{1}{d}|\xi_1 - z_1 - 1/5|^2.$$

Since $\xi \in \partial\{v_{\tau,z} > 0\} \subset \{-\varepsilon \le p(x) \le 0\}$, we know that $|\xi_1 - z_1 - 1/5| > 1/5 - l/2 > 0$ for $0 < l < 2/5$. On the other hand, by the prior discussions on $r(l)$, we know that on the nontrivial portion $\partial\{v_{\tau,z} > 0\} \setminus \{p(x) = 0\}$

$$|\xi'| = |x' - z'| \le r(l).$$

This means it suffices to set

$$r(l) = \sqrt{3l/5 + 3l^2/4} < \frac{1}{\sqrt{d}}(1/5 - l/2), \tag{9.14}$$

or simply choose $l = \frac{1}{100d}$.

By the previous paragraph, we deduce that there is at most one point $y^* = y^*(\tau, z)$ on the nontrivial portion of the barrier boundary $\partial\{v_{\tau,z} > 0\} \setminus \{p(x) = 0\}$ such that $\nabla v_{\tau,z}$ is parallel to e_1 at y^*. Let us now discuss the position of y^* for different z and τ. Observe that if one writes $1 = p \cdot p$, then we have

$$\begin{aligned} v_{\tau,z}(x) &= p(x) + m\varepsilon(w(x-z) - \tau) \\ &= p(x - m\varepsilon\tau p) + m\varepsilon w(x-z) \\ &= v_{0,z-m\varepsilon\tau p}(x - m\varepsilon\tau p) \\ &=: v_{0,z-\beta p}(x - \beta p), \end{aligned}$$

where we define $\beta = m\varepsilon\tau$. By this formula we have

$$y^*(\tau, z) = y^*(0, z - \beta p) + \beta p.$$

Furthermore, we have for $z' = z - p(z)p$

$$y^*(\tau, z) = y^*(0, z - \beta p) = y^*(0, (p(z) - \beta)p) + z' + \beta p.$$

Thus, we obtain that the positions of $y^*(\tau, z)$ are just translations of $y^*(0, \zeta p) =: y^*(\zeta)$. Using this one parameter family of y^*, we now consider the following subfamily of $v_{\tau,z}$: for each $\tau \ge 0$ such that if $y^*(\tau, 0) = y^*(-m\varepsilon\tau) + m\varepsilon\tau p$ exists in the nontrivial portion $\partial\{v_{\tau,0} > 0\} \setminus \{p(x) = 0\}$ and the tangential magnitude $|(y^*)'(\tau, 0)| = |(y^*)'(-m\varepsilon\tau)| \le l/4$, we choose $z = z'(\tau) \in B_{l/2}(0) \cap \{p(x) = 0\}$ so that

$$z(\tau) + (y^*)'(-m\varepsilon\tau) = (x^*)',$$

which means that we can make sure that $(y^*)'(\tau, z(\tau)) = (x^*)'$, and therefore the touching point can never be y^* by the assumptions on x^*; for the rest cases (either $\tau \ge 0$ doesn't correspond a y^* or $|(y^*)'| \ge l/4$) we choose $z(\tau) = 0$.

We are now able to complete the proof by simply considering the De Silva argument (DS) with $v_{\tau,z}$ replaced by $\tilde{v}_\tau := v_{\tau,z(\tau)}$ and hence showing the inequality (9.7) for some constant $l/2 > \mu > 0$ (even though we wrote $l/2$ in the De Silva argument (DS), we eventually obtain the improvement in a smaller ball of radius μ). Let $\tau^* \geq 0$ be the smallest number such that (DS) is satisfied for $\tilde{v}_{\tau^*}$. Because of our choice of $z = z(\tau)$, it suffices to consider the case that $y^*(\tau^*, z(\tau^*))$ exists but $|(y^*)'(\tau^*, z(\tau^*))| \geq l/4$ (in other cases, because the touching can never happen on y^* due to our choice of $z(\tau)$, this is a similar touching argument of De Silva, which shows that $\tau^* = 0$). In this case, $z(\tau^*) = 0$, and the boundary portion $\partial\{\tilde{v}_{\tau^*} > 0\}\backslash\{p(x) = 0\}$ is the function graph of a convex function $\xi_1 = g(\xi')$ on $B_{r(l)}(0) \cap \{p(x) = 0\}$, and $\nabla_{\xi'} g(0) = 0$. According to (9.11), (9.13) and (9.14), we know that the function g is in fact $\varepsilon m h(d)$-strictly-convex for some $h(d) > 0$, and hence

$$g(0) + \varepsilon m h(d) |(y^*)'(\tau^*, 0)|^2 \leq g\left((y^*)'(\tau^*, 0)\right) \leq 0,$$

which shows that

$$g(0) \leq -\frac{m h(d) l^2}{16} \varepsilon.$$

On the other hand, we have

$$g(0) + m\varepsilon \left(w(g(0), 0) - \tau^*\right) = 0,$$

which shows that

$$\tau^* \leq w(g(0), 0) - \frac{h(d) l^2}{16}.$$

This leads to the following inequality

$$u(x) \geq p(x) + m\varepsilon \left[w(x) - w(g(0), 0) + \frac{h(d) l^2}{16} \right].$$

Because the gradient of w is bounded near the origin and the point $(g(0), 0)$ is ε-close to the origin, we know that there is a constant $\mu(d) > 0$ such that for all $x \in B_{\mu(d)}(0)$, $|w(x) - w(g(0), 0)| \leq \frac{h(d) l^2}{32}$ and

$$u(x) \geq p(x) + m\varepsilon \left[\frac{h(d) l^2}{32} \right], \text{ for all } x \in B_{\mu(d)}(0),$$

where we can simply choose $\mu(d) = \frac{h(d) l^2}{64 d 1000^d}$. □

Corollary 9.5 *(Harnack Inequality). There is a universal constant $\overline{\varepsilon}$, such that if u is a viscosity solution to (9.1) and it satisfies at some point $x_0 \in \partial\{u > 0\}$ and for some $p \in \partial B_1(0)$*

$$(x \cdot p + a_0)_+ \leq u(x) \leq (x \cdot p + b_0)_+, \text{ in } B_r(x_0), \tag{9.15}$$

with

$$b_0 - a_0 \leq \varepsilon r, \; 0 < \varepsilon \leq \overline{\varepsilon}$$

then

$$(x.p + a_1)_+ \leq u(x) \leq (x \cdot p + b_1)_+, \text{ in } B_{r\mu}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0,\, b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ *universal.*

Proof. The proof is essentially the same as De Silva. □

The above corollary, by a similar argument in [11, Corollary 3.2], shows the following result.

Corollary 9.6. *The functions*

$$w_\varepsilon = \frac{u(x) - x \cdot p}{\varepsilon}$$

have a uniform Hölder modulus of continuity at 0 *in* B_1*, outside a ball of radius* $\varepsilon/\bar{\varepsilon}$*, i.e. for all* $x \in B_1$ *with* $|x| \geq \varepsilon/\bar{\varepsilon}$

$$|w_\varepsilon(x) - w_\varepsilon(0)| \leq C|x|^\gamma.$$

9.2. A compactness result

In this section, we show Proposition 1.5 by proving several lemmas.

Remark 9.7. We will also show that for $p \neq e_1$, the function w corresponds to a solution to the zero Neumann boundary condition.

To prove this proposition we would like to study the following blow-up families of functions

$$w_\varepsilon(x) := \frac{u(x) - x \cdot p}{\varepsilon}. \tag{9.16}$$

We argue by contradiction and assume that there is a subsequence (u_k, ε_k), where u_k is a solution to (9.1) in $B_1(0)$ with ε_k-flatness and $\varepsilon_k \to 0^+$, and we define

$$w_k := \frac{u_k - x \cdot p}{\varepsilon_k}. \tag{9.17}$$

By Corollary 9.6 and Arzela-Ascoli theorem, we know that w_k has compactness in $C(\overline{B_{1/2}})$, which allows us to assume that for some $w \in C^\alpha(\overline{B_{1/2}} \cap \{x \cdot p \geq 0\})$, the following uniform convergence

$$w_k \to w, \text{ uniformly on } B_{1/2}. \tag{9.18}$$

In particular, the free boundary $(\partial\{u_k > 0\}) \cap B_{1/2}$ converges in the Hausdorff sense to $\{x \cdot p = 0\} \cap B_{1/2}$.

Now we finish this section by characterizing the equation for w in the following lemmas. For notational convenience we define the following subsets of $\{x \cdot p \geq 0\} \cap B_{1/2}$:

$$B^+ := \{x \cdot p > 0\} \cap B_{1/2} \text{ and } B' := \{x \cdot p = 0\} \cap B_{1/2}.$$

Lemma 9.8. *For general* $p \in \partial B_1(0)$*, the limit function* w *is a viscosity subsolution to the following Neumann problem*

$$\begin{cases} \Delta w = 0 & \text{in } B^+, \\ \partial_p w \geq 0 & \text{on } B', \end{cases} \tag{9.19}$$

where $\partial_p = p \cdot \nabla$. Moreover, the function w is harmonic in B^+.

Proof. It suffices to check the condition on the flat boundary because harmonicity of w_k's are preserved under uniform convergence in the interior. Suppose there is a smooth function φ touching w from above at some $x_0 \in B'$, then by standard theory there are (c_k, x_k) such that $\varphi_k = \varphi + c_k$ touches w_k from above at $x_k \in B_{1/2} \cap (\{u_k > 0\} \cup \partial\{u_k > 0\})$, and $(c_k, x_k) \to (0^+, x_0)$ as $k \to \infty$. Denoting $\phi_k = x \cdot p + \varepsilon_k \varphi_k$, it is equivalent to say that ϕ_k touches u_k from above at x_k for each k. By performing the transformation $\varphi \mapsto \varphi + \eta(x \cdot p - x_0 \cdot p) - C(\eta)(x \cdot p - x_0 \cdot p)^2$ for suitably chosen $\eta, C(\eta) > 0$, we may assume without loss that $x_k \in (\partial\{u_k > 0\}) \cap B_{1/2}$. Now because $|\nabla u_k| \geq 1$ on the free boundary for each k, we have

$$1 + 2\varepsilon_k \partial_p \varphi(x_k) + o(\varepsilon_k) = |\nabla \phi_k|^2 \geq 1,$$

which implies that

$$\partial_p \varphi(x_0) \geq 0. \quad \square$$

Lemma 9.9. *In the case $p = e_1$, w satisfies the strong subsolution condition, Definition 8.1. That is, there are no C^1 up-to-boundary function of the form $\varphi(x_1, x') \equiv \psi(x_1)$ that touches w from above in $\Omega_h \cap \overline{B_1^+}$ at some $x_0 \in B'$ and $\varphi > w$ in $\overline{\Omega_h \setminus \Omega} \cap \overline{B_1^+}$ where Ω is an arbitrary open domain of $\mathbb{R}^d$ containing x_0 and $\Omega_h = \cup_{y \in \Omega} B_h(y)$ for some small $h > 0$ so that $\overline{\Omega_h} \cap \overline{B_1^+} \subset\subset B_1^+ \cup B_1'$.*

Proof. Similar as before, we have $\phi_k = x_1 + \varepsilon_k(\varphi + c_k)$ touching $u_k = x_1 + \varepsilon_k w_k$ from above at $x_k \in \partial\{u_k > 0\} \cap B_{1/2}$ that converges to x_0 as $k \to \infty$. Because of the strict inequality $\varphi > w$ in $\overline{\Omega_h \setminus \Omega} \cap \overline{B_1^+}$ and uniform convergence of w_k to w, we also have $\phi_k > u_k$ in $\overline{\Omega_h \setminus \Omega} \cap \{u_k > 0\}$. If $|\partial_1 \phi_k|(x_k) < 2$, then

$$\tilde{u}_k := \begin{cases} \min\{u_k, (\phi_k)_+(x - \eta e_1)\}, & \text{in } \Omega_h \cap B_1^+, \\ u_k, & \text{elsewhere} \end{cases}$$

will become a new supersolution that is strictly smaller than u_k for some small $0 < \eta \ll r_1 - r_2$, which is impossible because u_k is assumed to be a minimal supersolution to (1.11). Now, we have

$$1 + 2\varepsilon_k \partial_1 \varphi(x_k) + o(\varepsilon_k) = |\nabla \phi_k|^2 \geq 2,$$

which implies that

$$\psi'(0) \approx \partial_1 \varphi(x_k) \geq O(1/\varepsilon_k),$$

for any k large. $\square$

Lemma 9.10. *For general p, we extend w to the whole $B_{1/2}$ evenly, and then w is subharmonic in $B_{1/2}$.*

Proof. It suffices to check points $x \in B'_{1/2}$. Indeed, suppose φ touches w from above at $x \in B'_{1/2}$, then we may start with locally ($x_p = x \cdot p$, $x = (x_p, x')$)

$$\psi(x_p, x') = \frac{\varphi(x_p, x') + \varphi(-x_p, x')}{2}.$$

Now we have

$$\Delta \psi(x_p, 0) = \Delta \varphi(x_p, 0),\ \partial_p \psi(x_p, 0) = 0,$$

and ψ also touches w from above. Let $\psi_\varepsilon = \psi - \varepsilon x_p + C_\varepsilon$ for small $\varepsilon > 0$ and some $C_\varepsilon > 0$. By standard theory, we may choose C_ε so that ψ_ε also touches w from above at some $x_\varepsilon \in B_{1/2}$ and $x_\varepsilon \to x$ as $\varepsilon \to 0^+$. We claim that all $x_\varepsilon \notin B'_{1/2}$ because the function w satisfies $\partial_p w \geq 0$ in the viscosity sense, and hence if $x_\varepsilon \in B'_{1/2}$

$$\partial_p \psi_\varepsilon(x_\varepsilon) = \partial_p \psi(x_\varepsilon) - \varepsilon \geq 0,$$

which violates the definition of ψ. Therefore, $x_\varepsilon \in B_{1/2} \setminus B'_{1/2}$, and so $\Delta\varphi(x) = \Delta\psi(x) \geq 0$. □

Lemma 9.11. *In the case that $p = e_1$, we show that if a smooth function φ touches w from below at some $x_0 \in B'$ and satisfies $|\nabla'\varphi|(x_0) > 0$, then we have*

$$\partial_1 \varphi(x_0) \leq 0.$$

Proof. We follow a similar procedure as the proof for subsolution and obtain a converging sequence $(c_k, x_k) \to (0^+, x_0)$ such that $\phi_k := x_1 + \varepsilon_k(\varphi + c_k)$ touches u_k from below at $x_k \in (\partial\{u_k > 0\}) \cap B_{1/2}$. Since $|\nabla'\varphi(x_0)| > 0$, for large k we also have $|\nabla'\varphi(x_k)| > 0$, which implies that by the super-solution condition of u_k at x_k,

$$1 \geq |\nabla\phi_k|^2(x_k) = 1 + 2\varepsilon_k \partial_1\varphi(x_k) + o(\varepsilon_k),$$

and so we obtain $\partial_1\varphi(x_k) \leq 0$ for all large k. This completes the proof. □

Lemma 9.12. *In the case that $p \neq e_1$, then w is harmonic inside $B_{1/2}$.*

Proof. This is immediate by observing that for any touching function φ from below, there is some $\delta > 0$ such that

$$|p + C\varepsilon\nabla\varphi|^2 - |(p + C\varepsilon\nabla\varphi)\cdot e_1|^2 \geq \delta,$$

independent of small $\varepsilon > 0$. □

Proof of Proposition 1.5. The proof is done by combining the above lemmas. □

Appendix A. Classification of homogeneous solutions in 2D

In this section, we discuss the classification of homogeneous solutions of the form

$$u(r,\theta) = r^\kappa m(\theta),\ r > 0,\ \kappa \geq 0,\ \theta \in \partial B_1 \cap \{x_1 \geq 0\}$$

to the equation (1.1) in dimension $d = 2$. Before discussing the classification, let us notice that any homogeneous solutions as described above to the problem (1.1) are also homogeneous solutions to the no-sign Signorini problem (7.2). This is because, by the boundary condition of u, we know that m (similar to the boundary condition of h in (6.7)) satisfies

$$\min\left\{-\partial_{\vec{n}} m(\theta), \sqrt{|\nabla_\tau m(\theta)|^2 + m^2(\theta)}\right\} = 0 \quad \text{for} \quad \theta \in \partial' B'_1$$

which implies that m also satisfies

$$\min\{-\partial_{\vec{n}} m(\theta), |m|(\theta)\} = 0 \quad \text{for} \quad \theta \in \partial' B'_1. \tag{A.1}$$

Therefore, we only have to discuss the homogeneous solutions to (7.2). At this point, the analysis becomes very similar to the classical Signorini case, but we present the details anyway to be complete.

In dimension $d = 2$, we call $x_1 = y$ and $x_2 = x$, and take θ to be the standard polar coordinate, i.e. $\tan\theta = \frac{y}{x}$. We can further write

$$u(r,\theta) = r^\kappa m(\theta) \ \text{ for } \ r > 0,\, \theta \in [0,\pi].$$

Due to (A.1) and u being harmonic in B_1^+, we know that m satisfies

$$\begin{cases} m''(\theta) + \kappa^2 m(\theta) = 0, \ \text{for } \theta \in (0,\pi) \\ \min\{m'(0), |m|(0)\} = \min\{-m'(\pi), |m|(\pi)\} = 0. \end{cases}$$

The general solution to the equation can be written as

$$m_{\text{general}} = a\cos(\kappa\theta) + b\sin(\kappa\theta)$$

for some real numbers a, b. By the boundary condition, we have (we can without loss assume that $\kappa > 0$)

$$\min\{b, |a|\} = 0,$$

and

$$\min\{a\sin(\kappa\pi) - b\cos(\kappa\pi), |a\cos(\kappa\pi) + b\sin(\kappa\pi)|\} = 0.$$

Suppose $b > 0$ then $|a| = 0$ and we can further assume that $b = 1$ after normalization. This implies that

$$\min\{-\cos(\kappa\pi), |\sin(\kappa\pi)|\} = 0.$$

Since $\cos(\kappa\pi)$ and $\sin(\kappa\pi)$ can not be both zero at the same time, we obtain either

$$\cos(\kappa\pi) < 0,\, \sin(\kappa\pi) = 0,$$

or

$$\cos(\kappa\pi) = 0,\, |\sin(\kappa\pi)| > 0.$$

In the first case we have $\kappa = 2k - 1$, $k \in \mathbb{Z}_+$, and in the second case we have $\kappa = \frac{2k-1}{2}$, $k \in \mathbb{Z}_+$. From this we get

$$m_\kappa = \sin(\kappa\theta),\, \kappa = 2k - 1, \ \text{or } (2k-1)/2,\, k \in \mathbb{Z}_+. \tag{A.2}$$

In the case $b = 0$ and $|a| > 0$, we can normalize so that $|a| = 1$, and obtain

$$\min\{\pm\sin(\kappa\pi), |\cos(\kappa\pi)|\} = 0.$$

This gives in the case $a = 1$, $\kappa = \frac{4k-3}{2}$ and in the case $a = -1$, $\kappa = \frac{4k-1}{2}$, or in both cases $\kappa = k$ for $k \in \mathbb{Z}_+$. From this we get

$$m_\kappa = \cos(\kappa\theta),\, \kappa = \frac{4k-3}{2} \ \text{or } k,\, k \in \mathbb{Z}_+, \tag{A.3}$$

or

$$m_\kappa = -\cos(\kappa\theta),\ \kappa = \frac{4k-1}{2} \text{ or } k,\ k \in \mathbb{Z}_+. \tag{A.4}$$

Now combining (A.2), (A.3) and (A.4), we have that any κ-homogeneous solution u_κ to (7.2) takes one of the following form with $k \in \mathbb{Z}_+$

1. For $\kappa = 2k-1$ or $(2k-1)/2$

$$u_\kappa = \operatorname{Im}\left((x_2 + i|x_1|)^\kappa\right);$$

2. For $\kappa = \frac{4k-3}{2}$ or k

$$u_\kappa = \operatorname{Re}\left((x_2 + i|x_1|)^\kappa\right);$$

3. For $\kappa = \frac{4k-1}{2}$ or k

$$u_\kappa = -\operatorname{Re}\left((x_2 + i|x_1|)^\kappa\right).$$

Notice that the cases $\kappa = \frac{4k-2\pm1}{2}$ with $u_\kappa = \mp\operatorname{Re}\left((x_2 + i|x_1|)^\kappa\right)$ can be identified after reflections with the case $\kappa = (2k-1)/2$ with $u_\kappa = \operatorname{Im}\left((x_2 + i|x_1|)^\kappa\right)$.

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Chapter 5

(Non-)gradient flow structures of the mean curvature flow

5.1 Introduction

Mean Curvature Flow (MCF) is widely known to be a formal gradient flow of the perimeter functional. To be more specific, let $\Gamma_t \coloneqq J_t(\Gamma_0) \subset \mathbb{R}^d$ be a smooth family of hypersurfaces, where J_t satisfies the following ODE

$$\begin{cases} \dot{J}_t = \vec{V}(t, J_t), \quad t \in [0,1], \\ J_0(x) = x. \end{cases}$$

If the vector field $\vec{V}(t,\cdot) = \vec{H} = \kappa\vec{N}$ is the mean curvature vector field on Γ_t, then the trajectory Γ_t is called a mean curvature flow. On the other hand, by the first variation formula of the perimeter, a MCF satisfies

$$\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma_t) = -\int_{\Gamma_t} |\vec{H}|^2 d\mathcal{H}^{d-1}, \tag{5.1.1}$$

where $\mathcal{H}^{d-1}$ is the $d-1$-Hausdorff measure on $\mathbb{R}^d$. Formally speaking, we can naturally define a Riemannian structure on the space of hypersurfaces: at a hypersurface $\Gamma \subset \mathbb{R}^d$, the tangent space is defined as all the normal vector fields on Γ, and for $\vec{V}^\perp, \vec{W}^\perp$ in the tangent space, there is an inner product

$$(\vec{V}^\perp, \vec{W}^\perp)_\Gamma \coloneqq \int_\Gamma \vec{V}^\perp \cdot \vec{W}^\perp d\mathcal{H}^{d-1}. \tag{5.1.2}$$

In the above Riemannian structure a MCF is indeed a formal gradient flow of the perimeter $\mathcal{H}^{d-1}(\cdot)$.

This structure, however, turns out to be degenerate. By the work of Michor and Mumford [127, 128], the geodesic distance between any two hypersurfaces is zero. Now it is very natural to ask: is there an alternative gradient flow structure for MCF on a nondegenerate metric space of hypersurfaces?

To deal with the issue that pushforwards of Hausdorff measures do not preserve uniformness under the flow of general normal vector fields, Shi and Vorotnikov [162] proposed a new Riemannian structure with tangent space defined as a subspace of vector fields on Γ composed of $\vec{V}$ such that

$$\mathrm{div}_\Gamma \vec{V} \equiv Const., \tag{5.1.3}$$

where div_Γ is defined as the tangential divergence on Γ. For $\vec{V}, \vec{W}$ in the new tangent space, the inner product is defined as

$$(\vec{V}, \vec{W})_U \coloneqq \int_\Gamma \vec{V} \cdot \vec{W} d\mathcal{H}^{d-1}. \tag{5.1.4}$$

Notice that, different from (5.1.2), the right-hand side of (5.1.4) involves the tangential components of $\vec{V}$ and $\vec{W}$.

In the Riemannian structure described above, Shi and Vorotnikov discussed the gradient flow of perimeter functional and discovered a new geometric flow called *Uniformly Compressing Mean Curvature Flow* (UCMCF). The MCF itself can also be understood as a flow in the new structure by modifying the tangential components of the mean curvature vector fields [2], but because of the modifications in (5.1.4), the velocity field that drives UCMCF has a nontrivial difference from that of MCF in general.

Moreover, UCMCF is also a gradient flow of $\log$ perimeter in the space of normalized Hausdorff measures. This space is canonically embedded into the Wasserstein space (in the sense of Otto's formal Riemannian structure [139]) because the tangent space is defined as before and the inner product is simply a normalized version of the inner product (5.1.4)

$$(\vec{V}, \vec{W})_N \coloneqq \fint_\Gamma \vec{V} \cdot \vec{W} d\mathcal{H}^{d-1}. \tag{5.1.5}$$

This embedding ensures that the geodesic distance induced by (5.1.5) is nondegenerate because it is lower bounded by the Wasserstein distance. Similarly, tangentially modified MCF can also be discussed in this normalized structure. More detailed discussions of the structure and the flows are contained in Section 2.

In [128], Michor and Mumford also proposed a nondegenerate metric for plane curves by adding a term that involves curvatures to (5.1.2)

$$(\vec{V}^{\perp}, \vec{W}^{\perp})_M \coloneqq \int_{\Gamma} \left(1+\kappa^2\right) \vec{V}^{\perp} \cdot \vec{W}^{\perp} d\mathcal{H}^1, \tag{5.1.6}$$

where κ is the scalar curvature of the curve. In this structure, MCF can be understood as a flow directly without any further modifications.

The Riemannian structures (5.1.5) and (5.1.6) are appealing for various reasons. First and foremost they both result in nondegenerate geodesic distances on the space of embedded curves. The Shi-Vorotnikov uniformness-preserving metric is also appealing for preserving the "density of grid points" on the curve, a property that helps increase the computational stabilities of surface evolutions[132]. The tangential constraint (5.1.3) and the metric (5.1.4) also arise in evolution models for *incompressible membranes* [87, 132, 135]. Both metrics were introduced in the context of studying the MCF or MCF-like flows as gradient flows. It is natural to ask if the MCF itself is a gradient flow of some functional under these metric structures. In this paper we show that it is not in both cases:

Theorem 5.1.1. *The mean curvature flow for simple closed plane curves is not a gradient flow either in the Riemannian structure* (5.1.5) *proposed by Shi and Vorotnikov* [162] *or* (5.1.6) *proposed by Michor and Mumford* [128]*.*

As far as the author knows, no results concerning non-gradient-flow properties for MCF have been published (neither do those on rigorous gradient flow structures). On the other hand, many works over the past few decades indicate that MCF can be well approximated by true gradient flows. For example, it is known that MCF can be understood as the sharp interface limit of the Allen-Cahn equation, which is an L^2 gradient flow of a Ginzburg-Landau type functional [97]. The well-known MBO thresholding scheme [126] for MCF is proved to have a discrete gradient flow structure [76, 111].

5.2 Some preparations

5.2.1 On a submanifold of the Wasserstein space

In [162], Shi and Vorotnikov discussed a special Riemannian structure on the space

$$\mathcal{C}_d \coloneqq \left\{\Gamma \in C^\infty(\mathbb{S}^{d-1};\mathbb{R}^d)\,;\,\Gamma \text{ is an embedding of } \mathbb{S}^{d-1} \text{ in } \mathbb{R}^d\right\}/\sim,$$

where "$\sim$" is the equivalence relation that identifies any two embeddings with the same image. Although the arguments in this section can also be applied to higher codimensional cases, we will focus on hypersurfaces. The new structure naturally gives an embedding $i : \mathcal{C}_d \hookrightarrow \mathbb{W}_2(\mathbb{R}^d)$ to the 2-Wasserstein space of probability measures on $\mathbb{R}^d$, where the assignment i is defined as

$$i([\Gamma]) \coloneqq \frac{\mathcal{H}^{d-1}\big|_{\mathrm{Im}(\Gamma)}}{\mathcal{H}^{d-1}(\mathrm{Im}(\Gamma))} \tag{5.2.1}$$

We will not distinguish Γ, $[\Gamma]$, $i([\Gamma])$ and $\mathrm{Im}(\Gamma)$ when it is unambiguous. For example, "$d\Gamma$" will simply mean "$d\,(i([\Gamma]))$".

Definition 5.2.1. We call $\mathcal{C}_d$ endowed with the metric (5.1.5) the *Coherent Space* of hypersurfaces in $\mathbb{R}^d$. At each $\Gamma \in \mathcal{C}_d$, we would call $\mathbb{T}_\Gamma\mathcal{C}_d$ the space of all vector fields $\vec{V}$ on Γ satisfying (5.1.3) the *Coherent Tangent Space* at Γ.

We start by discussing the paths in $\mathcal{C}_d$. Let $\Phi_t(x)$ be a flow map of the following ODE

$$\begin{cases} \dot{\Phi}_t = \vec{V}(t,\Phi_t), & t \in [0,1], \\ \Phi_0(x) = x. \end{cases} \tag{5.2.2}$$

We consider $\Gamma_t \coloneqq \Phi_t(\Gamma_0)$ (flow of images) and observe according to first variation formula [97], for every smooth test function ζ, Γ_t (as normalized Hausdorff measures) should satisfy

$$\begin{aligned} \frac{d}{dt}\int \zeta d\Gamma_t &= \frac{d}{dt}\int_{\Phi_t(\Gamma_0)} \zeta d\frac{\mathcal{H}^{d-1}}{\mathcal{H}^{d-1}(\Gamma_t)} \\ &= \frac{1}{\mathcal{H}^{d-1}(\Gamma_t)}\int_{\Gamma_t} \nabla\zeta\cdot\vec{V}(t,\cdot)d\mathcal{H}^{d-1} + \frac{1}{\mathcal{H}^{d-1}(\Gamma_t)}\int_{\Gamma_t}\zeta\mathrm{div}_{\Gamma_t}\vec{V}(t,\cdot)d\mathcal{H}^{d-1} \\ &\quad - \frac{1}{\mathcal{H}^{d-1}(\Gamma_t)}\int_{\Gamma_t}\mathrm{div}_{\Gamma_t}\vec{V}(t,\cdot)d\mathcal{H}^{d-1}\cdot\frac{1}{\mathcal{H}^{d-1}(\Gamma_t)}\int_{\Gamma_t}\zeta d\mathcal{H}^{d-1} \\ &= \int \nabla\zeta\cdot\vec{V}(t,\cdot)d\Gamma_t + \int \zeta\left(\mathrm{div}_{\Gamma_t}\vec{V}(t,\cdot) - \int \mathrm{div}_{\Gamma_t}\vec{V}(t,\cdot)d\Gamma_t\right)d\Gamma_t. \end{aligned} \tag{5.2.3}$$

Observe that if $\vec{V}$ is smooth, then we have by Poincaré inequality,

$$\left|\frac{d}{dt}\int \zeta d\Gamma_t\right| \leqslant C\left(\|\vec{V}\|_{L^2_{\Gamma_t}} + \left\|\mathbf{div}_{\Gamma_t}\vec{V}(t,\cdot) - \int \mathbf{div}_{\Gamma_t}\vec{V}(t,\cdot)d\Gamma_t\right\|_{L^2_{\Gamma_t}}\right)\|\nabla\zeta\|_{L^2_{\Gamma_t}}, \qquad (5.2.4)$$

which means that the path Γ_t is in fact a Lipschitz path in the Wasserstein space. Let us now interpret $\Phi_t(\Gamma_0)$ (flow of images) as a pushforward of measures.

Proposition 5.2.2. *Let $\Gamma_t = \Phi_t(\Gamma_0)$ be defined as before with respect to a smooth vector field $\vec{V}$, then Γ_t as a family of probability measures should satisfy for all smooth test functions ζ,*

$$\frac{d}{dt}\int \zeta d\Gamma_t = \int \nabla\zeta\cdot(\vec{V}^\perp + \nabla_{\Gamma_t}U_t)d\Gamma_t,$$

where $\vec{V}^\perp$ is the normal component of $\vec{V}$ and U_t satisfies the following elliptic equation

$$\begin{cases}-\Delta_{\Gamma_t}U_t = \int \vec{V}\cdot\vec{H}d\Gamma_t - \vec{V}\cdot\vec{H},\\ \int U_t d\Gamma_t = 0.\end{cases} \qquad (5.2.5)$$

Moreover, if $\overline{\Phi}_t$ is the flow map of the vector field $\vec{V}^\perp + \nabla_{\Gamma_t}U_t$, then we have

$$\frac{\mathcal{H}^{d-1}\big|_{\Gamma_t}}{\mathcal{H}^{d-1}(\Gamma_t)} = \overline{\Phi}_t\#\frac{\mathcal{H}^{d-1}\big|_{\Gamma_0}}{\mathcal{H}^{d-1}(\Gamma_0)}. \qquad (5.2.6)$$

Remark 5.2.3. (i) The transformation

$$\begin{aligned} P = P_{\Gamma_t} : \mathbb{T}_{\Gamma_t}\mathbb{W}_2 &\to \mathbb{T}_{\Gamma_t}\mathbb{W}_2\\ \vec{V} &\mapsto \vec{V}^\perp + \nabla_{\Gamma_t}U_t\end{aligned}$$

is a projection, where $\mathbb{T}_{\Gamma_t}\mathbb{W}_2$ is the space of all smooth vector fields on Γ_t (we would call this the *Wasserstein tangent space*). Indeed, if we call $\mathrm{ndiv}_\Gamma\vec{W} := \mathrm{div}_\Gamma\vec{W} - \int \mathrm{div}_\Gamma\vec{W}d\Gamma$, then it can be seen by the following computation

$$\begin{aligned}\int \zeta\mathbf{ndiv}_{\Gamma_t}(\vec{V}^\perp + \nabla_{\Gamma_t}U_t)d\Gamma_t &= \int \zeta\mathbf{div}_{\Gamma_t}\vec{V}^\perp d\Gamma_t - \int \mathbf{div}_{\Gamma_t}\vec{V}^\perp d\Gamma_t\cdot\int\zeta d\Gamma_t\\ &\quad + \int \zeta\Delta_{\Gamma_t}U_t d\Gamma_t\\ &= \int\zeta\left(-\vec{V}\cdot\vec{H} + \int\vec{V}\cdot\vec{H}d\Gamma_t\right)d\Gamma_t\\ &\quad - \int\zeta\left(\int\vec{V}\cdot\vec{H}d\Gamma_t - \vec{V}\cdot\vec{H}\right)d\Gamma_t\\ &= 0.\end{aligned} \qquad (5.2.7)$$

(ii) Observe that the projection $P = P_{\Gamma_t}$ only relies on the information of the current surface Γ_t. Moreover, the image of the projection P_Γ is indeed the coherent tangent space because of the above remark. Since the structure defined in (5.1.5) is simply the restriction of Otto's formal Riemannian structure [139], we arrive at the conclusion that $\mathcal{C}_d$ is indeed a Riemannian submanifold of $\mathbb{W}_2(\mathbb{R}^d)$.

(iii) The vector field $\vec{V}^\perp + \nabla_{\Gamma_t} U_t$ can be written as the gradient of some function. Indeed, if we take an ε-tube neighborhood of Γ_t, then we may just find that $\vec{V}^\perp = \nabla V_t$ on Γ_t for some function V_t that is nonconstant only along the normal trajectories. Extending U_t as a constant along the normal trajectories, we then can write, at least in a ε-tube neighborhood of Γ_t there is a function $V_t + U_t$ such that $\vec{V}^\perp + \nabla_{\Gamma_t} U_t = \nabla(V_t + U_t)$ on Γ_t.

(iv) This proposition tells us that given any smooth path of curves Γ_t, $t \in [0,1]$, there is a smooth reparametrization $\bar{\Phi}_t(\theta) : \mathbb{S}^{d-1} \to \Gamma_t \subset \mathbb{R}^d$ satisfying

$$\sqrt{\det\left(D_\theta \bar{\Phi}_t^T D_\theta \bar{\Phi}_t\right)} \equiv l_t / |\mathbb{S}^{d-1}|,$$

where $l_t \coloneqq \mathcal{H}^{d-1}(\Gamma_t)$. In particular if $d = 2$ then we have $|\partial_\theta \bar{\Phi}_t| \equiv l_t/2\pi$.

(v) When $d = 2$, we know that $\mathcal{C}_2$ is connected by paths of the above type. This is indicated by the Whitney–Graustein theorem, which states that regular homotopy classes of plane curves can be classified by their turning numbers.

Proof. First observe that

$$\begin{aligned} \int \zeta \mathrm{div}_{\Gamma_t} \vec{V} d\Gamma_t &= \int \mathrm{div}_{\Gamma_t}\left(\zeta \vec{V}\right) - \nabla_{\Gamma_t}\zeta \cdot \vec{V} d\Gamma_t \\ &= -\int \zeta \vec{V} \cdot \vec{H} + \nabla_{\Gamma_t}\zeta \cdot \vec{V} d\Gamma_t. \end{aligned} \tag{5.2.8}$$

Now the last term in formula (5.2.3) becomes

$$\int \nabla\zeta \cdot \vec{V} d\Gamma_t + \int \zeta \mathrm{ndiv}_{\Gamma_t} \vec{V} d\Gamma_t \quad = \quad \int \zeta \left(\int \vec{V} \cdot \vec{H} d\Gamma_t - \vec{V} \cdot \vec{H} \right) + \nabla^\perp \zeta \cdot \vec{V} d\Gamma_t.$$

Using this equation and the estimate (5.2.4), we are able to solve

$$\frac{d}{dt} \int \zeta d\Gamma_t = \int \nabla\zeta \cdot \vec{G}_t d\Gamma_t = \int \nabla\zeta \cdot \vec{V} d\Gamma_t + \int \zeta \mathrm{ndiv}_{\Gamma_t} \vec{V} d\Gamma_t,$$

with $\vec{G}_t$ minimizing the L^2-energy $\int |\vec{G}_t|^2 d\Gamma_t$. This is equivalent to solve

$$\begin{aligned}\int \nabla\zeta \cdot \vec{G}_t d\Gamma_t &= \int \nabla\zeta \cdot \vec{V} d\Gamma_t + \int \zeta \mathbf{n}\mathrm{div}_{\Gamma_t}\vec{V} d\Gamma_t \\ &= \int \zeta\left(\int \vec{V}\cdot\vec{H} d\Gamma_t - \vec{V}\cdot\vec{H}\right) + \nabla^{\perp}\zeta\cdot\vec{V} d\Gamma_t.\end{aligned}$$

Fixing $\zeta \equiv 0$ on Γ_t, we observe that a solution $\vec{G}_t$ should share the same normal component with $\vec{V}$. Hence we need only compute the tangential component, which is equivalently solving

$$\int \nabla_{\Gamma_t}\zeta \cdot \vec{Q}_t d\Gamma_t = \int \zeta\left(\int \vec{V}\cdot\vec{H} d\Gamma_t - \vec{V}\cdot\vec{H}\right) d\Gamma_t. \tag{5.2.9}$$

Moreover, because the normal part of $\vec{G}_t$ is fixed, we just need to minimize the L^2-energy of its tangential part $\vec{Q}_t$. To that end, we recall that the space of tangential vector fields on Γ_t (which has the same topology as the unit sphere) can be decomposed as the direct sum of two orthogonal subspaces: gradients of functions and divergence-free vector fields. Here a tangential vector field $\vec{S}$ is called *divergence-free* if for all test functions $\zeta \in C_0^\infty(\mathbb{R}^d)$

$$\int \nabla_{\Gamma_t}\zeta \cdot \vec{S} d\Gamma_t = 0. \tag{5.2.10}$$

Observe that if $\vec{Q}_t$ is a solution to (5.2.9) and $\vec{S}$ is a divergence-free vector field, then $\vec{Q}_t + \vec{S}$ is still a solution to (5.2.9). Therefore, to minimize the L^2-energy of the tangential component it is equivalent to find $\vec{Q}_t^*$ satisfying

$$\int |\vec{Q}_t^* + \vec{S}|^2 d\Gamma_t \geq \int |\vec{Q}_t^*|^2 d\Gamma_t, \ \forall \text{ divergence-free } \vec{S},$$

which gives us the following variational formula

$$\int \vec{Q}_t^* \cdot \vec{S} d\Gamma_t = 0, \ \forall \text{ divergence-free } \vec{S}.$$

Now we may write for some function $U_t \in H^1(\Gamma_t)$

$$\vec{Q}_t^* = \nabla_{\Gamma_t} U_t,$$

and then we can modify (5.2.9) as

$$\int \nabla_{\Gamma_t}\zeta \cdot \nabla_{\Gamma_t} U_t d\Gamma_t = \int \zeta\left(\int \vec{V}\cdot\vec{H} d\Gamma_t - \vec{V}\cdot\vec{H}\right) d\Gamma_t,$$

which is equivalent to solving the elliptic equation (5.2.5). We immediately obtain the smoothness of $\vec{Q}_t^*$ from the standard elliptic theory. Equality (5.2.6) is obtained by observing that $\overline{\Phi}_t$ is a constant-speed reparametrization for Γ_t for all t. □

5.2.2 The uniformly compressing MCF

Let us now discuss a geometric flow that is very similar to MCF. We define the *log perimeter functional* $\mathcal{R}$ on $\mathcal{C}_d$ as $\mathcal{R}(\Gamma) := \log \mathcal{H}^{d-1}(\Gamma)$. Its differential can be written explicitly: Let Γ_t be a path with respect to some vector field $\vec{V}$ for $t \in [0,1]$. Then we have by the first variation formula

$$\begin{aligned} \frac{d}{dt}\mathcal{R}(\Gamma_t)\Big|_{t=0} &= \frac{d}{dt}\log \mathcal{H}^{d-1}(\Gamma_t)\Big|_{t=0} \\ &= -\int_{\Gamma_0} \vec{H}\cdot\vec{V} d\frac{\mathcal{H}^{d-1}}{\mathcal{H}^{d-1}(\Gamma_0)} \\ &= -\int \vec{H}\cdot\vec{V} d\Gamma_0. \end{aligned}$$

The (coherent) tangential derivative of $\mathcal{R}$ at Γ_0 is the realization of the above differential in the coherent tangent space $\mathbb{T}_{\Gamma_0}\mathcal{C}_d$, which means that there is a unique $\nabla_{\mathcal{C}_d}\mathcal{R}(\Gamma_0) \in \mathbb{T}_{\Gamma_0}\mathcal{C}_d$ such that

$$\int \nabla_{\mathcal{C}_d}\mathcal{R}(\Gamma_0)\cdot\vec{V} d\Gamma_0 = -\int \vec{H}\cdot\vec{V} d\Gamma_0, \ \forall \vec{V} \in \mathbb{T}_{\Gamma_0}\mathcal{C}_d.$$

The gradient flow of $\mathcal{R}$ with respect to the coherent metric (5.1.5) was first introduced by Shi and Vorotnikov [162] and is called the *uniformly compressing mean curvature flow*. It is the following interface evolution written in weak form: $\Gamma_t : [0,T] \to \mathcal{C}_d$ satisfies for all smooth test function ζ,

$$\frac{d}{dt}\int \zeta d\Gamma_t = -\int \nabla_{\mathcal{C}_d}\mathcal{R}(\Gamma_t)\cdot\nabla\zeta d\Gamma_t.$$

Let us now compute $\nabla_{\mathcal{C}_d}\mathcal{R}$. Recall that for all coherent vector field $\vec{V}$

$$\begin{aligned} \int \nabla_{\mathcal{C}_d}\mathcal{R}\cdot\vec{V} d\Gamma &= -\int \vec{H}\cdot\vec{V} d\Gamma \\ &= -\int \vec{H}\cdot\vec{V}^{\perp} d\Gamma, \end{aligned}$$

while on the right-hand side, if we denote the normal part of $\nabla_{\mathcal{C}_d}\mathcal{R}$ by $\vec{w}$, and its tangential part by $\nabla_\Gamma W$, then we have

$$\begin{aligned} -\int \vec{H}\cdot\vec{V}^{\perp} d\Gamma &= \int \vec{w}\cdot\vec{V}^{\perp} d\Gamma + \int \nabla_\Gamma W\cdot\nabla_\Gamma U d\Gamma \\ &= \int \vec{w}\cdot\vec{V}^{\perp} d\Gamma - \int W\Delta_\Gamma U d\Gamma \\ &= \int \vec{w}\cdot\vec{V}^{\perp} d\Gamma - \int \vec{V}^{\perp}\cdot W\vec{H} d\Gamma \\ &= \int \vec{V}^{\perp}\cdot(\vec{w} - W\vec{H}) d\Gamma. \end{aligned}$$

Since this holds for all normal velocity fields $\vec{V}^{\perp}$, we get the pointwise equality $\vec{H} = -\vec{w} + W\vec{H}$. On the other hand, by the coherence of the vector field $\nabla_{\mathcal{C}_d}\mathcal{R}$ we have the equation $-\Delta_\Gamma W = \int \vec{w}\cdot\vec{H}d\Gamma - \vec{w}\cdot\vec{H}$, then we get

$$\begin{aligned} -\Delta_\Gamma W &= \int \vec{w}\cdot\vec{H}d\Gamma - \vec{w}\cdot\vec{H} \\ &= \int (W-1)|\vec{H}|^2 d\Gamma - (W-1)|\vec{H}|^2. \end{aligned} \tag{5.2.11}$$

At this stage, we obtain the following lemma.

Lemma 5.2.4. *Let W be the unique solution to* (5.2.11) *such that $\int W d\Gamma = 0$, then the coherent tangential derivative of $\mathcal{R}$ at Γ takes the following form*

$$\nabla_{\mathcal{C}_d}\mathcal{R}(\Gamma) = (W-1)\vec{H} + \nabla_\Gamma W.$$

Remark 5.2.5. The above lemma implies that the uniformly compressing mean curvature flow is generally not mean curvature flow. Indeed, the UCMCF is driven by $(1-W)\vec{H}$ in the normal direction instead of simply $\vec{H}$, and when the surfaces are not of constant curvature, W is a non-trivial function, and hence the flows are distinct.

5.2.3 MCF as a flow on $\mathcal{C}_d$

In previous sections, we derived the gradient flow of log perimeter under the special submanifold structure induced by the embedding $\mathcal{C}_d \overset{i}{\hookrightarrow} \mathbb{W}_2(\mathbb{R}^d)$. Now we would like to interpret MCF as a flow on $\mathcal{C}_d$. We may write for some surface Γ and its mean curvature vector field $\vec{H} = \vec{H}_\Gamma$

$$P_\Gamma(\vec{H}) = \vec{H} + \nabla_\Gamma \Sigma,$$

where Σ satisfies

$$\begin{cases} -\Delta_\Gamma \Sigma = \int |\vec{H}|^2 d\Gamma - |\vec{H}|^2, \\ \int \Sigma d\Gamma = 0. \end{cases} \tag{5.2.12}$$

Thinking of $P_.(\vec{H}_.)$ as a vector field on $\mathcal{C}_d$ we can see that a MCF Γ_t (viewed as normalized Hausdorff measures) satisfies for all test function ζ

$$\frac{d}{dt}\int \zeta d\Gamma_t = \int \nabla\zeta \cdot P_{\Gamma_t}(\vec{H})d\Gamma_t.$$

That is to say, the MCF is the flow of the vector field $P.(\vec{H}.)$ on $\mathcal{C}_d$. This flow may be referred to as "tangentially modified" MCF. It is equivalent to MCF when we look at the support of the flow.

5.3 The proof of Theorem 5.1.1

The proof of the first part is composed of Sections 3.1 and 3.2. The proof of the second part with respect to the structure proposed by Michor and Mumford follows the same outline and is given in Section 3.3.

5.3.1 The hypothesis and a criterion

We are interested in the existence of an energy functional $\mathcal{F}$ on $\mathcal{C}_2$ such that its gradient flow in the coherent space is exactly the mean curvature flow (also known as the curve-shortening flow). If there were such an energy functional $\mathcal{F}$, then for any closed path of simple closed plane curves Γ_t driven by some vector field $\vec{V}_t \in \mathbb{T}_{\Gamma_t}\mathbb{W}_2$ for $t \in [0,1]$, we would have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\Gamma_t) &= -\left(P_{\Gamma_t}(\vec{H}_{\Gamma_t}), P_{\Gamma_t}(\vec{V}_t)\right)_{\mathbb{T}_{\Gamma_t}\mathbb{W}_2}\\ &= -\int \left(\vec{H}_t \cdot \vec{V}_t + \nabla_{\Gamma_t}\Sigma_t \cdot \nabla_{\Gamma_t}U_t\right) d\Gamma_t,\end{aligned}$$

where Σ_t satisfies (5.2.12), and U_t satisfies (5.2.5). Integrating both sides with respect to $t \in [0,1]$ we have

$$0 = \mathcal{F}(\Gamma_1) - \mathcal{F}(\Gamma_0) = -\int_0^1 \int \left(\vec{H}_t \cdot \vec{V}_t + \nabla_{\Gamma_t}\Sigma_t \cdot \nabla_{\Gamma_t}U_t\right) d\Gamma_t dt.$$

Thus if the MCF were a gradient flow in $\mathcal{C}_2$, then the vector field $P.(\vec{H}.)$ would be "conservative": its integral along any closed path in $\mathcal{C}_2$ would be zero.

Hypothesis 5.3.1. For all closed paths Γ_t driven by a vector field $\vec{V}_t = P_{\Gamma_t}(\vec{V}_t)$,

$$\int_0^1 \int \left(\vec{H}_t \cdot \vec{V}_t + \nabla_{\Gamma_t}\Sigma_t \cdot \nabla_{\Gamma_t}U_t\right) d\Gamma_t dt = 0, \tag{5.3.1}$$

with Σ_t and U_t defined the same as above.

We present the proof by giving a contradiction of Hypothesis 5.3.1. Observe that the first term satisfies

$$\begin{aligned}\int_0^1 \int \vec{H}_t \cdot \vec{V}_t d\Gamma_t dt = &\int_0^1 \frac{d}{dt}\mathcal{R}(\Gamma_t)dt\\ &= \log l_1 - \log l_0\\ &=0,\end{aligned}$$

and hence we only have to compute

$$\begin{aligned}\int_0^1 \int \nabla_{\Gamma_t}\Sigma_t \cdot \nabla_{\Gamma_t} U_t d\Gamma_t dt &= -\int_0^1 \int \Sigma_t \Delta_{\Gamma_t} U_t d\Gamma_t dt\\ &= -\int_0^1 \int \Sigma_t \vec{H}_t \cdot \vec{V}_t d\Gamma_t dt.\end{aligned} \tag{5.3.2}$$

According to Remark 5.2.3 (iv), we have a smooth (piecewise smooth in time) constant speed reparametrization $\Phi_t(\theta) : \mathbb{S}^1 \to \Gamma_t$ such that $|\partial_\theta \Phi_t| \equiv l_t/2\pi$. This implies that for every $t \in [0,1]$ we have

$$\int \Sigma_t \vec{H}_t \cdot \vec{V}_t d\Gamma_t = \frac{1}{2\pi}\int_0^{2\pi} \Sigma_t(\Phi_t(\theta))\vec{H}_t(\Phi_t(\theta)) \cdot \vec{V}_t(\Phi_t(\theta))d\theta. \tag{5.3.3}$$

Assuming that $s = s_t \in [0, l_t]$ is a unit speed reparametrization of Γ_t, then we have $\partial_s = \frac{2\pi}{l_t}\partial_\theta$, and hence we have

$$\vec{V}(\Phi_t(\theta)) = \dot{\Phi}_t(\theta), \quad \vec{H}(\Phi_t(\theta)) = \frac{4\pi^2}{l_t^2}\partial_\theta^2 \Phi_t(\theta),\ \theta \in [0,2\pi]. \tag{5.3.4}$$

Moreover, the differential equation for $\bar{\Sigma}_t(\theta) \coloneqq \Sigma_t(\Phi_t(\theta))$ is

$$\begin{cases}-\partial_\theta^2 \bar{\Sigma}_t = \dfrac{4\pi^2}{l_t^2}\left[\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |\partial_\theta^2 \Phi_t|^2 d\theta - |\partial_\theta^2 \Phi_t|^2\right], \quad \theta \in [0,2\pi],\\ \displaystyle\int_0^{2\pi} \bar{\Sigma}_t d\theta = 0.\end{cases} \tag{5.3.5}$$

We solve equation (5.3.5) via Green's kernel $G(\theta,\xi) : [0,2\pi]^2 \to \mathbb{R}$ satisfying

$$\begin{cases}-\partial_\theta^2 G(\theta,\xi) = \delta_\xi(\theta) - \dfrac{1}{2\pi}, \quad \xi,\theta \in [0,2\pi],\\ \displaystyle\int_0^{2\pi} G(\theta,\xi)d\theta = 0.\end{cases} \tag{5.3.6}$$

Green's kernel G to (5.3.6) has the following form, although we will not need it,

$$G(\theta,\xi) = \frac{1}{4\pi}(\xi-\theta)^2 + \min(\xi,\theta) - \frac{1}{2}(\xi+\theta) + \frac{1}{3}\pi. \tag{5.3.7}$$

Now, the solution $\bar{\Sigma}_t$ can be written as

$$\bar{\Sigma}_t(\theta) = -\frac{4\pi^2}{l_t^2}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\Phi_t(\xi)|^2 d\xi. \tag{5.3.8}$$

Collecting (5.3.4) and (5.3.8), we may rewrite (5.3.3) as

$$\int \Sigma_t \vec{H}_t \cdot \vec{V}_t d\Gamma_t = -\frac{8\pi^3}{l_t^4}\int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\Phi_t(\xi)|^2 \partial_\theta^2\Phi_t \cdot \dot{\Phi}_t(\theta) d\xi d\theta. \tag{5.3.9}$$

Introducing $\tilde{\Phi}_t \coloneqq \Phi_t/l_t$, we have, ignoring the constants

$$\begin{aligned} &\frac{1}{l_t^4}\int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\Phi_t(\xi)|^2 \partial_\theta^2\Phi_t \cdot \dot{\Phi}_t(\theta) d\xi d\theta \\ &\qquad = \int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2 \partial_\theta^2\tilde{\Phi}_t \cdot \dot{\Phi}_t(\theta)/l_t d\xi d\theta \\ &\qquad = \int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2 \partial_\theta^2\tilde{\Phi}_t \cdot \dot{\tilde{\Phi}}_t(\theta) d\xi d\theta \\ &\qquad\quad + \partial_t \log l_t \int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2 \partial_\theta^2\tilde{\Phi}_t \cdot \tilde{\Phi}_t(\theta) d\xi d\theta \end{aligned} \tag{5.3.10}$$

Observe that $\tilde{\Phi}_t$ is also a constant speed parametrization of another closed path $\tilde{\Gamma}_t \coloneqq \mathrm{Im}(\tilde{\Phi}_t)$. Moreover, we have that $\mathcal{H}^{d-1}(\tilde{\Gamma}_t) \equiv 1$.

Lemma 5.3.2. *If Hypothesis 5.3.1 holds true for all closed paths, then given any smooth curve* $\Gamma \in \mathcal{C}_2$ *and its unit speed parametrization* $\Phi : \mathbb{S}^1 \to \Gamma \subset \mathbb{R}^2$,

$$\int_0^{2\pi} |\Phi|^2 |D_\theta^2\Phi|^2 d\theta - \frac{1}{2\pi}\int_0^{2\pi} |\Phi|^2 d\theta \int_0^{2\pi} |D_\theta^2\Phi|^2 d\theta = 0. \tag{5.3.11}$$

In particular, for any $\vec{u} \in \mathbb{R}^2$,

$$\int_0^{2\pi} \Phi\cdot\vec{u}\,|D_\theta^2\Phi|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \Phi\cdot\vec{u}\,d\theta \int_0^{2\pi} |D_\theta^2\Phi|^2 d\theta. \tag{5.3.12}$$

Proof. Integrating both sides of (5.3.10) with respect to time $t \in [0,1]$ we have, according to Hypothesis 5.3.1,

$$\int_0^1 \partial_t \log l_t \int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2 \partial_\theta^2\tilde{\Phi}_t \cdot \tilde{\Phi}_t(\theta) d\xi d\theta dt = 0. \tag{5.3.13}$$

Fixing $\tilde{\Phi}_t$, we observe that equation (5.3.10) still holds if we replace Φ_t by $\tilde{l}_t\tilde{\Phi}_t$ for any smooth positive 1-periodic function $\tilde{l}_t$. This implies that

$$\int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2 \partial_\theta^2\tilde{\Phi}_t \cdot \tilde{\Phi}_t(\theta) d\xi d\theta \tag{5.3.14}$$

is independent of time. Since by Remark 5.2.3 (v) $\mathcal{C}_2$ is path-connected, the above quantity should be a constant for all elements in $\mathcal{C}_2$ that have perimeter 1. On the other hand, we have

$$\begin{aligned}\partial_\theta^2|\tilde{\Phi}_t|^2 &= 2\partial_\theta^2\tilde{\Phi}_t\cdot\tilde{\Phi}_t + 2|\partial_\theta\tilde{\Phi}_t|^2\\ &= 2\partial_\theta^2\tilde{\Phi}_t\cdot\tilde{\Phi}_t + 2,\end{aligned}$$

and then

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} G(\theta,\xi)|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2\partial_\theta^2\tilde{\Phi}_t\cdot\tilde{\Phi}_t(\theta)d\xi d\theta &= \frac{1}{2}\int_0^{2\pi}|\partial_\theta^2\tilde{\Phi}_t(\xi)|^2\int_0^{2\pi}G(\theta,\xi)\partial_\theta^2|\tilde{\Phi}_t|^2 d\theta d\xi\\ &= -\frac{1}{2}\int_0^{2\pi}|\tilde{\Phi}_t|^2|\partial_\theta^2\tilde{\Phi}_t|^2 d\theta\\ &\quad + \frac{1}{4\pi}\int_0^{2\pi}|\tilde{\Phi}_t|^2 d\theta\int_0^{2\pi}|\partial_\theta^2\tilde{\Phi}_t|^2 d\theta.\end{aligned} \tag{5.3.15}$$

We have shown (5.3.11) by observing that the quantity above is clearly 0 for any circle with perimeter 1. Equation (5.3.12) is derived by variation of (5.3.11) in the direction $\vec{u}$. □

5.3.2 The construction of a counter-example

Although (5.3.12) seems unlikely to hold for all curves in $\mathcal{C}_2$, we provide here a counter-example to exhibit precisely the contradiction.

Lemma 5.3.3. *There exists a smooth constant speed embedding $\Phi : \mathbb{S}^1 \to \Gamma \subset \mathbb{R}^2$ such that there is a $\vec{u}^* \in \mathbb{R}^2$*

$$\int_0^{2\pi}\Phi\cdot\vec{u}^*\,|D_\theta^2\Phi|^2 d\theta \neq \frac{1}{2\pi}\int_0^{2\pi}\Phi\cdot\vec{u}^*\,d\theta\int_0^{2\pi}|D_\theta^2\Phi|^2 d\theta.$$

Proof. We would like to construct an example of the shape illustrated in Figure 5.1. Let Γ_ε, $0<\varepsilon\ll 1$ denote a family of curves that are smooth in a neighborhood of scale $O(\varepsilon)$ at each node A, B and C, and coinciding with the triangle ΔABC elsewhere, and p_ε is defined as their arc-lengths. To illustrate the shape of Γ_ε, we translate one of A, B, C to the origin and rotate the triangle so that the triangle can be locally written as the function graph of

$$\psi(x) = \begin{cases}\cot(\alpha/2)x, & x\in[0,\varepsilon],\\ -\cot(\alpha/2)x, & x\in[-\varepsilon,0),\end{cases} \tag{5.3.16}$$

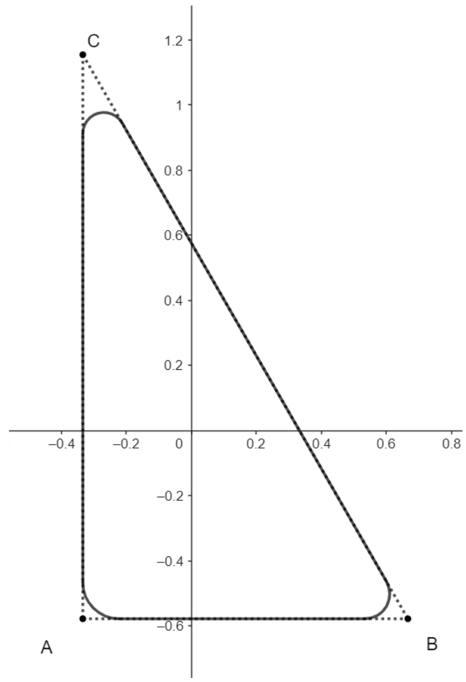

Figure 5.1: The example curve Γ is close to a right triangle with mass center 0; $A = \left(-\frac{1}{3}, -\frac{\sqrt{3}}{3}\right)$, $B = \left(\frac{2}{3}, -\frac{\sqrt{3}}{3}\right)$, $C = \left(-\frac{1}{3}, \frac{2\sqrt{3}}{3}\right)$.

where $\alpha \in [0, 2\pi]$ denotes the open angle of the cone at the chosen node. We would like to construct Γ_ε's by replacing ψ by some function u satisfying

$$\begin{cases} u''(x) = \phi_\varepsilon(x), & x \in [-\varepsilon, \varepsilon], \\ u'(-\varepsilon) = -u'(\varepsilon) = -\cot(\alpha/2), \\ u(-\varepsilon) = u(\varepsilon) = \cot(\alpha/2)\varepsilon, \end{cases} \tag{5.3.17}$$

where we would like to choose $\phi_\varepsilon \in C_0^\infty(-\varepsilon, \varepsilon)$ to be an even function satisfying $\phi_\varepsilon(x) \equiv K > 0$ on $x \in (-\varepsilon + \varepsilon^2, \varepsilon - \varepsilon^2)$ and $0 \le \phi_\varepsilon \le K$ elsewhere. Observe that the replacement of ψ by u gives us a smooth curve near the chosen node.

Before computing (5.3.12), let us compute some basic qualities of u. By compatibility condition, we have

$$(2\varepsilon + O(\varepsilon^2))K \approx \int_{-\varepsilon}^{\varepsilon} \phi_\varepsilon(x)dx = 2\cot(\alpha/2), \tag{5.3.18}$$

which implies that

$$K = \frac{\cot(\alpha/2)}{\varepsilon} + O(1). \tag{5.3.19}$$

On the other hand, we have the derivative of u has the form

$$\begin{aligned} u'(x) &= \int_{-\varepsilon}^{x} \phi_\varepsilon(y)dy - \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} \phi_\varepsilon(z)dz \\ &= Kx + O(\varepsilon) \\ &= \frac{\cot(\alpha/2)}{\varepsilon}x + O(\varepsilon). \end{aligned} \tag{5.3.20}$$

Collecting these values, we are now able to compute (5.3.12). Observe that because our refinement of triangle ΔABC is only at scale ε, we have the following asymptotics (where $\Phi_\varepsilon(s)$ is some unit speed reparametrization of Γ_ε)

$$p_\varepsilon \approx 3 + \sqrt{3}, \int \Phi_\varepsilon ds \int |D_s^2 \Phi_\varepsilon|^2 ds = O(K^2\varepsilon^2) = O(1), \tag{5.3.21}$$

and

$$\int \Phi_\varepsilon |D_s^2 \Phi_\varepsilon|^2 ds = \int_{B(A,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^2 dsA + \int_{B(B,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^2 dsB + \int_{B(C,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^2 dsC + O(1). \tag{5.3.22}$$

Using (5.3.20) and (5.3.17), we have

$$\begin{aligned} \int_{B(A,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^2 ds &= \int_{-\varepsilon}^{\varepsilon} \frac{|u''|^2}{(1+|u'|^2)^3}\sqrt{1+|u'|^2}dx \\ &= \int_{-\varepsilon}^{\varepsilon} \frac{|\phi_\varepsilon|^2}{(1+|u'|^2)^{5/2}}dx \\ &= K^2 \int_{-\varepsilon}^{\varepsilon} \frac{dx}{\left(1+(Kx)^2\right)^{5/2}} + O(1) \\ &= \left(\frac{\cot(\alpha_A/2)}{\varepsilon}\right)^2 \int_{-\varepsilon}^{\varepsilon} \frac{dx}{\left(1+\left(\frac{\cot(\alpha_A/2)}{\varepsilon}x\right)^2\right)^{5/2}} + O(1) \\ &= \frac{\cot(\alpha_A/2)}{\varepsilon} \int_{-\cot(\alpha_A/2)}^{\cot(\alpha_A/2)} \frac{dx}{(1+x^2)^{5/2}} + O(1). \end{aligned} \tag{5.3.23}$$

Plugging $\alpha_A = \pi/2$, we have

$$\begin{aligned} \int_{B(A,10\varepsilon)} \kappa^2 ds &= \frac{\cot(\pi/4)}{\varepsilon} \int_{-\cot(\pi/4)}^{\cot(\pi/4)} \frac{dx}{(1+x^2)^{5/2}} + O(1) \\ &= \frac{1}{\varepsilon}\int_{-1}^{1} \frac{dx}{(1+x^2)^{5/2}} + O(1) \\ &= \frac{5\sqrt{2}}{6}\cdot\frac{1}{\varepsilon} + O(1). \end{aligned} \tag{5.3.24}$$

Similarly, we have

$$\begin{aligned}\int_{B(B,10\varepsilon)} \kappa^2 ds &= \frac{\cot(\alpha_B/2)}{\varepsilon} \int_{-\cot(\alpha_B/2)}^{\cot(\alpha_B/2)} \frac{1}{(1+x^2)^{5/2}} dx + O(1) \\ &= \frac{\cot(\pi/6)}{\varepsilon} \int_{-\cot(\pi/6)}^{\cot(\pi/6)} \frac{dx}{(1+x^2)^{5/2}} + O(1) \\ &= \frac{\sqrt{3}}{\varepsilon} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+x^2)^{5/2}} + O(1) \\ &= \frac{9}{4} \cdot \frac{1}{\varepsilon} + O(1),\end{aligned} \tag{5.3.25}$$

and

$$\begin{aligned}\int_{B(C,10\varepsilon)} \kappa^2 ds &= \frac{\cot(\alpha_C/2)}{\varepsilon} \int_{-\cot(\alpha_C/2)}^{\cot(\alpha_C/2)} \frac{1}{(1+x^2)^{5/2}} dx + O(1) \\ &= \frac{\cot(\pi/12)}{\varepsilon} \int_{-\cot(\pi/12)}^{\cot(\pi/12)} \frac{dx}{(1+x^2)^{5/2}} + O(1) \\ &= \frac{2+\sqrt{3}}{\varepsilon} \int_{-\sqrt{3}-2}^{\sqrt{3}+2} \frac{dx}{(1+x^2)^{5/2}} + O(1) \\ &= \frac{41\sqrt{2}+25\sqrt{6}}{24} \cdot \frac{1}{\varepsilon} + O(1).\end{aligned} \tag{5.3.26}$$

Combining (5.3.21), (5.3.22), (5.3.24), (5.3.25) and (5.3.26), we may make the conclusion that when $\varepsilon > 0$ is very small,

$$\begin{aligned}\int \Phi_\varepsilon |D_s^2 \Phi_\varepsilon|^2 ds - \frac{1}{p_\varepsilon} \int \Phi_\varepsilon\, ds \int |D_s^2 \Phi_\varepsilon|^2 ds &= \frac{1}{\varepsilon} \left(\frac{5\sqrt{2}}{6} A + \frac{9}{4} B + \frac{41\sqrt{2}+25\sqrt{6}}{24} C \right) + O(1) \\ &= \frac{1}{\varepsilon} \left(-\frac{1}{3} \cdot \frac{5\sqrt{2}}{6} + \frac{2}{3} \cdot \frac{9}{4} - \frac{1}{3} \cdot \frac{41\sqrt{2}+25\sqrt{6}}{24}, \star \right) + O(1) \\ &\approx \frac{1}{\varepsilon} (0.5487, \star) \neq 0.\end{aligned} \tag{5.3.27}$$

The proof is done by choosing $\Phi(\theta) = \Phi_\varepsilon\left(\frac{p_\varepsilon \theta}{2\pi}\right)$ for a small $\varepsilon > 0$.

□

5.3.3 The proof of the second part of Theorem 5.1.1

Similar to Section 3, we prove the theorem by giving a contradiction to the following hypothesis:

Hypothesis 5.3.4. For all closed paths Γ_t in the space of plane curves driven by a vector field $\vec{V}_t = \vec{V}_t^{\perp}$,

$$\int_0^1 \int_{\Gamma_t} \left(1 + |\vec{H}_t|^2\right) \vec{H}_t \cdot \vec{V}_t \, d\mathcal{H}^1 dt = 0. \tag{5.3.28}$$

Observe that the first term on the left-hand side of Hypothesis (5.3.28) is 0 by using the first variation formula of perimeters. By using (5.3.4) we can rewrite the second term of the left-hand side as (where C is a computable constant)

$$\int_0^1 \int_{\Gamma_t} |\vec{H}_t|^2 \vec{H}_t \cdot \vec{V}_t \, d\mathcal{H}^1 dt = C \int_0^1 \frac{1}{l_t^5} \int_0^{2\pi} |\partial_\theta^2 \Phi_t(\theta)|^2 \partial_\theta^2 \Phi_t(\theta) \cdot \dot{\Phi}_t(\theta) d\theta dt. \tag{5.3.29}$$

Introducing $\tilde{\Phi}_t = \Phi_t / l_t$, we have

$$\begin{aligned}\frac{1}{l_t^5} \int_0^{2\pi} |\partial_\theta^2 \Phi_t(\theta)|^2 \partial_\theta^2 \Phi_t(\theta) \cdot \dot{\Phi}_t(\theta) d\theta &= \frac{1}{l_t} \int_0^{2\pi} |\partial_\theta^2 \tilde{\Phi}_t(\theta)|^2 \partial_\theta^2 \tilde{\Phi}_t(\theta) \cdot \dot{\tilde{\Phi}}_t(\theta) d\theta \\ &\quad - \partial_t \left(\frac{1}{l_t}\right) \int_0^{2\pi} |\partial_\theta^2 \tilde{\Phi}_t(\theta)|^2 \partial_\theta^2 \tilde{\Phi}_t(\theta) \cdot \tilde{\Phi}_t(\theta) d\theta.\end{aligned} \tag{5.3.30}$$

Lemma 5.3.5. *If Hypothesis 5.3.4 holds true for all closed paths in the space of plane curves, then given any smooth curve* $\Gamma \in \mathcal{C}_2$ *and its unit speed parametrization* $\Phi : \mathbb{S}^1 \to \Gamma \subset \mathbb{R}^2$, *we have*

$$\int_0^{2\pi} |D_\theta^2 \Phi|^2 D_\theta^2 \Phi d\theta = 0. \tag{5.3.31}$$

Proof. Integrating both sides of (5.3.30) with respect to time $t \in [0,1]$ we obtain by Hypothesis 5.3.4

$$\int_0^1 \frac{1}{l_t} \int_0^{2\pi} |\partial_\theta^2 \tilde{\Phi}_t(\theta)|^2 \partial_\theta^2 \tilde{\Phi}_t(\theta) \cdot \dot{\tilde{\Phi}}_t(\theta) d\theta - \partial_t \left(\frac{1}{l_t}\right) \int_0^{2\pi} |\partial_\theta^2 \tilde{\Phi}_t(\theta)|^2 \partial_\theta^2 \tilde{\Phi}_t(\theta) \cdot \tilde{\Phi}_t(\theta) d\theta dt = 0. \tag{5.3.32}$$

Fixing $\tilde{\Phi}_t$, and replacing Φ_t in (5.3.30) by $\tilde{l}_t \tilde{\Phi}_t$ for an arbitrary smooth 1-periodic function $\tilde{l}_t > 0$, we observe by integration by parts in t in (5.3.32) that $\tilde{\Phi}_t$ should satisfy

$$\partial_t \int_0^{2\pi} |\partial_\theta^2 \tilde{\Phi}_t|^2 \partial_\theta^2 \tilde{\Phi}_t \cdot \tilde{\Phi}_t(\theta) d\theta + \int_0^{2\pi} |\partial_\theta^2 \tilde{\Phi}_t|^2 \partial_\theta^2 \tilde{\Phi}_t \cdot \dot{\tilde{\Phi}}_t(\theta) d\theta = 0. \tag{5.3.33}$$

Replacing $\tilde{\Phi}_t$ above by $\tilde{\Phi}_t + \vec{u}$ for some $\vec{u} \in \mathbb{R}^2$, we see that $\tilde{\Phi}_t$ should satisfy

$$\partial_t \int_0^{2\pi} |\partial_\theta \tilde{\Phi}_t|^2 \partial_\theta^2 \tilde{\Phi}_t d\theta = 0. \tag{5.3.34}$$

Similar to Section 3, path-connectivity of $\mathcal{C}_2$ and the computation about circles imply (5.3.31).

□

Now let us construct a counterexample to (5.3.31).

Lemma 5.3.6. *There exists a smooth constant speed embedding* $\Phi : \mathbb{S}^1 \to \Gamma \subset \mathbb{R}^2$ *such that*

$$\int_0^{2\pi} |D_\theta^2 \Phi|^2 D_\theta^2 \Phi d\theta \neq 0. \tag{5.3.35}$$

Proof. We would like to construct a curve of the shape in Figure 5.2.

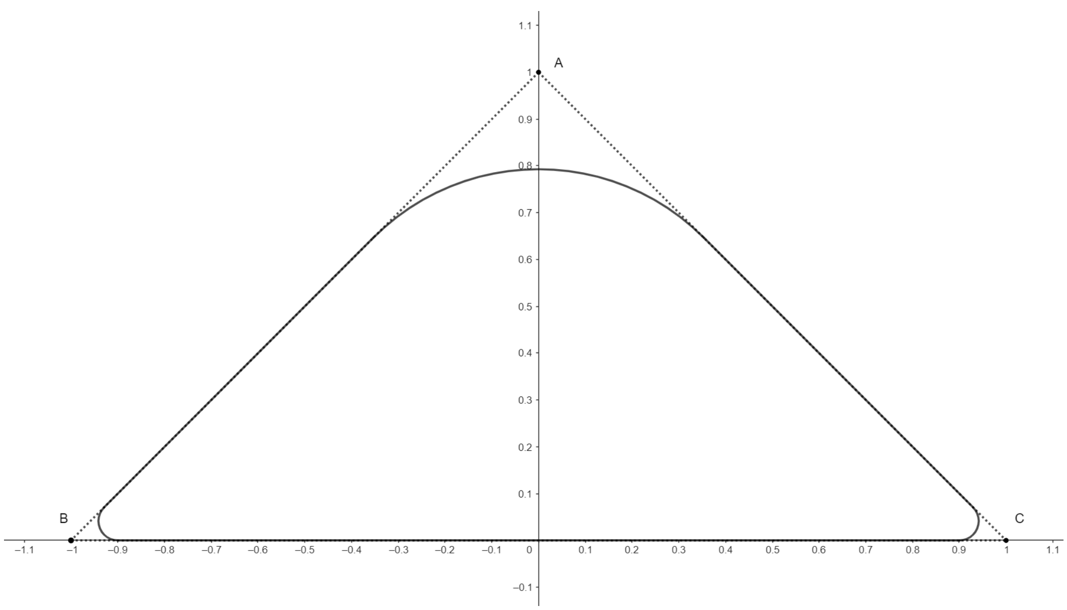

Figure 5.2: The curve is close to the right triangle ΔABC, with $O(1)$ mollification at $A = (0,1)$, and $O(\varepsilon)$ mollification at $B = (-1,0)$ and $C = (1,0)$ for some $\varepsilon > 0$ small.

Similar to the proof of lemma (5.3.3), we mollify the conic points A, B and C by locally replacing the curve by the function graph of the solution to equation (5.3.17). Near point A, the mollification is at scale $O(1)$, but near B and C the mollifications are at scale $O(\varepsilon)$. Let Γ_ε denote the family of the mollified curves, $4.2022 \lessapprox p_\varepsilon < 2 + 2\sqrt{2}$ be their arc-lengths and $\Phi^\varepsilon(s) : [0, p_\varepsilon] \to \Gamma_\varepsilon \subset \mathbb{R}^2$ be a family of unit speed reparametrization of the mollified curves at scale ε, we have the following asymptotics

$$\begin{aligned}\int |D_s^2\Phi^\varepsilon|^2 D_s^2\Phi^\varepsilon ds &= \int_{B(B,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^3\vec{N}ds + \int_{B(C,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^3\vec{N}ds + O(1) \\ &=: \tilde{B} + \tilde{C} + O(1).\end{aligned} \tag{5.3.36}$$

Because the curve is symmetric with respect to the vertical axis, we observe that $\tilde{B}$ and $\tilde{C}$ share the same vertical component and $\tilde{B}+\tilde{C}$ has zero horizontal component. Moreover, since the mollified curve near C (or B) is locally symmetric with respect to the middle-angle line passing through C (or B), the vector $\tilde{C}$ has a fixed direction that is not parallel to the horizontal line, and hence it suffices to show that

$$\left|\int_{B(C,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^3\vec{N}ds\right| \gg O(1). \tag{5.3.37}$$

In fact, after rotation and translation of the curve so that C is at the origin and the curve is locally of the form (5.3.16), we have

$$\begin{aligned}\left|\int_{B(C,10\varepsilon)\cap\Gamma_\varepsilon} \kappa^3\vec{N}ds\right| &= \int_{-\varepsilon}^{\varepsilon} \frac{(u'')^3}{(1+(u')^2)^{9/2}}dx \\ &\approx \left(\frac{\cot(\alpha_C/2)}{\varepsilon}\right)^2 \int_{-\cot(\alpha_C/2)}^{\cot(\alpha_C/2)} \frac{dx}{(1+x^2)^{9/2}} \\ &\gg O(1).\end{aligned} \tag{5.3.38}$$

The proof is done by choosing $\Phi(\theta) = \Phi^\varepsilon\left(\frac{p_\varepsilon\theta}{2\pi}\right)$ for a small $\varepsilon > 0$.

□

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